Elements of Machine Design I

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Cover design: *Vijay K. Goyal*
Editor-in-chief: *Vijay K. Goyal*

Printed at University of Puerto Rico at Mayagüez Press.
Mayagüez, Puerto Rico
USA
5–15–2012

Type Set in Times Roman 12 pt using LaTeX.

Dedication

To the Almighty God: Father, Son Yeshua, and the Holy Spirit;
Math, Science and Engineering show the Wonders of God, who revealed himself through Yeshua;
If God cannot be found then all the knowledge is meaningless.
"Through Him all things were made; without Him nothing was made that has been made."
– BIBLE: John 1:3

To my wife Maricelis, my son Jeremiah, and my daughter Naarah;
They are God's blessing to my life.

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Preface

There are many books on Elements of Machine Design but none with the unique approach enclosed in this book. This book has the purpose of creating a unique blend of mechanical engineering analysis and design using commonly used theories. By no means this book is intended to substitute other textbooks, rather to complement the existing textbook. Many thanks to Instructor Ezequiel Medici for his inputs for this book, they were invaluable. Also many thanks to Dr. David Serrano for providing problems for dynamic loading failure theories.

Course Syllabus

1. Instructor

1a. Dr. Vijay K. Goyal, Associate Professor of the Mechanical Engineering Department

1b. Office: L-207

1c. Office Hours: M W F: 11:30a - 1:30p; or by appointment

1d. Office Phone: (787) 832-4040 ext. 2111/3659 (Please do not call at home nor at my cell phone)

1e. E-mail: vijay.goyal@upr.edu

2. General Information

2a. Course Number: INME 4011

2b. Course Title: Design of Machine Elements I

2c. Credit-Hours: Three of lecture and computation included

2d. Classroom: L-242A

3. Course Description

Application of the fundamentals of statics, dynamics, strength of material, and material science to the design of machine members and other mechanical elements.

4. Pre/Co-requisites

4a. Material Sciences (INME 4107)

4b. Mechanics of Materials (INGE 4019)

4c. Mechanism Design (INME 4005)

5. Textbook, Supplies and Other Resources

5a. Class notes are posted on the class website. The official course textbook is the course website: http://www.me.uprm.edu/vgoyal/inme4011.html

5b. Collins, J. A., *Mechanical Design of Machine Elements and Machines*, 2003, John Wiley and Sons, New York, NY.

5c. Hamrock, B. J., Schmid, S. R., and Jacobson, B., *Fundamentals of Machine Elements*, 2005, Second Edition, Mc-Graw Hill, New York, NY.

5d. Juvinall, R. C., and Marsheck, K. A., *Fundamentals of Machine Component Design*, 2000, John Wiley and Sons, New York, NY.

5e. *Mischke, C. R., and Budynas, R, G., *Shigley's Mechanical Engineering Design*, 2011, Ninth Edition, Mc-Graw Hill, New York, NY.

5f. Thomas, G. B., Finney R. L., Weir, M. D., and Giordano F. R., *Thomas Calculus, Early Transcendental Update*, 2003, Tenth Edition, Addison-Wesley, Massachusetts.

*The topics covered in this course mainly follow the material from this book.

6. Purpose

The purpose of this course is to teach students how to apply the fundamental knowledge acquired during their mechanical engineering courses, combined with fundamentals of the basic sciences, and determine the dimensions of the machine elements to sustain the given load so that the composite machine can function without failure.

7. Course Goals

The course will be divided into seven specific topics divided into chapters. Each unit has the purpose to help the student understand and grasp the basic concept in mechanical engineering problems.

7a. Applied Elasticity. After completing this topic students should be able to:

- i) Understand what are stresses
- ii) Find principal stresses: Eigenvalue
- iii) Find principal stresses: Mohr's circle
- iv) identify and solve plane stress problems
- v) Derive Octahedral and Von Mises Stresses
- vi) Understand what are strains
- vii) find the principal strains
- viii) identify and solve plane strain problem
- ix) Learn the stress-strain relationship for isotropic elastic materials
- x) Understand the Saint-Venant's Principle
- xi) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project.

7b. Material Selection. After completing this topic students should be able to:

- i) Use Ranked-ordered table for best material selection
- ii) Ashby Charts for best material selection
- iii) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project

7c. Design and Analysis of Beams. After completing this topic students should be able to:

- i) Euler-Bernoulli Beam Theory
- ii) Stresses for a flexure beam under pure bending
- iii) Stresses for a flexure beam under axial load
- iv) Stresses for a flexure beam under both pure bending and axial load
- v) Sign convention for stress resultants on a beam cross section
- vi) Shear, Moment and Load diagrams
- vii) Slope and deflection diagrams
- viii) Shear stress due to torsional loading
- ix) Both combined torsional and shear loading
- x) Stress concentration factors

xi) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project

7d. Load and Deflection Analysis. After completing this topic students should be able to:

i) Obtain the exact deflection and slope profiles.
ii) Apply Castigliano's Second Theorem to solve statically determinate problems
iii) Apply Castigliano's Second Theorem to Solve statically indeterminate problems
iv) Obtain the effective stiffness to obtain load and static deflection relationships.

7e. Uncertainty Analysis. After completing this topic students should be able to:

i) Identify the various uncertainties in designing machine components.
ii) Obtain design safety factors
iii) Understand what is margins of safety, and how it is used
iv) Reliability in engineering design (optional)
v) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project

7f. Failure theories for Static Loading. After completing this topic students should be able to:

i) Static theories for ductile and brittle materials.
ii) Basics of fracture mechanics
iii) Stability of Beam-Columns
iv) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project

7g. Failure theories for Dynamic Loading. After completing this topic students should be able to:

i) Free Vibrations: natural frequencies
ii) Impact Loading
iii) Fatigue: Cyclic Loading
iv) Different fatigue theories
v) S-N Diagram
vi) Cumulative Damage
vii) Apply concepts of this unit to the most critical location of the most critical machine element of their course design project

In addition to the above units, all students will demonstrate the ability to describe the context of the report (introduction), describe clearly and precisely the procedures used (methodology), report verbally and visually the findings (results), interpret the findings (analysis of results), justify the solutions persuasively (conclusions), and propose recommendations. The students will demonstrate the ability to make effective oral presentations and written reports using appropriate computer tools.

8. Requirements

8a. Requirements: In order to succeed in the course students are expected to:

- should attend all class sessions and be punctual
- on a daily basis check the class website

- use a non-programmable calculator
- do all homework
- practice all suggested problems
- take all exams
- submit all work in English
- be ready to ask any questions at the beginning of every class session
- and obtain a minimum of 69.5% in the course

8b. **Grading Distribution:** Total course points are 100% and are distributed as follows:

Homework and Quizzes	30%
Mid-Term 1	20%
Mid-Term 2	25%
Mid-Term 3	25%
Final Examination	20%

** Final grade will be the sum of all homework, Midterms (I and II) and Final Examination minus the lowest grade from Mid-terms and Final examination. Students with a grade of “A” may be waived from the final exam.

Students should take advantage of bonus homework and projects to improve their grade because there will be no “grade curving” at the end of the semester. Your grade will be determined by the following fixed grade scale:

A	89.500 – 100+
B	79.500 – 89.499
C	69.500 – 79.499
D	49.500 – 69.499
F	0 – 49.499

Your final grade will be scaled based on the attendance. For an example, if you miss 3 classes and your final grade is 100% then your official final grade will be $100 * (42/45) = 93\%$. (NOT APPLIED TO OFF-CAMPUS STUDENTS).

8c. **Passing Criteria:** Students failing to provide a successful, high-standard, computer projects may not pass the course, as they are entitled to a grade of IF or ID, regardless of their progress in the mid-term examinations, homework, small projects, among other evaluation criteria. By successful we mean obtaining a percentage higher than 80% in overall projects. Moreover, a successful projects do not entitle the student to pass the course either.

8d. **Homework and Tests:** Only your own handwritten solutions, written legibly on one side of an $8.5'' \times 11''$ sheet of paper will be accepted for grading. In the case of computer assignment, a computer print out is acceptable whenever a copy of the code is included and well documented by hand. Students are encouraged to work together on the homework, but submissions must be the students own work. NO LATE HOMEWORK WILL BE ACCEPTED.

9. Laboratory/Field Work (If applicable)

9a. Cell phones/pagers: All students MUST turn off their cell phones and pagers at the beginning of each class session. By not doing so it is considered disrespectful and students will be asked to leave the class. Students who need to have their cell phones or pagers on at all time must inform the instructor at the beginning of the academic semester.

9b. Smoking: Smoking is not permitted in any area other than those areas designated for smoking.

9c. Electronic Devices: Radios, tape recorders, and other audio or video equipment are not permitted in the lab or classroom at any time. Students must consult with the professor at the beginning of the academic semester.

9d. Laptop Computers, Notebooks, PC-Tablets: Students can bring their personal computers to classroom. However this must not interfere with other student's work nor with the class session. Students with their personal computers are responsible for any problems with software versions or differences with the one available in the classroom.

10. Department/Campus Policies

10a. Class attendance: Class attendance is compulsory. The University of Puerto Rico at Mayagüez reserves the right to deal at any time with individual cases of non attendance. Professors are expected to record the absences of their students. Absences affect the final grade, and may even result in total loss of credits. Arranging to make up work missed because of legitimate class absence is the responsibility of the student. (Bulletin of Information Graduate Studies)

Students with three unexcused absences or more may be subject to a one or two final grade letter drop, according to the UPRM Rules and Regulations.

10b. Absence from examinations: Students are required to attend all examinations. If a student is absent from an examination for a justifiable and acceptable reason to the professor, he or she will be given a special examination. Otherwise, he or she will receive a grade of zero of "F" in the examination missed. (Bulletin of Information Graduate Studies)

In short, any student missing a test without prior notice or unexcused absence will be required to drop the course. There will be no reposition exam. At professor's judgment, those students with a genuine excuse will be given an oral 15–20 minutes oral comprehensive final exam and it will substitute the missed examination(s).

Under no circumstances should the students schedule interviews during previously set dates for examinations.

10c. Final examinations: Final written examinations must be given in all courses unless, in the judgment of the Dean, the nature of the subject makes it impracticable. Final examinations scheduled by arrangements must be given during the examination period prescribed in the Academic Calendar, including Saturdays. (see Bulletin of Information Graduate Studies).
Final examination in this course is used to substitute any mid-term grade up to 20%.

10d. Partial withdrawals: A student may withdraw from individual courses at any time during the term, but before the deadline established in the University Academic Calendar. (see Bulletin of Information Graduate Studies).

10e. Complete withdrawals: A student may completely withdraw from the University of Puerto Rico, Mayaguüez Campus, at any time up to the last day of classes. (see Bulletin of Information Graduate Studies).

10f. Disabilities: All the reasonable accommodations according to the Americans with Disability Act (ADA) Law will be coordinated with the Dean of Students and in accordance with the particular needs of the student.

Those students with special needs must identify themselves at the beginning of the academic semester (with the professor) so that he/she can make the necessary arrangements according to the Office of Affairs for the Handicap. (Certification #44)

10g. Ethics: Any academic fraud is subject to the disciplinary sanctions described in article 14 and 16 of the revised General Student Bylaws of the University of Puerto Rico contained in Certification 018-1997-98 of the Board of Trustees. The professor will follow the norms established in articles 1-5 of the Bylaws.

The honor code will be strictly enforced in this course. Students are encouraged to review the honor system policy which has been placed on the class website. All assignments submitted shall be considered graded work unless otherwise noted. Thus all aspects of the course work are covered by the honor system. Any suspected violations of the honor code will be promptly reported to the honor system. Honesty in your academic work will develop into professional integrity. The faculty and students of UPRM will not tolerate any form of academic dishonesty. MUST BE TAKEN SERIOUSLY. Any violation may result in an automatic "F" in the course and such behavior will be reported to the Dean's office of the College of Engineering.

11. General Topics

11a. Exam and Presentation Dates: (These dates may be subject to change)

Mid-Term 1:
Topics 1–3
Review session: Class Time
Exam Date: Posted on class website

Mid-Term 2: Topics 4–5
Review session: Class Time
Exam Date: Posted on class website

Mid-Term 3:
Topics 6–8
Review session: Class Time
Exam Date: Posted on class website

Final Examination:
Comprehensive
Review session: Class Time
Exam Date: Posted on class website

Syllabus is subject to changes

Acknowledgments

There are many people who have made this work possible. First and foremost, I am mostly thankful to Yeshua, for giving me the opportunity to live in this time. All my success I give to him for He has been my strength and inspiration at all times.

Secondly, I express my special appreciation to my wife Maricelis, my son Jeremiah, and my daughter Naarah for their support and inspiration behind this effort. I could not have completed this task without their prayers, love, understanding, encouragement, and support.

Thirdly, I would like to thank all the graduate students who collaborated to complete this book. In addition, many thanks to the invaluable inputs from the students who used the manuscript form of this book during the 2002–2012period at the University of Puerto Rico at Mayagüez (UPRM).

Authors want to thank The MathWorks™ Book Program for providing us a complementary recent version of MATLAB® to complete this book and allowing us use their software as the means of problem solving tool.

Lastly, I cannot leave behind all the people who have given their suggestions to this work, such as Dr. Paul A. Sundaram, Professor of Department of Mechanical Engineering at UPRM, whom I consider my mentor. Special thanks to all the friends who encouraged and helped me achieve this goal.

God bless and thank you all,

Vijay K. Goyal

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Chapter 1
Engineering Design

Most of the engineering problems involve design and analysis. Some to a greater extend than others. Thus here we briefly discuss the difference between design and analysis, followed by a description of the design process itself. The design process is an event and every event should be planned; thus, we include a description of how to build and use a Gantt Chart. In the last two section a description on how to write a proposal and a report is included.

It should be highlighted that design is creative process: Albert Einstein said: "Imagination is more important than knowledge, for knowledge is finite whereas imagination is infinite."

1.1 Mechanical Design

The main objective of any engineering design project is the fulfillment of some human need or desire. Broadly, *engineering* may be described as a judicious blend of science and art in which natural resources, including energy sources, are transformed into useful products, structures, or machines that benefit humankind. *Science* may be defined as any organized body of knowledge. *Art* may be thought of as a skill or set of skills acquired through a combination of study, observation, practice, and experience, or by intuitive capability or creative insight. Thus engineers utilize or apply scientific knowledge together with artistic capability and experience to produce products or plans for products.

Mechanical design is creating new devices or improving existing ones in an attempt to provide the "best", or "optimum" design consistent with the constraints of time, money, and safety, as dictated by the application and the market place. In other words, mechanical design may be defined as an iterative decision-making process that has as its objective the creation and optimization of a new or improved mechanical engineering system or device for the fulfillment of a human need or desire, with due regard for conservation of resources and environmental impact.

1.2 Design Process

1.2.1 Design vs. Analysis

Analysis and design, although closely related, are different in nature. The analysis problem is concerned with determining behavior of an existing system, or trial system being designed for the given task.

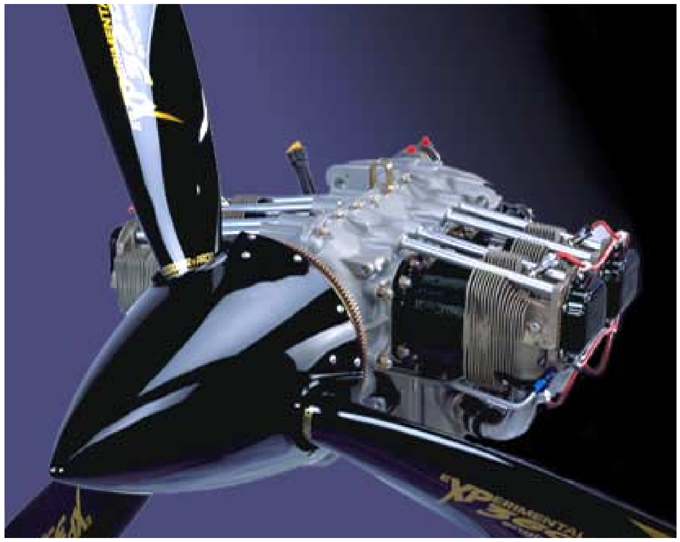


Figure 1.1: Aircraft engine by superior air parts.

Determination of the behavior of the system implies calculations of its response under the specified inputs. On the other hand, the design problem is concerned in calculating sizes and shapes of various parts of the system to meet performance requirements.

The design process of any given system can be a trial and error procedure. First we estimate a design and then analyze it to verify if it performs according to the specifications. If it does, we have a feasible design (acceptable design). We may still want to change it to improve its performance. If the trial design does not work, we need to change it to come up with an acceptable system. In both these cases, we must be able to analyze designs to make further decisions. Thus analyzing capability must be available in the design process.

1.2.2 Design Levels

There are different levels of challenges that the engineering faces when designing. These stages are adaptive design, development design, and new design.

(1) *Adaptive design* is concerned with those design activities that require no special knowledge or skill because the design is a minor modification to an existing design. The problems presented are easily solved by a designer with ordinary technical training.

(2) *Development design* is concerned with those design activities that require considerable scientific training. The engineer must have a design ability because the design starts from an existing design but the final outcome may differ markedly from the initial product.

(3) *New design* is heavily based on innovation because the design is completely new in its nature. Only a small number of designs are new designs.

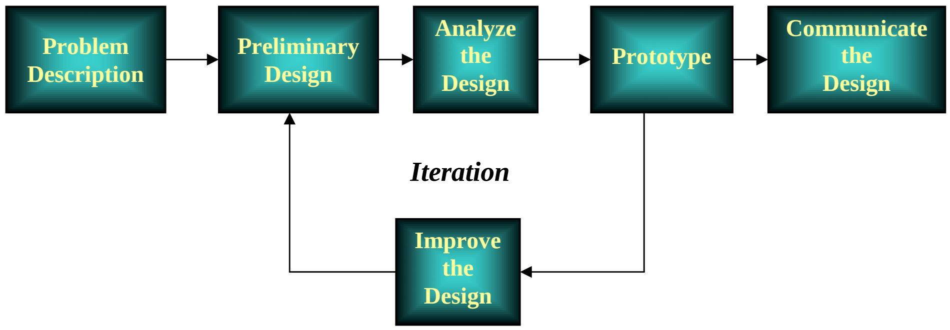


Figure 1.2: Six Stages of the design process

1.2.3 Design Process

The design process usually results in a set of drawings, calculations and reports, and the system can be built based on these. We shall use a systems engineering model to describe the design process. Fig. 1.2 shows that the design is an iterative process. By iterative we mean analyzing several trial systems in a sequence before an acceptable design is obtained. Thus designer's experience, intuition and ingenuity are required in the design of systems in most fields of engineering. Engineers, usually strive to different connotations for different systems. The design process should be a well-organized activity and this process involves nine steps. Here we regroup these nine steps into five major stages.

I. **PROJECT DESCRIPTION.** The first stage is to define precisely specifications for the system. Considerable interaction between the engineer and the sponsor is usually necessary to quantify the system specifications. This stage can be summarized into four steps:

   (1) Recognize needs and set goals – *what must be done to resolve the need*

   (2) Perform market analysis – *what is already available in the market and what they have to offer*

   (3) Establish a function analysis – *need statement and where the problem/need stands in the whole system*

   (4) Identify all specifications and constraints – *list of all pertinent data and parameters that tend to control the design and guide it toward the desired goal.*

II. **PRELIMINARY DESIGN.** The second important stage in the process is to come up with a preliminary design of the system. Various concepts for the system are studied. Since this must be done in a relatively short time, highly idealized models are used. Various subsystems are identified and their preliminary designs estimated. This stage can be summarized as follows:

   (5) Brainstorming – *Generation of alternative solution to the stated goal. (Brainstorming)*

III. **ANALYSIS.** The third stage in the process is to carry out a detailed design for all subsystems. Decisions made at this stage generally affect the final appearance and performance of the system. At the end of the preliminary design phase, a few promising concepts needing further analysis are identified. To evaluate various possibilities, this must be done for all the promising concepts identified in the previous step. Systematic optimization methods can aid the designer in accelerating the detailed design process. At the end of the process, a description of the system is available in the form of reports and drawings. This stage can be summarized in in three steps:

(6) Evaluation of alternatives – *Through a discussion and evaluation process, the design is selected.*

(7) Analysis and Optimization – *Test the selected design against the physical laws.*

IV. **PROTOTYPE.** The following stage may or may not be necessary for all systems. These involve a prototype system fabrication and its testing. The steps are necessary when the system has to be mass produced or human lives are involved. These may appear to be the final steps in the design process. However, they are not, because during tests the system may not perform according to specifications. Therefore, specifications may have to be modified or, other concepts may have to be studied. In fact, this re-examination may be necessary at any step of the design process.

(8) Experiment and/or Testing – *The design on paper is transformed into a physical reality.*

The iterative process has to be continued until an acceptable system has evolved. Depending on the complexity of the system, the process may take anywhere from a few days to several months or even years.

In most practical problems, the designers play a key role in guiding the process to acceptable regions. They must be an integral part of the process and use their intuition and judgment in obtaining the final design.

V. **COMMUNICATION.** Lastly, we have to be able to communicate our idea. One can come up with the most outstanding-innovative idea but if we are not able to sell the idea then we have failed to complete the design process. Thus when the final product is obtained, the last step consists in:

(9) Marketing – *Selling the idea to management or the client.*

1.2.4 Problem Description

According to the Webster's dictionary a problem is a question raised for consideration or solution, a question or situation that requires further investigation and a reason for conducting the experiment. According to the Google's dictionary definition is the act or process of stating a precise meaning or significance; formulation of a meaning. A definition is designed to settle a thing in its compass and extent; an explanation is intended to remove some obscurity or misunderstanding, and is therefore more extended and minute; a description enters into striking particulars with a view to interest or impress by graphic effect. It is not therefore true, though often said, that description is only an extended definition. "Logicians distinguish definitions into essential and accidental. An essential definition states what are regarded as the constituent parts of the essence of that which is to be defined; and an accidental definition lays down what are regarded as circumstances belonging to it, viz., properties or accidents, such as causes, effects, etc."–Whately. According to the Webster's dictionary description is the act of describing something; in other words, it is a full and detailed explanation of something.

Thus a problem description is one of the most important steps in engineering. Here the researcher should clearly describe the entire project in as much detail as possible. This scope of the project might be considered to be too broad in nature, and so the researcher may impose limitations or restrictions on the study.

A good problem description will help the employee to produce what we are interested in the most efficient time-frame. A poor job could lead to undesirable results. During the formulation of the problem description one uses the first stage of the design process. This description should include all the necessary information to complete a given task.

In short, we need to clearly state the "WHAT IS THE PROBLEM". When formulating the problem description make sure to keep in mind the following:

1. The engineer is a person who applies scientific knowledge to satisfy humankind's needs.
2. The first task of the engineer involves determining the real problems.
3. It is necessary to formulate a clear, exact statement of the problem in engineering words and symbols.
4. Vague statements from the costumer usually result in bad design.
5. Before an engineer can define the problem properly, he or she must recognize all the problems that exist.

1.2.5 Decision Making

During the design process, one may generate more than one possible design to meet the customer needs. Many methods exist but the decision matrix method, or Pugh method, is one of the most common ones. This is a method for concept selection using a scoring matrix. It is implemented by establishing an evaluation team, and setting up a matrix of evaluation criteria versus alternative embodiments.

For an alternative to be considered it should meet the customer needs and seem feasible to the engineering group. Should it fail any of the above criteria it must be dropped. Those concepts that pass this screening process need to be evaluated with respect to each other, using a fixed criteria.

Usually, the options are scored relative to criteria using a symbolic approach (one symbol for better than, another for neutral, and another for worse than baseline). These get converted into scores and combined in the matrix to yield scores for each option. Effective for comparing alternative concepts Scores concepts relative to one another Iterative evaluation method Most effective if each member of a design team performs it independently and results are compared. Comparison of the scores generated gives insight into the best alternatives.

Now we shall highlight the steps in constructing the Pugh matrix:

1. **Choose or develop the criteria for comparison.** It is extremely important to examine the

customer requirements to generate a list of criteria for comparison. The criteria is a set of engineering requirements and targets and may have different levels of importance. Different weighing schemes, can be used.

The *first method* is based on assigning a scale from 0% to 100% to each criteria. Remember each criteria is assigned a weigh factor that corresponds to its importance relative to other criteria. However, when the weighing factors are added they should equal to 100%.

The *second method* is based on assigning an absolute factor, where each criteria is evaluated individually on a scale from 0 to 10. Other criteria will not interfere with the weigh factor used.

2. **Select the Alternatives to be compared.** The alternatives are the different ideas developed during concept generation. All concepts should be compared at the same level of generalization and in similar language. Some of the alternatives will be dropped because they do not satisfy the customer demands or they are not feasible. Through initial screening stages try to reduce your alternatives up to six, whenever possible. Remember all alternatives must be feasible candidates to enter the final stage.

3. **Generate Scores.** After careful consideration, the design team chooses a concept to become the benchmark or datum[1] against which all other concepts are rated.

   For each comparison, the product should be evaluated as being[2] better (+1), the same (0), or worse (−1) than the datum. A number of variations on scoring Pughs method exist. For example a seven level scale could be used for a finer scoring system where:

   +3 meets criterion extremely better than datum

   +2 meets criterion much better than datum

   +1 meets criterion better than datum

   0 meets criterion as well as datum

   −1 meets criterion not as well as datum

   −2 meets criterion much worse then the datum

   −3 meets criterion far worse than the datum

   If it is impossible to make a comparison, more information should be developed.

4. **Compute the total score.** Four scores will be generated, the number of plus scores, minus scores, the overall total and the weighted total. The overall total is the number of plus scores and the number of minus scores. The weighted total is the scores times their respective weighting factors, added up. The totals should not be treated as absolute in the decision making process but as guidance only.

   If the two top scores are very close or very similar, then they should be examined more closely to make a more informed decision.

[1] The datum is something known or assumed, an information from which conclusions can be inferred. Thus, a datum could be an existing design or one of the same alternatives under consideration. This reference could be an industry standard, an obvious solution to the problem, the most favorable concept, or a combination of subsystems.

[2] The matrix can be developed with a spreadsheet like Excel.

Example 1.1.

Toyota is one of the most popular brand cars in our society today. Prof. Goyal wants to buy a Toyota Car but has a hard time making-up his mind. Use the Pugh Matrix Method to help him decide between a Corolla, Prius and MR2 Spyder. Take Toyota Corolla as the Datum, Prius as the first car, and MR2 Spyder as the second car. Develop as least 15 different criteria and use both techniques (0%–100% and 0–10) discussed in class. Use criteria in safety, economy, environment, design, comfort, among others.[3]

(a)

(b)

(c)

Figure 1.3: Toyota Corolla, Toyota Prius and Toyota MR2 Spyder.

The following are the criteria definitions along with a description used to compare the three Toyota models on a -2 to $+2$ scale and the explanation of the nature of the scores given to each car. These were the parameters used in conjunction with Pugh's matrix method to compare the cars and be able to make a justified decision as to which model satisfies best Professor Goyal's needs. As a team we have considered the eighteen different criteria and these are:

1. **Fuel economy** This criterion is based on the average miles per gallon obtained by each car, both on the highway and the city. On a scale of 0–10 we gave this criterion a 10 and on a percentage basis we gave it a 20% weight due to the fact that ever increasing

[3] Solution thanks to Samuel Medina; Jose Borrero; Omar Bravo; and José Ortiz. First course in Machine Design, Fall 2004.

gas prices demands better technologies that reduce fuel consumption. On this test the Prius obtained the maximum rating of +2, due to the fact that it offers around 60mpg on the highway and 51mpg on the city, versus the MR2 Spyder's 32mpg and 26mpg and the Corolla's 38mpg and 29mpg. Since the MR2-S obtained a little bit less miles per gallon than our datum we gave it a score of −1.

2. **Base Price** The price category is as obvious as it sounds. Here we compare the three vehicles based on the price of their basic unit. This is a priority on every consumer's lists, except for those who earn in the six or seven figure salary range. On both scales base price was rated at 8. The Prius because of all of its technological and standard features it starts out at $20,295, which is about $4,500 more than the base price of the Corolla. For this reason we gave the Prius a score of −1. Meanwhile the MR2-S starts at $24,895 which is about $9,000 more than the Corolla, still not that expensive for the performance you can buy, but expensive enough to drive you away from it and make you want to drive a Corolla or a Prius, its score was −2.

3. **People capacity** The people capacity is the amount of passengers than can be accommodated in the vehicle at one time without folding any seats. On a scale of 0–10 we rated this criterion as a 6 and on a percentage basis we assigned to it 5%. This category does not rate that high because usually when people want to buy a new car all that they want is a car that can accommodate more than two people, this means no roadsters and no coupes. Here the Prius tied our datum with a maximum seating capacity of five that is why it obtained a score of 0, meanwhile the MR2-S only has seating for two so it got a score of −2.

4. **Styling** Looks or appearance of the car, consumer appeals. This is a subjective category which is solely based on our teams opinion of which car we would choose based on exterior appearance. Here we have to say that roadsters are sexier that anything else on the road, (well maybe not sexier than a Ferrari) that is why we gave a score of +2 to the MR2-S; on the other hand the Prius looks interesting enough to make you want to drive it so we gave it a score of +1.

5. **Safety (crash testing)** This criterion is based on the score given by the National Highway Traffic and Safety Administration (NHTSA) on the tests performed on the vehicles. This is a very important category that is most of the time overlooked by car buyers, but since we are very aware of its importance we gave it a score of 8, both on a scale of 0–10 and percentage wise. The Prius and the Corolla were basically tied in this category, with excellent protection on the driver and good protection on the passenger, the front and the rear of the car, for this reason we gave the Prius a score of 0. Meanwhile our team was not able to find any crash test data from any source for any of the production years of the MR2-S, so we were unable to compare this car to the other two under this criterion.

6. **Fit and finish** This criterion refers to the quality and craftsmanship demonstrated in the construction of the cars interior. This is one very important aspect that is certainly overlooked. As a car buyer you do not want to go and buy a vehicle that has poor quality interior materials, for example: cup holders that can not hold a soda can, a steering wheel whose color fades after one use, etc. On a scale of 0–10 we rated this category as a 7 and on a percentage basis as a 5%. One thing I have to say here and it is that all Toyota's are very well built and they have superb quality. Here we gave the

slight edge to the MR2-S due to the fact that it uses materials that look and feel just a little bit more expensive than those on the Prius and Corolla, this is why we gave the Prius a 0 and the MR2-S a +1.

7. **Comfort (Head room, Leg room, etc)** The criterion of comfort is based on the amount of space available to the driver and passenger's which makes the ride comfortable to the average Puerto Rican in height. This test was rated as a 7 in both of the scales, because you can never sacrifice comfort, unless you are in a race car and were are not evaluating any. The Prius offered just a little bit more space everywhere than the Corolla, but when you add little by little it amounts to a lot and that is why it obtained a score of +2. Meanwhile sports cars are generally a bit uncomfortable and the MR2-S is not the exception there is just enough space for two here and no space to stretch out, so we gave it a −2.

8. **Suspension type** The suspension type of the vehicle not only determines the handling characteristics of the car but it also determines the ride quality. This criterion is generally not important to 99% of the car buyers out there, but to us as mechanical engineers and performance car aficionados it is an important criterion, nonetheless we rate it as normal people and on a 0–10 scale we rated it as a 5 and on a percentage basis we rated it as 2%. Here the Prius and the Corolla are tied once again, they both have independent front suspension and front are rear sway bars; since there is no improvement from the Corolla to the Prius we gave it a 0. Meanwhile the MR2-S is a sports car, because of this it has independent four wheel suspension and well as front and rear sway bars, these features give this car much better handling characteristics than the other two vehicles, because of this we gave the MR2-S a score of +2.

9. **Power to weight ratio** The power to weight ratio measures of the cars ability to accelerate; it is obtained by dividing the cars horsepower by its weight. This criterion is especially important to people who care a bit about how well their machine stacks up against other cars in terms of the cars ability to use its power effectively, on a 0–10 scale we rated is as a 6 and on a percentage basis we rated it as 3%. Surprisingly the Prius did not fare so well here it scored a -2, because of the fact that it produces just a little over 110hp and weights almost 2900lbs, while the Corolla produces 130hp and weights just a bit over 2600lbs. On the other hand the MR2-S produces 138hp and only weights 2200lbs, this earned it a score of +2.

10. **Power train** The power train category is a comparison between the horsepower and torque of the engines of the trim level vehicles selected. The car with the better power train offers more speed and acceleration to its driver. This category is not important to all car buyers out there, most people just want the power to haul some stuff around or maybe even go up a hill, this is why it is scored at 7 on a 0–10 scale and as a 3% on the percentage scale. The MR2-S with its 138hp and 125ft-lbs of torque gets a score of +2, while the Prius with its 110hp combination (70hp from the gasoline engine and about 40hp from the electric motor) gets a score of +1, because the cvt extracts every bit of power available of the engines.

11. **Performance** The performance criterion was based on the average time that it takes each of the vehicles being evaluated to accelerate from 0-60mph. This category does not matter much to all people, most car buyers want a car that can get them from point A to point B, for this reason we gave it a score of 5 on a 0–10 scale and 4% on the

percentage scale. Once again the MR2-S wins with a time around 7.0 secs. The Corolla hits 60mph around 9.0 seconds, while the Prius does this around 10.37 seconds. The score here was +2 for the MR2-S and −1 for the Prius.

12. **Braking distance** This criterion is based on the amount of feet that it takes the car to stop from 60-0mph. This aspect is especially important because of safety reasons, because good brakes always come in handy whenever you have to execute an unexpected stop we assigned a score of 8 on a 0–10 scale and a score of 2% on a percentage basis. Since the MR2-S has four wheel disk brakes it outperforms both the Prius and the Corolla by 12 to 30 feet. The MR2-S stops in 119ft, while the Corolla and the Prius stop at 150ft and 131.65ft respectively. Because of the facts presented above we gave the MR2-S a score of +2 and the Prius a score of +1.

13. **Turning cycle** This category measures the manoeuvrability of the car in tight spaces. This category was rated at 4 on a 0–10 scale and as a 1% on the percentage scale. Car buyers certainly do not go into the market looking for the car with the tightest turning radius but they certainly do not want the turning radius of a Ford F-350. This category was won by the Toyota Prius which has a turning circle of 34.1ft and the MR2-S came in second with a turning circle of 34.8ft. The scores here were +2 and +1 respectively.

14. **Storage capacity** The storage capacity is the space available to carry luggage. This category was rated as a 5 on a 0–10 scale and as a 3% on the percentage scale, due to the fact that when people go out there on the market to look for a car they are not looking for 30 cubic feet of cargo space, instead all that they want is enough space to haul around the groceries. In spite of its small looking body the Prius offers almost three more cubic feet of cargo space than the Corolla and almost 15 more than the MR2-S. The score here was +2 for the Prius and −2 for the MR2-S because of its insignificant 1.9 cubic feet of cargo space.

15. **Standard features and technologies** This category refers to the amenities you receive with the purchase of the base model of each one of the cars. On a score of 0–10 we rated this test as a 7 and on a percentage scale we rated it as a 5%. The Prius just had tons of standard features that were either optional or not available on the other two cars such as: audio control on the steering wheel and a continuously variable transmission among others. The scoring here was Prius +2 and MR2-S 0.

16. **Availability of spare parts** This criterion refers to the easiness with which you may find or source parts on the market for any of the three vehicles being compared. It was rated as a 5 on a 0–10 scale and as a 3% on a percentage scale, because the truth is that just a tiny fraction of car buyers out take into consideration this aspect whenever they are searching for a new car. Right now we gave the Prius a score of −2 here because of the fact that the technologies of the car are relatively new and there are few technicians and parts that can be replaced without having to take the car to the dealer and leave it there for a couple of weeks. Meanwhile the MR2-S has been on the market for five years now and there are enough parts on the market to fix any problem without having to take it to the dealer, because of this we gave it a score of +2.

17. **Warranty** This category refers to the basic manufactures warranty offered when you by the vehicle brand view from an authorized Toyota dealer. The vehicle's warranty or for that matter any type of warranty is very important to the consumer, because it gives you a certain sense of confidence about the product that you are purchasing it

also serves to tell you how reliable the product will tend to be. On the 0–10 scale we rated warranty as a 9 and 15% on the percentage scale. All Toyota cars are backed up by excellent warranties, 3 years or 36,000 miles is their basic warranty; 5 years or 60,000 miles on the drive train and a 5 year unlimited mileage warranty against corrosion. Both the Prius and the MR2-S got the same score here 0.

Table 1.1: Rank-Ordered

Criteria	Score (0–10)	A Prius	B MR2 Spyder	DATUM Corolla
1. Fuel Economy	10	2	−1	0
2. Base Price	8	−1	−2	0
3. People Capacity	6	0	−2	0
4. Styling	7	1	2	0
5. Safety (Crash Testing)	8	0	0	0
6. Fit & Finish (Quality)	7	0	1	0
7. Comfort	7	2	−2	0
8. Suspension Type	5	0	2	0
9. Power to Weight Ratio	6	−2	2	0
10. Powertrain	7	1	2	0
11. Performance	5	−1	2	0
12. Braking Distance	8	1	2	0
13. Turning Circle	4	2	1	0
14. Storage Capacity	5	2	−1	0
15. Standard Features	7	2	0	0
16. Availability of spare parts	5	−2	1	0
17. Warranty	9	0	0	0
Total	114	7	6	0
Total Positive	—	13	15	0
Total Negative	—	−6	−9	0
Weighted Total	—	52	30	0

Table 1.2: Weighted Method

Criteria	Weight (%)	A Prius	B MR2 Spyder	DATUM Corolla
1. Fuel Economy	20	2	−1	0
2. Base Price	8	−1	−2	0
3. People Capacity	5	0	−2	0
4. Styling	6	1	2	0
5. Safety (Crash Testing)	8	0	0	0
6. Fit & Finish (Quality)	5	0	1	0
7. Comfort	7	2	−2	0
8. Suspension Type	2	0	2	0
9. Power to Weight Ratio	3	−2	2	0
10. Powertrain	3	1	2	0
11. Performance	4	−1	2	0
12. Braking Distance	2	1	2	0
13. Turning Circle	1	2	1	0
14. Storage Capacity	3	2	−1	0
15. Standard Features	5	2	0	0
16. Availability of spare parts	3	−2	1	0
17. Warranty	15	0	0	0
Total	100	7	6	0
Total Positive	—	13	15	0
Total Negative	—	−6	−9	0
Weighted Total	—	59	−17	0

According to the results obtained using Pugh's Matrix Method, the Toyota Prius should be the car bought by Professor Goyal due to the high score obtained in the test. This car not only give our outstanding fuel economy but it also offer cutting edge styling and tons of standard features not available on the other two cars such as traction control and braking assistance, among others.

End Example □

1.2.6 Scheduling: Gantt Chart

Development of a new product according to the design process is always limited by the time available for the entire process. Thus for most engineering problems scheduling is an essential part of design. With a careful planning, deadlines are met and the customer satisfaction increases. Although many techniques exist for scheduling, here one of the most common methods will be briefly discuss. The method makes use of a Gantt Chart.

History

Henry Laurence Gantt (1861–1919) was a mechanical engineer, management consultant and industry advisor. Henry Laurence Gantt developed Gantt charts in the second decade of the 20th century. Gantt charts were used as a visual tool to show scheduled and actual progress of projects. Accepted as a commonplace project management tool today, it was an innovation of world-wide importance in the 1920s. Gantt charts were used on large construction projects like the Hoover Dam started in 1931 and the interstate highway network started in 1956.

Overview

A ***Gantt chart*** is a graphical representation of the duration of a project or tasks against the progression of time. A ***project*** is a set of activities which ends with specific accomplishment and which has (1) Non-routine tasks, (2) Distinct start/finish dates, and (3) Resource constraints (time/money/people/equipment).

Tasks are activities which must be completed to acheive project goal. Break the project into tasks and subtasks. Tasks have start and end points, are short relative to the project and are significant (not *going to library*, but rather, *search literature*). Use verb-noun form for naming tasks, e.g. *create drawings* or *build prototype*. Use action verbs such as *create*, *define* and *gather* rather than *will be made*. Each task has a duration. Very difficult to estimate durations accurately. Doubling your best guess usually works well.

Milestones are important checkpoints or interim goals for a project. Can be used to catch scheduling problems early. Name by noun-verb form, e.g. *report due*, *parts ordered*, *prototype complete*.

Your plan will evolve so be flexible and update on a regular basis. It also helps to identify risk areas for project, for example, things you don't know how to do but will have to learn. These are risky because you may not have a good sense for how long the task will take. Or, you may not know how long it will take to receive components you purchased for a project.

Basics

Gantt charts are a project planning tool that can be used to represent the timing of tasks required to complete a project. Because Gantt charts are simple to understand and easy to construct, they are used by most project managers for all but the most complex projects. Gantt charts are used for planning and monitoring progress.

The reason to use Gantt charts as a useful tools for planning and scheduling projects is that they allow to assess how long a project should take, lay out the order in which tasks need to be carried out, help manage the dependencies between tasks, and determine the resources needed. The reason to use Gantt charts when a project is under way is that they monitor progress, can immediately see what should have been achieved at a point in time, help manage the dependencies between tasks, and allow to see how remedial action may bring the project back on course.

In a Gantt chart, each task takes up one row. Dates run along the top in increments of days, weeks or months, depending on the total length of the project. The expected time for each task is represented by a horizontal bar whose left end marks the expected beginning of the task and whose right end marks the expected completion date. Tasks may run sequentially, in parallel or overlapping. Some Gantt charts include two extra columns between the task and the dates: the first column holds the duration of the activity and the second one which team members are responsible for such task. MS Excel is a good tool to create Gantt charts.

The first Gantt chart represents the planning of the events. A second one helps to compare the progress with the scheduled timing. Thus as the project progresses, the chart is updated by filling in the bars to a length proportional to the fraction of work that has been accomplished on the task. This way, one can get a quick reading of project progress by drawing a vertical line through the chart at the current date. Completed tasks lie to the left of the line and are completely filled in. Current tasks cross the line and are behind schedule if their filled-in section is to the left of the line and ahead of schedule if the filled-in section stops to the right of the line. Future tasks lie completely to the right of the line.

In constructing a Gantt chart, keep the tasks to a manageable number (no more than 15 or 20) so that the chart fits on a single page. More complex projects may require subordinate charts which detail the timing of all the subtasks which make up one of the main tasks. For team projects, it often helps to have an additional column containing numbers or initials which identify who on the team is responsible for the task.

Often the project has important events which you would like to appear on the project timeline, but which are not tasks. For example, you may wish to highlight when a prototype is complete or the date of a design review. You enter these on a Gantt chart as milestone events and mark them with a special symbol, often an upside-down triangle.

How to build a Gantt chart?

Basically, there are four steps in preparing a Gantt chart:

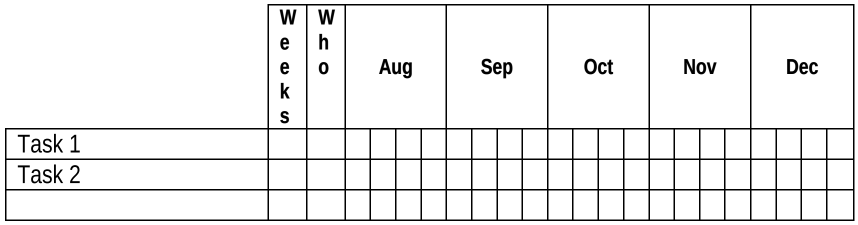
1. List all events or milestones in an ordered list, whenever possible.

2. Estimate the time required to establish each event (remember it is an estimate).

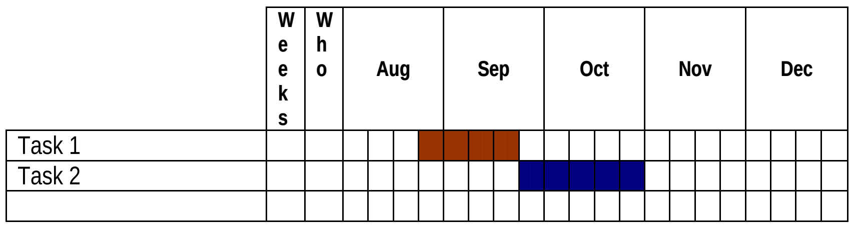
3. List the starting time and end time for each event.

4. Represent the information in a bar chart.

A Gantt chart is a matrix and it is:



(a) Setting up the Gantt chart.



(b) Using the Gantt chart.

Figure 1.4: Gantt chart in planning.

- constructed with a horizontal axis representing the total time span of the project, broken down into increments (days, weeks, or months)
- constructed with a vertical axis representing the tasks that make up the project.
- constructed with a graph area which contains horizontal bars for each task connecting the period start and period ending symbols.
- has variants such as:
  - Milestones: important checkpoints or interim goals for a project
  - Resources: for team projects, it often helps to have an additional column containing numbers or initials which identify who on the team is responsible for the task
  - Status: the projects progress, the chart is updated by filling in the task's bar to a length proportional to the amount of work that has been finished
  - Dependencies: an essential concept that some activities are dependent on other activities being completed first

It is greatly used in all project management groups and here is an example on how to use it. Figure 1.4(a) shows the planning of a task that starts in the August and ends in December.

For task 1 mark the time period when you will be working on it. Only shade that time interval. For instance if task one will begin at the end of August and conclude by the third week of September then represent this using a bar, as shown in Fig. 1.4(b).

In constructing a Gantt chart, keep the tasks to a manageable number (no more than 15 or 20) so that the chart fits on a single page. More complex projects may require subordinate charts which detail the timing of all the subtasks which make up one of the main tasks. For team projects, it often helps to have an additional column containing numbers or initials which identify who on the team is responsible for the task.

Often the project has important events which you would like to appear on the project timeline, but which are not tasks. For example, you may wish to highlight when a prototype is complete or the date of a design review. You enter these on a Gantt chart as milestone events and mark them with a special symbol, often an upside-down triangle.

1.3 Technical Communication

Communication is one of the most important skills every engineer should develop. The best idea or design might never be implemented, or even be considered, unless the designer can communicate the idea or design to the proper people, in a clear and appropriate way. The quality of your communication, be either oral or written, does not necessarily determine the quality of your work. However, the quality of your work is best understood through the communication process.

Mainly two ways exist through which we all communicate: Writing and oral presentations. Let us first talk about the major steps in the writing process and then we shall discuss about oral presentation techniques. These techniques are thoroughly discussed in Appendix **??**.

1.4 References

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1.5 Suggested Problems

Problem 1.1.

Professor Goyal prefers all the course homework be done by hand, with the exception of simple arithmetic calculations. Mr. Juan González is taking his Machine Design course this semester and needs to purchase a calculator. His is on a tight-budget constraint and would pay for what is only necessary for the course. He is very picky regarding the colors: Juan prefers dark color calculators only. He wants to purchase a calculator that would last him for the next 20-years at least. In addition, the guarantee will also play a big role. Using the Pugh's method help him decide from the following options:

1. HP 48gII graphing calculator
2. HP 30s scientific calculator
3. Texas Instrument TI-92 Plus Graphic Calculator
4. DATUM: Casio FX-250

The following are the criteria definitions used to compare the models: -3 to +3 scale and the explanation of the nature of the scores given to each car. Use the ranked-order method and compare the results with those obtained using the weighted method.

□

Chapter 2
Applied Linear Elasticity

Instructional Objectives of Chapter 2

After completing this chapter, the reader should be able to:

1. Explain and apply the concepts of stress and strain.
2. Determine the principal stresses and strains, and their principal planes.
3. Identify the various stress and strain measures.
4. Understand linear elasticity as applied to aerospace structures.

The stress and strain states at critical locations in a structural component are extremely important to evaluate the safety of structural components. Most of the concepts covered in this chapter are no longer solved by hand but with the use of computer software. However, a theoretical understanding how the state of stress is expressed at a point and state of strain at the neighborhood of the same point may be crucial and important. Thus, we will begin our discussion with the theory of stresses, followed by the theory of strains. In the next chapter, we will discuss how both stresses and strains are related.

2.1 Theory of Stresses

The concept of stress began with the study of strength and failure of solids. The state of stress in a solid body can be defined as a measure of force intensity, at a point, acting within the solid. It has units of:

$$\frac{[\text{Force}]}{[\text{Length}]^2}$$

We should handle stress at a point carefully because it is directionally dependent. By directionally dependent we do not meant stress depends on the direction of the but that it depends on the coordinate system.

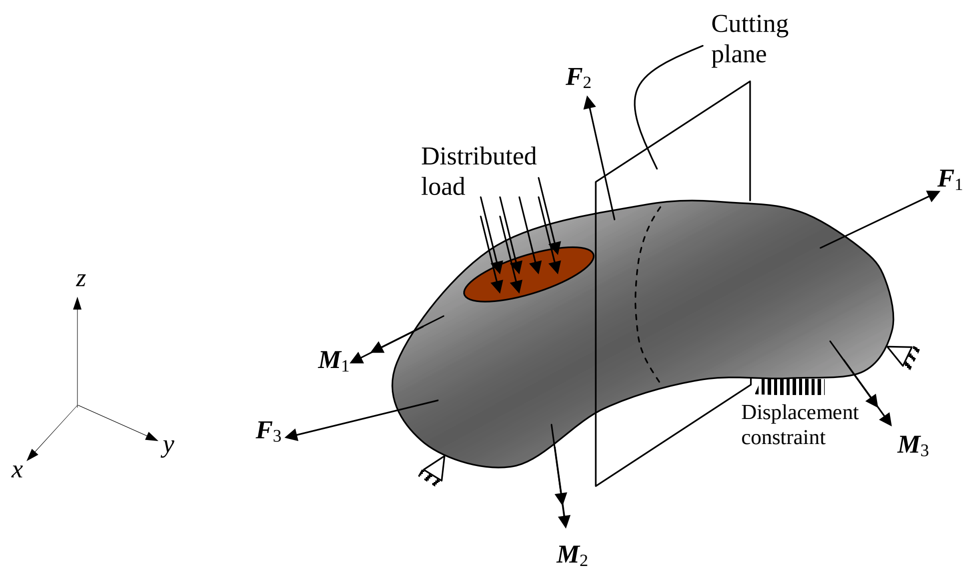


Figure 2.1: Solid body in equilibrium.

2.1.1 State of Stress at a Point

In order to better understand the concept of stress, let us consider a solid body in equilibrium, loaded and constrained in an arbitrary fashion, as shown in Fig. 2.1. Let an arbitrary plane cut the solid body as shown in Fig. 2.2. We define the small element of area of a cutting plane through point P in the solid be defined as δA. The infinitesimal plane has a unit normal $\underline{\hat{\mathbf{n}}}$ and encloses the point of interest.

Let us denote $\delta\underline{\mathbf{F}}^{(\mathrm{n})}$ as the force exerted by the rest of the body on δA of a cutting plane through point P. Likewise, the couple exerted at point P will be denoted as $\delta\underline{\mathbf{M}}^{(\mathrm{n})}$. Both $\delta\underline{\mathbf{F}}^{(\mathrm{n})}$ $\delta\underline{\mathbf{M}}^{(\mathrm{n})}$ are resultants loads and are, in general, different in magnitude and orientation from the corresponding resultants acting

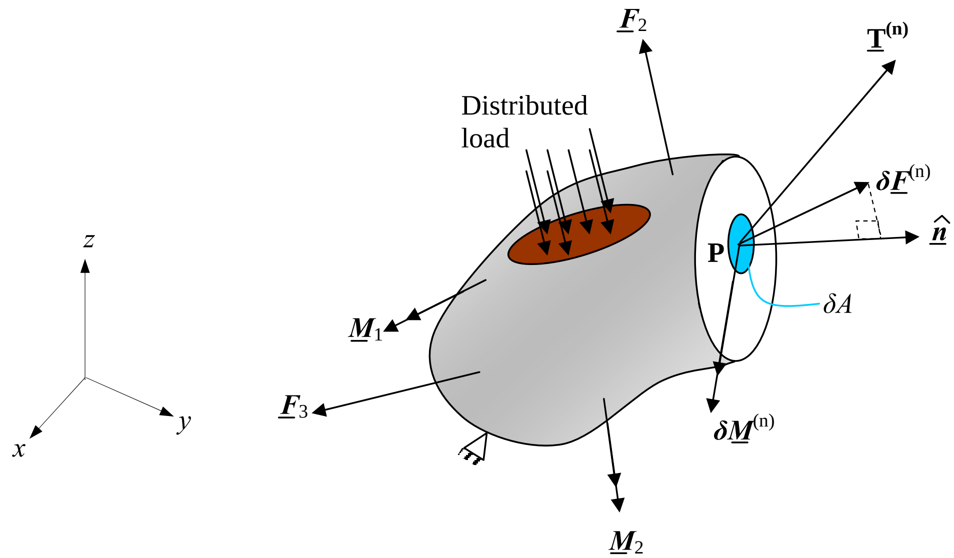


Figure 2.2: Solid body in equilibrium sliced with an arbitrary plane.

on the entire surface of the cut. Now, the average stress vector on this area can be obtained by

$$\underline{\mathbf{T}}^{(\mathrm{n})}_{\mathrm{ave}} = \frac{\delta \underline{\mathbf{F}}^{(\mathrm{n})}}{\delta A_{(\mathrm{n})}} \tag{2.1}$$

As we continuously reduce the small surface area, $\delta A_{(\mathrm{n})}$, the area becomes an element of infinitesimal area $\delta A_{(\mathrm{n})} \rightarrow 0$ and we approach the point P. As the small surface becomes a point, the force and couple acting on the element keep decreasing in magnitude and changing in orientation whereas the normal to the surface remains the unit vector, $\hat{\underline{\mathbf{n}}}$ of constant direction in space. This limiting process gives place to the concept of stress vector, which is defined as

$$\lim_{\delta A_{(\mathrm{n})} \rightarrow 0} \frac{\delta \underline{\mathbf{F}}^{(\mathrm{n})}}{\delta A_{(\mathrm{n})}} = \underline{\mathbf{T}}^{(\mathrm{n})} \tag{2.2}$$

where $\underline{\mathbf{T}}^{(\mathrm{n})}$ is called the stress vector and in the cartesian coordinate is defined as

$$\underline{\mathbf{T}}^{(\mathrm{n})} = T_{\mathrm{x}}\,\hat{\underline{\mathbf{i}}} + T_{\mathrm{y}}\,\hat{\underline{\mathbf{j}}} + T_{\mathrm{z}}\,\hat{\underline{\mathbf{k}}} = \begin{Bmatrix} T_{\mathrm{x}} \\ T_{\mathrm{y}} \\ T_{\mathrm{z}} \end{Bmatrix} \tag{2.3}$$

The existence of the stress vector is a fundamental assumption of continuum mechanics. In this limiting process, we assume that the couple becomes smaller and smaller:

$$\lim_{\delta A_{(\mathrm{n})} \rightarrow 0} \frac{\delta \underline{\mathbf{M}}^{(\mathrm{n})}}{\delta A_{(\mathrm{n})}} = 0 \tag{2.4}$$

The above is also a logical assumption of continuum mechanics because in the limiting process, both forces and moment arms become increasingly small. Forces decrease because the area on which they act decreases, and thus moment arms decrease because the dimensions of the surface decrease. At the limit, the couple is the product of a differential element of force by a differential element of moment arm, giving rise to a negligible, second order quantity.

We use the notation $\underline{\mathbf{T}}^{(\mathrm{n})}$ to emphasize the fact that the stress vector at a given point $\mathbf{P}$ in the continuum depends explicitly upon the particular surface, which is represented by the unit normal $\hat{\underline{\mathbf{n}}}$. Thus, the superscript $^{(\mathrm{n})}$ refers the normal of the surface where the stress vector is acting. We can obtain the normal, tangential and resultant components of the stress vector using vector algebra[1]. Hence we obtain the normal stress component as follows

$$\sigma_{\mathrm{nn}} = \underline{\mathbf{T}}^{(\mathrm{n})} \cdot \hat{\underline{\mathbf{n}}} \tag{2.5}$$

The stress vector in the normal stress component can be found by:

$$\underline{\mathbf{S}}_{\mathrm{nn}} = \sigma_{\mathrm{nn}} \hat{\underline{\mathbf{n}}} \tag{2.6}$$

The tangential component as follows

$$\sigma_{\mathrm{tt}} = \sqrt{\left\| \underline{\mathbf{T}}^{(\mathrm{n})} \right\|^2 - \sigma_{\mathrm{nn}}^2} \tag{2.7}$$

where

$$\left\| \underline{\mathbf{T}}^{(\mathrm{n})} \right\| = \sqrt{\underline{\mathbf{T}}^{(\mathrm{n})} \cdot \underline{\mathbf{T}}^{(\mathrm{n})}} \tag{2.8}$$

An alternative to the above approach can be to find the stress vector in the tangential directional and then taking the magnitude of the vector:

$$\underline{\mathbf{S}}_{\mathrm{tt}} = \underline{\mathbf{T}}^{(\mathrm{n})} - \underline{\mathbf{S}}_{\mathrm{nn}} \qquad \rightarrow \qquad \sigma_{\mathrm{tt}} = \left\| \underline{\mathbf{S}}_{\mathrm{tt}} \right\|, \qquad \left\| \underline{\mathbf{S}}_{\mathrm{tt}} \right\| = \sqrt{\underline{\mathbf{S}}_{\mathrm{tt}} \cdot \underline{\mathbf{S}}_{\mathrm{tt}}}$$

For an infinite number of cutting planes through point $\mathbf{P}$, each identified by a specific $\hat{\underline{\mathbf{n}}}$, there will be an infinite associated stress vectors $\underline{\mathbf{T}}^{(\mathrm{n})}$ for a given loading of the body. This total pair of the companion vectors $\underline{\mathbf{T}}^{(\mathrm{n})}$ and $\hat{\underline{\mathbf{n}}}$ at $\mathbf{P}$ define the state of stress at that point.

Now that we have defined the concept of stress, let us proceed to obtain the state of stress for the cartesian coordinate system: x-y-z. Let the stress vector act on three mutually orthogonal planes described by the unit base vectors $\hat{\underline{\mathbf{i}}}$, $\hat{\underline{\mathbf{j}}}$, and $\hat{\underline{\mathbf{k}}}$. Consider the x-plane (plane normal to the x-axis), with a differential area δA with the unit normal $\hat{\underline{\mathbf{i}}}$. Let $\delta\underline{\mathbf{F}}^{(\mathrm{x})}$ be the force acting by the rest of the body on

[1]For a review in vector algebra see Appendix A.

the enclosing x-plane. Then the stress vector $\underline{\mathbf{T}}^{(\mathrm{x})}$ is

$$\begin{aligned}
\underline{\mathbf{T}}^{(\mathrm{x})} &= \lim_{\delta A \to 0} \frac{\delta \underline{\mathbf{F}}^{(\mathrm{x})}}{\delta A} \\
&= \lim_{\delta A \to 0} \left\{ \frac{\delta F_x^{(\mathrm{x})}\, \hat{\underline{\mathbf{i}}} + \delta F_y^{(\mathrm{x})}\, \hat{\underline{\mathbf{j}}} + \delta F_z^{(\mathrm{x})}\, \hat{\underline{\mathbf{k}}}}{\delta A} \right\} \\
&= \lim_{\delta A \to 0} \left\{ \frac{\delta F_x^{(\mathrm{x})}}{\delta A}\, \hat{\underline{\mathbf{i}}} + \frac{\delta F_y^{(\mathrm{x})}}{\delta A}\, \hat{\underline{\mathbf{j}}} + \frac{\delta F_z^{(\mathrm{x})}}{\delta A}\, \hat{\underline{\mathbf{k}}} \right\} \qquad (2.9) \\
&= \lim_{\delta A \to 0} \left\{ \frac{\delta F_x^{(\mathrm{x})}}{\delta A} \right\} \hat{\underline{\mathbf{i}}} + \lim_{\delta A \to 0} \left\{ \frac{\delta F_y^{(\mathrm{x})}}{\delta A} \right\} \hat{\underline{\mathbf{j}}} + \lim_{\delta A \to 0} \left\{ \frac{\delta F_z^{(\mathrm{x})}}{\delta A} \right\} \hat{\underline{\mathbf{k}}} \\
&= \sigma_{\mathrm{xx}}\, \hat{\underline{\mathbf{i}}} + \sigma_{\mathrm{xy}}\, \hat{\underline{\mathbf{j}}} + \sigma_{\mathrm{xz}}\, \hat{\underline{\mathbf{k}}}
\end{aligned}$$

The component σ_{xx} is called the normal stress component in the x-direction. The other two components σ_{xy} and σ_{xz} act tangential to the plane and are called shear components. Using a similar process, we can obtain the stress in the y and z directions. Thus, the components of the stresses acting the x, y, and z direction are

$$\begin{aligned}
\underline{\mathbf{T}}^{(\mathrm{x})} = &\quad = \sigma_{\mathrm{xx}}\, \hat{\underline{\mathbf{i}}} + \sigma_{\mathrm{xy}}\, \hat{\underline{\mathbf{j}}} + \sigma_{\mathrm{xz}}\, \hat{\underline{\mathbf{k}}} \\
\underline{\mathbf{T}}^{(\mathrm{y})} = &\quad = \sigma_{\mathrm{yx}}\, \hat{\underline{\mathbf{i}}} + \sigma_{\mathrm{yy}}\, \hat{\underline{\mathbf{j}}} + \sigma_{\mathrm{yz}}\, \hat{\underline{\mathbf{k}}} \qquad (2.10) \\
\underline{\mathbf{T}}^{(\mathrm{z})} = &\quad = \sigma_{\mathrm{zx}}\, \hat{\underline{\mathbf{i}}} + \sigma_{\mathrm{zy}}\, \hat{\underline{\mathbf{j}}} + \sigma_{\mathrm{zz}}\, \hat{\underline{\mathbf{k}}}
\end{aligned}$$

All three stress vectors describe the state of stress at a given point.

Now, we express the unit vector acting on $\mathbf{P}$, for any arbitrary orthogonal planes, as follows:

$$\hat{\underline{\mathbf{n}}} = n_x\, \hat{\underline{\mathbf{i}}} + n_y\, \hat{\underline{\mathbf{j}}} + n_z\, \hat{\underline{\mathbf{k}}}$$

Thus, if we want the stresses acting in the x-direction:

$$T_x = \underline{\mathbf{T}}^{(\mathrm{x})} \cdot \hat{\underline{\mathbf{n}}} = \sigma_{\mathrm{xx}}\, n_x + \sigma_{\mathrm{xy}}\, n_y + \sigma_{\mathrm{xz}}\, n_z$$

Similarly, we can obtain the stresses acting in the y and z directions:

$$T_y = \underline{\mathbf{T}}^{(\mathrm{y})} \cdot \hat{\underline{\mathbf{n}}} = \sigma_{\mathrm{yx}}\, n_x + \sigma_{\mathrm{yy}}\, n_y + \sigma_{\mathrm{yz}}\, n_z$$

$$T_z = \underline{\mathbf{T}}^{(\mathrm{z})} \cdot \hat{\underline{\mathbf{n}}} = \sigma_{\mathrm{zx}}\, n_x + \sigma_{\mathrm{zy}}\, n_y + \sigma_{\mathrm{zz}}\, n_z$$

This shows that we need a total of nine stress components to completely describe the state of stress at a given point. Furthermore, the shear stresses will be expressed using τ instead of σ and σ will be used for normal stress only.

Cauchy's Stress Tensor

We can express the equations of stress quantities in each mutually orthogonal planes in a matrix form as follows

$$\begin{Bmatrix} T_{\mathrm{x}} \\ T_{\mathrm{y}} \\ T_{\mathrm{z}} \end{Bmatrix} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.11}$$

$$\underline{\mathbf{T}} = \underline{\boldsymbol{\sigma}} \cdot \underline{\hat{\mathbf{n}}}$$

where $\underline{\boldsymbol{\sigma}}$ is called the stress tensor and $\underline{\hat{\mathbf{n}}}$ is the unit normal to the plane. The stress tensor contains the nine stress components and we defined it as follows

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} \end{bmatrix} \tag{2.12}$$

As we can see, all normal and shear stress components fully characterize the state of stress at a point. We will later show that if the stress components acting on three orthogonal faces are known, it is possible to compute the stress components acting at the same point, on any face with an arbitrary orientation. Hence, the fact that the state of stress at a point is a complex concept: its complete definition requires the knowledge of nine stress components acting on three mutually orthogonal faces. This is quite different form the concept of a force: (i) a force is vector quantity that is characterized by its magnitude and orientation, (ii) a force can be defined by the three components of the force vector in a given coordinate system. Thus, the definition of a force requires three quantities, whereas the definition of the stress state requires nine quantities. In this context, a force is a vector, i.e. a first order tensor, whereas a state of stress is a second order tensor.

2.1.2 Stress Convention and Signs

We usually classify stresses into normal stresses and shear stresses: Normal stress, σ, are stress perpendicular (normal) to the plane on which they act; and shear stresses, τ, are stresses parallel to the plane on which they act.

Fig. 2.3 shows the complete definition of the state of stress at a point. Note that the positive direction of each stress component are given and the gray dashed arrows and numbers are on hidden faces. The stresses have two subscripts and these are interpreted as follows:

$$\tau_{\text{[plane where stress acts][direction of the stress]}}$$

The first of which indicates the direction of the plane on which it acts and the second of which indicates the direction of the stress in the plane. For an example, τ_{zx} is the shear stress acting on the z-plane

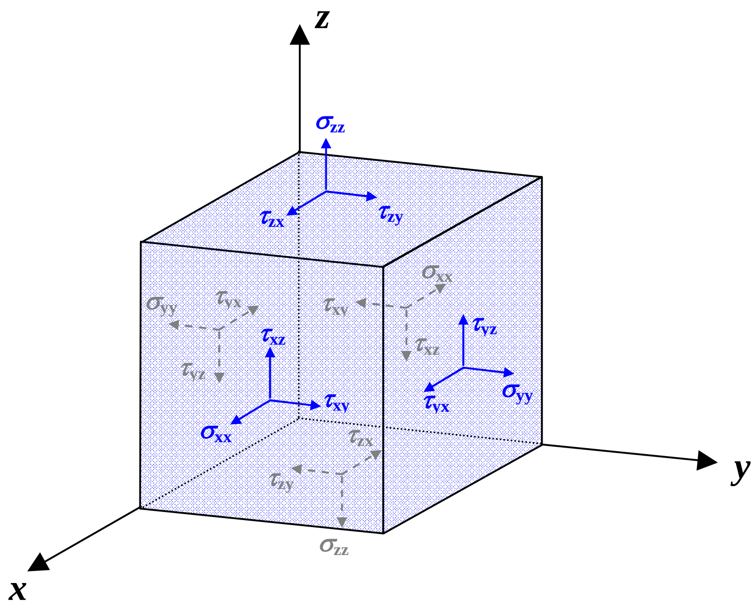


Figure 2.3: Complete definition of the state of stress at a point.

in the direction of x. In short, the convention we use is such that a stress component, acting on the x-plane, is positive if acting toward the positive x-axis.

If any stress component is positive the direction of the outer normal will be given as shown in Fig. 2.3. Based on our convention, if a normal stress is positive in its value then it will be in tension; if it is negative then it will be in compression. Positive (+) shear stresses act in the direction of an axis whose sign is the same as the sign of the axis in the direction of the outward drawn normal to the plane on which the shear stresses act.

2.1.3 Equilibrium

Volume Equilibrium

In general, the state of stress varies throughout the solid body, and hence, it is clear that the stresses acting on two parallel faces located a small distance apart are not equal. To better understand this, let us consider a small differential element with only shear stresses acting about the z-axis, as shown in Fig. 2.4.

In Fig. 2.4, the two faces of a differential volume element that are normal to y-axis. The normal stress component on the negative face at coordinate y is σ_{yy}, but the stress components on the positive face at coordinate $y + dy$ will be slightly different and we can write it as $\sigma_{yy}(y + dy)$. If $\sigma_{yy}(y)$ is an analytic function, it is then possible to express $\sigma_{yy}(y + dy)$ in terms of $\sigma_{yy}(y)$ using a Taylor series expansion to

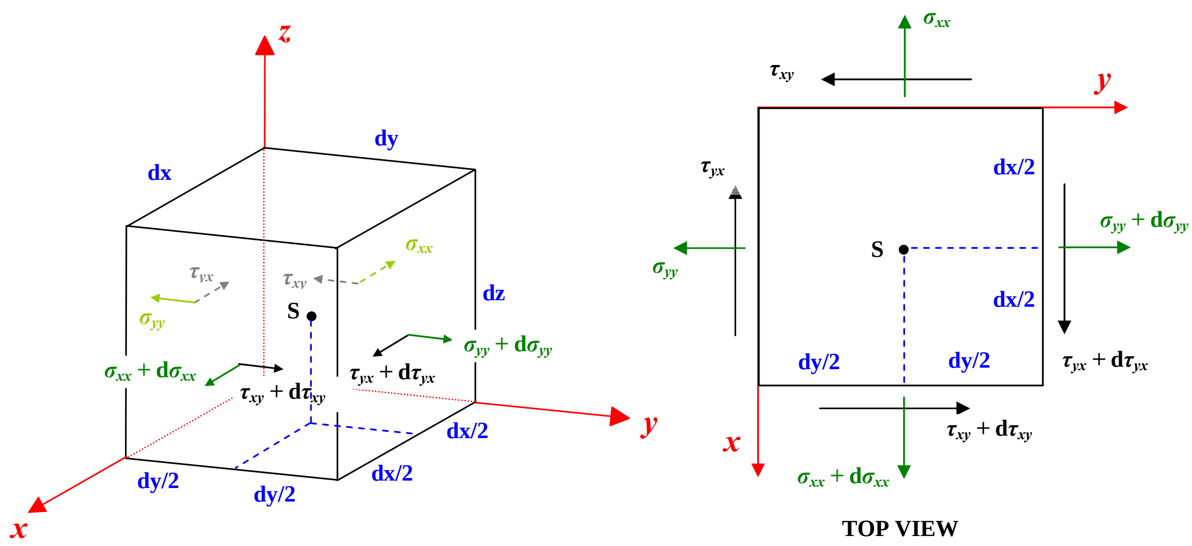


Figure 2.4: Shear stresses on the faces of an element at a point in an elastic body about the z-axis.

obtain

$$\sigma_{\rm yy}(y+dy) = \sigma_{\rm yy}\Big|_{\rm y+dy} = \sigma_{\rm yy} + {\rm d}\,\sigma_{\rm yy} = \sigma_{\rm yy}(y) + \frac{\partial \sigma_{\rm yy}}{\partial y}\Big|_{\rm y}\,{\rm dy} + \cdots + \text{higher order terms}$$

This expansion is a essential in the deriving the differential equations governing the behavior of a continuum such as a solid body. Since all differentials are infinitesimally small, we neglect all higher order terms. Hence, we can write the stress components on the positive face at coordinate $y + dy$ as

$$\sigma_{\rm yy}(y+dy) = \sigma_{\rm yy}(y) + \frac{\partial \sigma_{\rm yy}}{\partial y}\Big|_{\rm y}\,{\rm dy}$$

The same series expansion technique can be applied to all other directions and shear stress components.

Force Equilibrium

Let us assume that in addition to the internal loads, the solid is subject to body forces per unit volume, represented by a vector $\underline{\mathbf{b}}$ acting about its centroid. These body forces can be gravity forces, inertial forces, or forces of an electric or magnetic origin; the components of this body force vector resolved in the cartesian coordinate system as

$$\underline{\mathbf{b}} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

The units of the force vector are force per unit volume.

Now, let us consider the differential element of volume subjected to stress components acting on its six external faces and to body forces per unit volume. According to Newton's law, static equilibrium requires the sum of all the forces acting on this differential element to vanish. Considering all the forces acting along the direction of x-axis, the equilibrium condition is

$$-\sigma_{\rm xx}\,\mathrm{dy}\,\mathrm{dz} + \left(\sigma_{\rm xx} + \frac{\partial \sigma_{\rm xx}}{\partial x}\,\mathrm{dx}\right)\mathrm{dy}\,\mathrm{dz} - \tau_{\rm yx}\,\mathrm{dx}\,\mathrm{dz} + \left(\tau_{\rm yx} + \frac{\partial \tau_{\rm yx}}{\partial y}\,\mathrm{dy}\right)\mathrm{dx}\,\mathrm{dz} +$$

$$-\tau_{\rm zx}\,\mathrm{dx}\,\mathrm{dy} + \left(\tau_{\rm zx} + \frac{\partial \tau_{\rm zx}}{\partial z}\,\mathrm{dz}\right)\mathrm{dx}\,\mathrm{dy} + b_x\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = 0$$

The above equation represents an equilibrium of forces. Hence, the stress components must be multiplied by the surface area on which they act to yield the corresponding force; and the components of the body force per unit volume must be multiplied by the volume of the differential element, $dx\,dy\,dz$. Now simplifying the equilibrium condition, we get

$$\left(\frac{\partial \sigma_{\rm xx}}{\partial x}\,\mathrm{dx}\right)\mathrm{dy}\,\mathrm{dz} + \left(\frac{\partial \tau_{\rm yx}}{\partial y}\,\mathrm{dy}\right)\mathrm{dx}\,\mathrm{dz} + \left(\frac{\partial \tau_{\rm zx}}{\partial z}\,\mathrm{dz}\right)\mathrm{dx}\,\mathrm{dy} + b_x\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = 0$$

$$\left(\frac{\partial \sigma_{\rm xx}}{\partial x} + \frac{\partial \tau_{\rm yx}}{\partial y} + \frac{\partial \tau_{\rm zx}}{\partial z} + b_x\right)\mathrm{dx}\,\mathrm{dy}\,\mathrm{dz} = 0$$

Taking the limit as dx, dy, and dz approach zero (the differential volume approaches to a point)

$$\frac{\partial \sigma_{\rm xx}}{\partial x} + \frac{\partial \tau_{\rm yx}}{\partial y} + \frac{\partial \tau_{\rm zx}}{\partial z} + b_x = 0$$

Similarly reasoning along the y and z axis, we obtain the three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial \sigma_{\rm xx}}{\partial x} + \frac{\partial \tau_{\rm yx}}{\partial y} + \frac{\partial \tau_{\rm zx}}{\partial z} + b_x = 0$$

$$\frac{\partial \tau_{\rm xy}}{\partial x} + \frac{\partial \sigma_{\rm yy}}{\partial y} + \frac{\partial \tau_{\rm zy}}{\partial z} + b_y = 0$$

$$\frac{\partial \tau_{\rm xz}}{\partial x} + \frac{\partial \tau_{\rm yz}}{\partial y} + \frac{\partial \sigma_{\rm zz}}{\partial z} + b_z = 0$$

What the above three equations tell us is that a state of stress within an elastic continuum is statically admissible if and only if it satisfies the equilibrium equations.

Example 2.1.

Determine under what conditions the following state of stress within an elastic solid is statically admissible. Note that A, B, C are constants.

$$\begin{aligned}
\sigma_{xx} &= 2\,A\,x^2 \\
\sigma_{yy} &= 2\,C\,\left(4\,x^2 + y^2\right) \\
\tau_{xy} &= -4\,B\,x\,y \\
\tau_{yx} &= \tau_{xy} \\
\tau_{xz} &= \tau_{yz} = \tau_{zx} = \tau_{zy} = \sigma_{zz} = 0
\end{aligned}$$

Ignore all body forces.

Any state of stress satisfying the equilibrium equations is statically admissible. Assuming that body forces are negligible, the equilibrium equations can be written as follows

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \frac{\partial (2\,A\,x^2)}{\partial x} + \frac{\partial (-4\,B\,x\,y)}{\partial y} + \frac{\partial (0)}{\partial z} \\
&= 4\,A\,x - 4\,B\,x + 0 = 4\,x\,(A - B)
\end{aligned}$$

For all possible values of x, the first equilibrium equation is satisfied only if $A = B$.

$$\begin{aligned}
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \frac{\partial (-4\,B\,x\,y)}{\partial x} + \frac{\partial (8\,C\,x^2 + 2\,C\,y^2)}{\partial y} + \frac{\partial (0)}{\partial z} \\
&= -4\,B\,y + 4\,C\,y + 0 = 4\,y\,(B - C)
\end{aligned}$$

For all possible values of y, the first equilibrium equation is satisfied only if $B = C$.

$$\begin{aligned}
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \frac{\partial (0)}{\partial x} + \frac{\partial (0)}{\partial y} + \frac{\partial (0)}{\partial z} \\
&= 0 + 0 + 0 \\
&= 0 \qquad \text{satisfies third equilibrium equation}
\end{aligned}$$

Also, note that

$$\tau_{xy} = \tau_{yx}$$

for moment equilibrium. Hence, if $A = B = C$, then the state of stress satisfies all three equilibrium equations and also $\tau_{xy} = \tau_{yx}$ (for moment equilibrium). Under these conditions, the given stress state is a statically admissible one.

End Example □

Moment Equilibrium

In addition, the requirement of equilibrium of moments implies that the shear stresses on orthogonal planes at a point are equal in magnitude. To better understand this concept, consider a small differential element with only shear forces acting about the z-axis, as shown in Fig. 2.5.

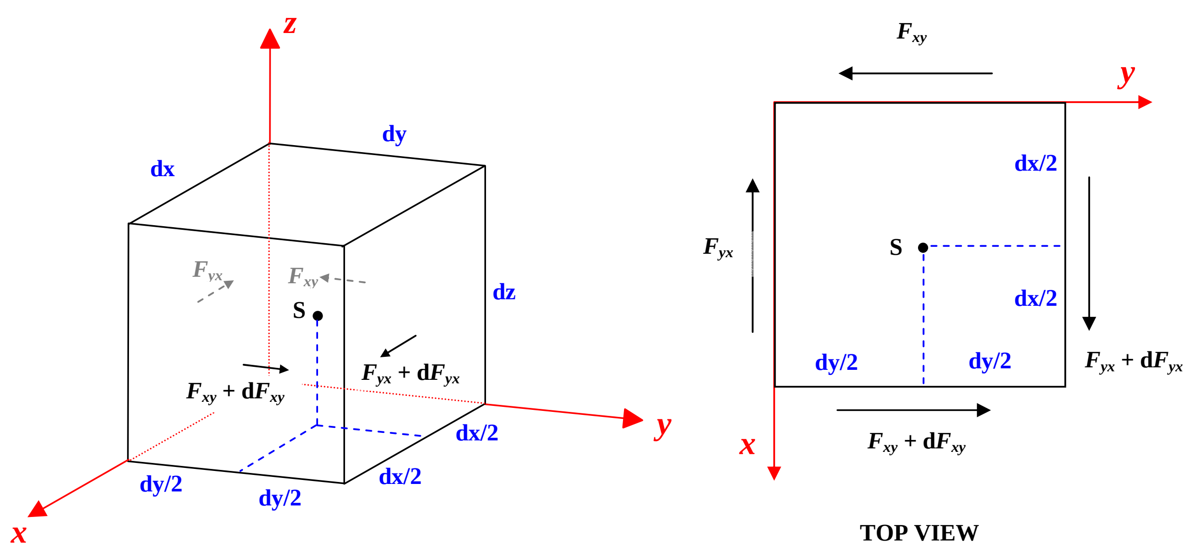


Figure 2.5: Shear forces on the faces of an element at a point in an elastic body about the z-axis.

Note that when taking moment about the z-axis at a point **S** in the middle of the differential element, all normal forces vanish. The small variation in the loads can be obtained as a result of a Taylor's series expansion,

$$F_{xy}\Big|_{x+dx} = F_{xy} + dF_{xy} = F_{xy} + \frac{\partial F_{xy}}{\partial x}\,dx$$

$$F_{yx}\Big|_{y+dy} = F_{yx} + dF_{yx} = F_{yx} + \frac{\partial F_{yx}}{\partial y}\,dy$$

Now, the moment about the z-axis at a point **S** in the middle of the differential element gives:

$$\left\{F_{\rm xy}\right\}\frac{\rm dx}{2}+\left\{F_{\rm xy}+{\rm d}\,F_{\rm xy}\right\}\frac{\rm dx}{2}-\left\{F_{\rm yx}\right\}\frac{\rm dy}{2}-\left\{F_{\rm yx}+{\rm d}\,F_{\rm xy}\right\}\frac{\rm dy}{2}=0$$

$$\left\{\tau_{\rm xy}\,{\rm dz}\,{\rm dy}\right\}\frac{\rm dx}{2}+\left\{\tau_{\rm xy}\,{\rm dz}\,{\rm dy}+\frac{\partial\tau_{\rm xy}}{\partial x}\,{\rm dx}\,{\rm dz}\,{\rm dy}\right\}\frac{\rm dx}{2}$$

$$-\left\{\tau_{\rm yx}\,{\rm dz}\,{\rm dx}\right\}\frac{\rm dy}{2}-\left\{\tau_{\rm yx}\,{\rm dz}\,{\rm dx}+\frac{\partial\tau_{\rm yx}}{\partial y}\,{\rm dy}\,{\rm dz}\,{\rm dx}\right\}\frac{\rm dy}{2}=0$$

Regrouping and simplifying,

$$\left\{\tau_{\rm xy}+\frac{\partial\tau_{\rm xy}}{\partial x}\,{\rm dx}\right\}{\rm dx}\,{\rm dy}\,{\rm dz}-\left\{\tau_{\rm yx}+\frac{\partial\tau_{\rm yx}}{\partial y}\,{\rm dy}\right\}{\rm dx}\,{\rm dy}\,{\rm dz}=0$$

$$\left\{\tau_{\rm xy}+\frac{\partial\tau_{\rm xy}}{\partial x}\,{\rm dx}-\tau_{\rm yx}-\frac{\partial\tau_{\rm yx}}{\partial y}\,{\rm dy}\right\}{\rm dx}\,{\rm dy}\,{\rm dz}=0$$

For all differential volume ($d{\rm Vol}={\rm dx}\,{\rm dy}\,{\rm dz}$) different from zero,

$$\tau_{\rm xy}+\frac{\partial\tau_{\rm xy}}{\partial x}\,{\rm dx}-\tau_{\rm yx}-\frac{\partial\tau_{\rm yx}}{\partial y}\,{\rm dy}=0$$

Taking the limit as dx, dy, and dz approach zero (the differential volume approaches to a point)

$$\tau_{\rm xy}-\tau_{\rm yx}=0\qquad\rightarrow\qquad\tau_{\rm xy}=\tau_{\rm yx}\tag{2.13}$$

Similarly, moment equilibrium about the y- and x-axis leads to

$$\tau_{\rm zx}=\tau_{\rm xz}\qquad\tau_{\rm zy}=\tau_{\rm yz}\tag{2.14}$$

Hence, this lead to a symmetric stress tensor, known as the stress tensor,

$$\underline{\boldsymbol{\sigma}}=\begin{bmatrix}\sigma_{\rm xx}&\tau_{\rm xy}&\tau_{\rm xz}\\ \tau_{\rm yx}&\sigma_{\rm yy}&\tau_{\rm yz}\\ \tau_{\rm zx}&\tau_{\rm zy}&\sigma_{\rm zz}\end{bmatrix}=\begin{bmatrix}\sigma_{\rm xx}&\tau_{\rm xy}&\tau_{\rm xz}\\ \tau_{\rm xy}&\sigma_{\rm yy}&\tau_{\rm yz}\\ \tau_{\rm xz}&\tau_{\rm yz}&\sigma_{\rm zz}\end{bmatrix}\tag{2.15}$$

The implication of these equalities is summarized by the principal of reciprocity of shearing stresses:

> *The shearing stresses acting in the direction normal to the common edge of two orthogonal faces must be equal in magnitude and be simultaneously oriented toward or away from the common edge.*

2.1.4 Surface Equilibrium: Cauchy's Stress Relation

We use the Cauchy's stress relation to relate the surface tractions at a point on the surface of the body to the inner stresses, or to determine the stress boundary conditions which must be satisfied at those points on the boundary where the tractions or surface forces are specified. Cauchy's relationship can be expressed as,

$$\left\{ \begin{array}{c} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{array} \right\} = \left[\begin{array}{ccc} \sigma_{\rm xx} & \tau_{\rm xy} & \tau_{\rm xz} \\ \tau_{\rm xy} & \sigma_{\rm yy} & \tau_{\rm yz} \\ \tau_{\rm xz} & \tau_{\rm yz} & \sigma_{\rm zz} \end{array} \right] \left\{ \begin{array}{c} n_x \\ n_y \\ n_z \end{array} \right\}$$

$$\underline{\mathbf{T}}^{(\rm s)} = \underline{\boldsymbol{\sigma}} \cdot \underline{\hat{\mathbf{n}}}_{(\rm s)} \tag{2.16}$$

where $\underline{\hat{\mathbf{n}}}_{(\rm s)}$ is the unit normal to the plane $\mathbf{s}$, $\underline{\boldsymbol{\sigma}}$ the stress tensor at the point in the plane $\mathbf{s}$, and $\underline{\mathbf{T}}^{(\rm s)}$ the total stress acting on $\underline{\hat{\mathbf{n}}}_{(\rm s)}$. It should be clear that $\underline{\mathbf{T}}^{(\rm s)}$ does not necessarily act in the direction of $\underline{\hat{\mathbf{n}}}_{(\rm s)}$.

Example 2.2.

The state of stress at a point in a structural component is given as

$$\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa}$$

(a) Show this state of stress on a differential element.

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} \end{bmatrix} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{xy}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{xz}} & \tau_{\mathrm{yz}} & \sigma_{\mathrm{zz}} \end{bmatrix} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa}$$

Must work in the following convention:

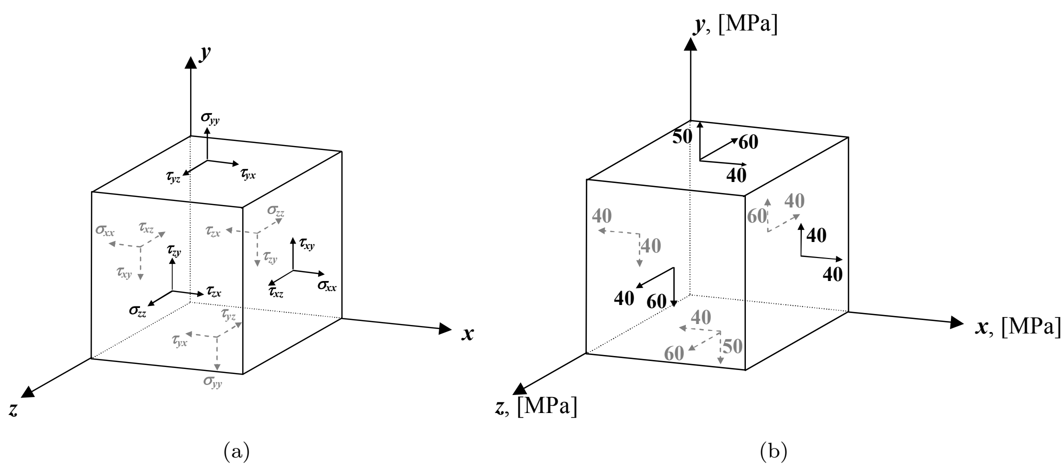
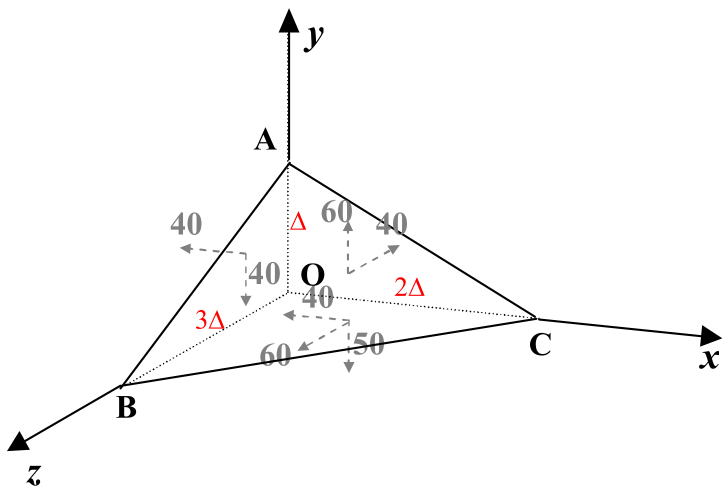


Figure 2.6: This is an infinitesimal element representing the state of stress for the given problem (NOTE: Units are part of the answer)

(b) Determine the stress vectors acting on the mutually orthogonal face **OAC**, **OCB**, **OBA**. Use both static equilibrium of stresses and Cauchy's relation (Cauchy's formula).

METHOD ONE: Using static equilibrium

The infinitesimal element in Fig. 2.6 is in equilibrium. Therefore, the sum of all forces



should be zero:

$$A_{(j)}\,\underline{\mathbf{T}}^{(j)} + A_{(-j)}\,\underline{\mathbf{T}}^{(-j)} = 0 \quad \rightarrow \quad A_{(j)}\,\underline{\mathbf{T}}^{(-j)} = -A_{(-j)}\,\underline{\mathbf{T}}^{(j)}$$

where j represents the direction

Since the area $A_{(-j)} = A_{(j)}$,

$$\underline{\mathbf{T}}^{(-j)} = -\underline{\mathbf{T}}^{(j)} \quad \text{where } j \text{ represents the direction}$$

Stress vector on face **OAC**

$$\underline{\mathbf{T}}^{(\text{-z})} = -\underline{\mathbf{T}}^{(\text{z})} = -\underline{\boldsymbol{\sigma}}\,\hat{\underline{\mathbf{n}}}_{(\text{z})} = -\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\begin{aligned}
\underline{\mathbf{T}}^{(\text{OAC})} &= -\tau_{\text{xz}}\,\hat{\underline{\mathbf{i}}} - \tau_{\text{yz}}\,\hat{\underline{\mathbf{j}}} - \sigma_{\text{zz}}\,\hat{\underline{\mathbf{k}}} \\
&= -(0)\,\hat{\underline{\mathbf{i}}} - (-60)\,\hat{\underline{\mathbf{j}}} - (40)\,\hat{\underline{\mathbf{k}}} \\
&= 60\,\hat{\underline{\mathbf{j}}} - 40\,\hat{\underline{\mathbf{k}}} \qquad \text{MPa}
\end{aligned}$$

Stress vector on face **OBA**

$$\underline{\mathbf{T}}^{(\text{-x})} = -\underline{\mathbf{T}}^{(\text{x})} = -\underline{\boldsymbol{\sigma}}\,\hat{\underline{\mathbf{n}}}_{(\text{x})} = -\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{\mathbf{T}}^{(\text{OBA})} = -\sigma_{\text{xx}}\,\hat{\underline{\mathbf{i}}} - \tau_{\text{yx}}\,\hat{\underline{\mathbf{j}}} - \tau_{\text{zx}}\,\hat{\underline{\mathbf{k}}}$$

$$= -(40)\,\hat{\underline{\mathbf{i}}} - (40)\,\hat{\underline{\mathbf{j}}} - (0)\,\hat{\underline{\mathbf{k}}}$$

$$= -40\,\hat{\underline{\mathbf{i}}} - 40\,\hat{\underline{\mathbf{j}}} \qquad \text{MPa}$$

Stress vector on face **OCB**

$$\underline{\mathbf{T}}^{(\text{-y})} = -\underline{\mathbf{T}}^{(\text{y})} = -\underline{\boldsymbol{\sigma}}\,\hat{\underline{\mathbf{n}}}_{(\text{y})} = -\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\underline{\mathbf{T}}^{(\text{OCB})} = -\tau_{\text{xy}}\,\hat{\underline{\mathbf{i}}} - \sigma_{\text{yy}}\,\hat{\underline{\mathbf{j}}} - \tau_{\text{zy}}\,\hat{\underline{\mathbf{k}}}$$

$$= -(40)\,\hat{\underline{\mathbf{i}}} - (50)\,\hat{\underline{\mathbf{j}}} - (-60)\,\hat{\underline{\mathbf{k}}}$$

$$= -40\,\hat{\underline{\mathbf{i}}} - 50\,\hat{\underline{\mathbf{j}}} + 60\,\hat{\underline{\mathbf{k}}} \qquad \text{MPa}$$

METHOD TWO: Using Cauchy's Relation

Find unit vectors on faces on which traction forces are desired

$$\hat{\underline{\mathbf{n}}}_{(\text{OAC})} = -\hat{\underline{\mathbf{k}}} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} \quad \hat{\underline{\mathbf{n}}}_{(\text{OCB})} = -\hat{\underline{\mathbf{j}}} = \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} \quad \hat{\underline{\mathbf{n}}}_{(\text{OBA})} = -\hat{\underline{\mathbf{i}}} = \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix}$$

Now the stress vectors are found using Cauchy's formula

$$\underline{\mathbf{T}}^{(j)} = \underline{\boldsymbol{\sigma}} \cdot \hat{\underline{\mathbf{n}}}_j$$

Stress vector on face **OAC**

$$\underline{\mathbf{T}}^{(\text{OAC})} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} \text{MPa} = \begin{Bmatrix} 0 \\ 60 \\ -40 \end{Bmatrix} \text{MPa}$$

$$= 60\,\hat{\underline{\mathbf{j}}} - 40\,\hat{\underline{\mathbf{k}}} \qquad \text{MPa}$$

Stress vector on face **OBA**

$$\underline{\mathbf{T}}^{(\mathrm{OBA})} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} \text{ MPa} = \begin{Bmatrix} -40 \\ -40 \\ 0 \end{Bmatrix} \text{ MPa}$$

$$= -40\,\hat{\underline{\mathbf{i}}} - 40\,\hat{\underline{\mathbf{j}}} \qquad \text{MPa}$$

Stress vector on face **OCB**

$$\underline{\mathbf{T}}^{(\mathrm{OCB})} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} \text{ MPa} = \begin{Bmatrix} -40 \\ -50 \\ 60 \end{Bmatrix} \text{ MPa}$$

$$= -40\,\hat{\underline{\mathbf{i}}} - 50\,\hat{\underline{\mathbf{j}}} + 60\,\hat{\underline{\mathbf{k}}} \qquad \text{MPa}$$

(c) Determine the total force vectors acting on the mutually orthogonal face **OAC**, **OCB**, **OBA**. Note that **OA** $= \Delta$, **OB** $= 3\Delta$, **OC** $= 2\Delta$ (where Δ is given in meters).

Since force is equal to stress multiplied by area, we proceed to calculate the area of faces **OAC**, **OCB**, **OBA**

$$\begin{aligned} A &= \frac{1}{2}\,(\text{base} \cdot \text{height}) \\ A_{\mathrm{OAC}} &= \frac{1}{2}\,(\Delta)\,(2\,\Delta) = \Delta^2 \ \text{meters}^2 \\ A_{\mathrm{OCB}} &= \frac{1}{2}\,(2\,\Delta)\,(3\,\Delta) = 3\,\Delta^2 \ \text{meters}^2 \\ A_{\mathrm{OBA}} &= \frac{1}{2}\,(3\,\Delta)\,(\Delta) = \frac{3}{2}\Delta^2 \ \text{meters}^2 \end{aligned}$$

Total force acting on face OAC: *(Note that distance Δ is given in meters)*

$$\begin{aligned} \underline{\mathbf{F}}^{(\mathrm{OAC})} &= \underline{\mathbf{T}}^{(\mathrm{OAC})}\,A_{\mathrm{OAC}} \\ &= (60\,\hat{\underline{\mathbf{j}}} - 40\,\hat{\underline{\mathbf{k}}})\,(\Delta^2) \text{ MPa-m}^2 \\ &= 60\,\Delta^2\,\hat{\underline{\mathbf{j}}} - 40\,\Delta^2\,\hat{\underline{\mathbf{k}}} \text{ MPa-m}^2 \\ &= 60\,\Delta^2\,\hat{\underline{\mathbf{j}}} - 40\,\Delta^2\,\hat{\underline{\mathbf{k}}} \text{ MN} \end{aligned}$$

Total force acting on face OBA: *(Note that distance Δ is given in meters)*

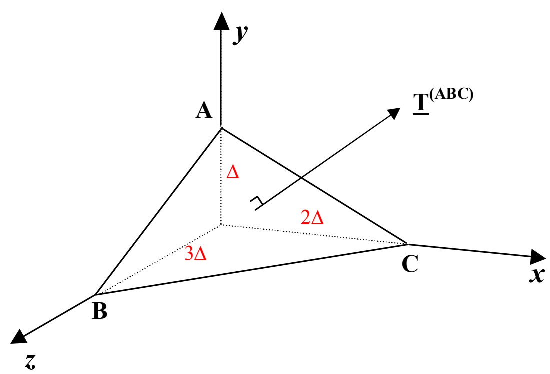
$$\begin{aligned} \underline{\mathbf{F}}^{(\mathrm{OBA})} &= \underline{\mathbf{T}}^{(\mathrm{OBA})}\,A_{\mathrm{OBA}} \\ &= (-40\,\hat{\underline{\mathbf{i}}} - 40\,\hat{\underline{\mathbf{j}}})\left(\frac{3}{2}\Delta^2\right) \text{ MPa-m}^2 \\ &= -60\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 60\,\Delta^2\,\hat{\underline{\mathbf{j}}} \text{ MPa-m}^2 \\ &= -60\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 60\,\Delta^2\,\hat{\underline{\mathbf{j}}} \text{ MN} \end{aligned}$$

Total force acting on face OCB: *(Note that distance Δ is given in meters)*

$$\begin{aligned}\underline{\mathbf{F}}^{(\text{OCB})} &= \underline{\mathbf{T}}^{(\text{OCB})}\, A_{\text{OCB}}\\ &= (-40\,\hat{\underline{\mathbf{i}}} - 50\,\hat{\underline{\mathbf{j}}} + 60\,\hat{\underline{\mathbf{k}}})(3\,\Delta^2)\ \text{MPa-m}^2\\ &= -120\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 150\,\Delta^2\,\hat{\underline{\mathbf{j}}} + 180\,\Delta^2\,\hat{\underline{\mathbf{k}}}\ \text{MPa-m}^2\\ &= -120\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 150\,\Delta^2\,\hat{\underline{\mathbf{j}}} + 180\,\Delta^2\,\hat{\underline{\mathbf{k}}}\ \text{MN}\end{aligned}$$

(d) Determine the stress vector acting on the face **ABC**. Use both static equilibrium and Cauchy's relation.

METHOD ONE: STATIC EQUILIBRIUM



The stress vector is obtain by dividing the total force acting on face **ABC** by area of face **ABC**

$$\underline{\mathbf{T}}^{(\text{ABC})} = \frac{\underline{\mathbf{F}}^{(\text{ABC})}}{A_{\text{ABC}}}$$

Thus the stress vector acting in the **ABC** plane can be found by first obtaining the force in the **ABC** then diving the force by the area enclosed by **ABC**. Note that element OABC is in static equilibrium, therefore

$$\sum \underline{\mathbf{F}} = \underline{\mathbf{0}} \quad \rightarrow \quad \underline{\mathbf{F}}^{(\text{ABC})} + \underline{\mathbf{F}}^{(\text{OAC})} + \underline{\mathbf{F}}^{(\text{OCB})} + \underline{\mathbf{F}}^{(\text{ OBA})} = \underline{\mathbf{0}}$$

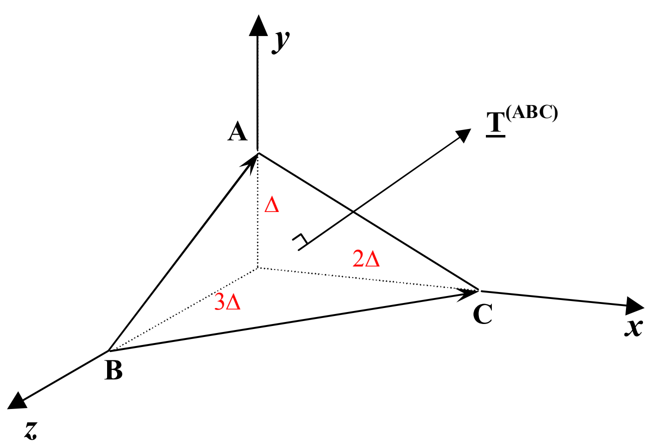
Thus, the total force acting on face **ABC** is

$$\underline{\mathbf{F}}^{(\mathrm{ABC})} = -\underline{\mathbf{F}}^{(\mathrm{OAC})} - \underline{\mathbf{F}}^{(\mathrm{OCB})} - \underline{\mathbf{F}}^{(\mathrm{OBA})}$$

$$\begin{aligned}\underline{\mathbf{F}}^{(\mathrm{ABC})} = &-\Big(60\,\Delta^2\,\hat{\underline{\mathbf{j}}} - 40\,\Delta^2\,\hat{\underline{\mathbf{k}}}\Big)\\ &-\Big(-120\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 150\,\Delta^2\,\hat{\underline{\mathbf{j}}} + 180\,\Delta^2\,\hat{\underline{\mathbf{k}}}\Big)\\ &-\Big(-60\,\Delta^2\,\hat{\underline{\mathbf{i}}} - 60\,\Delta^2\,\hat{\underline{\mathbf{j}}}\Big)\end{aligned}$$

$$\underline{\mathbf{F}}^{(\mathrm{ABC})} = 180\,\Delta^2\,\hat{\underline{\mathbf{i}}} + 150\,\Delta^2\,\hat{\underline{\mathbf{j}}} - 140\,\Delta^2\,\hat{\underline{\mathbf{k}}}\ \mathrm{MN}$$

Since stresses are obtained dividing forces by the area, we proceed to find the magnitude of area $\underline{\mathbf{A}}_{\mathrm{ABC}}$. The area of **ABC** is calculate using analytical geometry as follows:



Area of triangle **ABC** can be found by using the following equation

$$\underline{\mathbf{A}}_{\mathrm{ABC}} = \frac{1}{2}\,\underline{\mathbf{r}}_{(\mathrm{BC})} \times \underline{\mathbf{r}}_{(\mathrm{BA})}$$

where

$$\underline{\mathbf{r}}_{(\mathrm{BC})} = \underline{\mathbf{C}} - \underline{\mathbf{B}} = \left\{\begin{array}{c} 2\,\Delta \\ 0 \\ 0 \end{array}\right\} - \left\{\begin{array}{c} 0 \\ 0 \\ 3\,\Delta \end{array}\right\} = \left\{\begin{array}{c} 2\,\Delta \\ 0 \\ -3\,\Delta \end{array}\right\}\ \text{meters}$$

$$= 2\,\Delta\,\hat{\underline{\mathbf{i}}} - 3\,\Delta\,\hat{\underline{\mathbf{k}}}\ \text{meters}$$

$$\underline{\mathbf{r}}_{(BA)} = \underline{\mathbf{A}} - \underline{\mathbf{B}} = \begin{Bmatrix} 0 \\ \Delta \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 3\,\Delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ -3\,\Delta \end{Bmatrix} \text{ meters}$$

$$= \Delta\,\hat{\underline{\mathbf{j}}} - 3\,\Delta\,\hat{\underline{\mathbf{k}}} \text{ meters}$$

Thus the area is then

$$\begin{aligned} \underline{\mathbf{A}}_{ABC} = & \frac{1}{2}\,\underline{\mathbf{r}}_{(BC)} \times \underline{\mathbf{r}}_{(BA)} = \frac{1}{2} \begin{vmatrix} \hat{\underline{\mathbf{i}}} & \hat{\underline{\mathbf{j}}} & \hat{\underline{\mathbf{k}}} \\ 2\,\Delta & 0 & -3\,\Delta \\ 0 & \Delta & -3\,\Delta \end{vmatrix} \\ = & \frac{3}{2}\,\Delta^2\,\hat{\underline{\mathbf{i}}} + 3\,\Delta^2\,\hat{\underline{\mathbf{j}}} + \Delta^2\,\hat{\underline{\mathbf{k}}} \text{ meters}^2 \end{aligned} \tag{2.17}$$

We could have also used the following (check it!)

$$\underline{\mathbf{A}}_{ABC} = \frac{1}{2}\,\underline{\mathbf{r}}_{(CA)} \times \underline{\mathbf{r}}_{(CB)} = \frac{1}{2}\,\underline{\mathbf{r}}_{(AB)} \times \underline{\mathbf{r}}_{(AC)} = \frac{3}{2}\,\Delta^2\,\hat{\underline{\mathbf{i}}} + 3\,\Delta^2\,\hat{\underline{\mathbf{j}}} + \Delta^2\,\hat{\underline{\mathbf{k}}} \text{ meters}^2$$

Thus the magnitude value of $\underline{\mathbf{A}}_{ABC}$ is

$$\begin{aligned} \|\underline{\mathbf{A}}_{ABC}\| &= \sqrt{\underline{\mathbf{A}}_{ABC} \cdot \underline{\mathbf{A}}_{ABC}} = \sqrt{A_x^2 + A_y^2 + A_z^2} \\ &= \sqrt{(\frac{3\,\Delta^2}{2})^2 + (3\,\Delta^2)^2 + (\Delta^2)^2} = \frac{7}{2}\Delta^2 \text{ meters}^2 \end{aligned} \tag{2.18}$$

Now the stress vector is obtain by dividing the total force acting on face ABC by area of face ABC

$$\underline{\mathbf{T}}^{(ABC)} = \frac{\underline{\mathbf{F}}^{(ABC)}}{\|\underline{\mathbf{A}}_{ABC}\|} = \frac{180\,\Delta^2\,\hat{\underline{\mathbf{i}}} + 150\,\Delta^2\,\hat{\underline{\mathbf{j}}} - 140\,\Delta^2\,\hat{\underline{\mathbf{k}}}}{\frac{7}{2}\Delta^2}$$

$$= \frac{360}{7}\,\hat{\underline{\mathbf{i}}} + \frac{300}{7}\,\hat{\underline{\mathbf{j}}} - 40\,\hat{\underline{\mathbf{k}}} = \begin{Bmatrix} \frac{360}{7} \\ \frac{300}{7} \\ -40 \end{Bmatrix} \text{ MPa}$$

METHOD TWO: CAUCHY'S FORMULA

We can use Cauchy's relationship to obtain the stress vector,

$$\underline{\mathbf{T}}^{(ABC)} = \underline{\boldsymbol{\sigma}} \cdot \hat{\underline{\mathbf{n}}}_{ABC}$$

where $\underline{\mathbf{T}}^{(ABC)}$ is the stress vector acting on the face ABC and the stress tensor $\underline{\boldsymbol{\sigma}}$ is known. Thus in order to calculate the stress vector acting on the face ABC, we need to calculate the unit vector. The unit vector can be obtained using

$$\hat{\underline{\mathbf{n}}}_{\text{ABC}} = \frac{\underline{\mathbf{A}}_{\text{ABC}}}{\|\underline{\mathbf{A}}_{\text{ABC}}\|}$$

The unit normal to face **ABC** is found using Eqs. (2.17) and (2.18)

$$\hat{\underline{\mathbf{n}}}_{\text{ABC}} = \frac{\underline{\mathbf{A}}_{\text{ABC}}}{\|\underline{\mathbf{A}}_{\text{ABC}}\|} = \frac{\frac{3}{2}\,\Delta^2\,\hat{\underline{\mathbf{i}}} + 3\,\Delta^2\,\hat{\underline{\mathbf{j}}} + \Delta^2\,\hat{\underline{\mathbf{k}}}}{\frac{7}{2}\Delta^2} = \frac{3}{7}\hat{\underline{\mathbf{i}}} + \frac{6}{7}\hat{\underline{\mathbf{j}}} + \frac{2}{7}\hat{\underline{\mathbf{k}}}$$

Stress vector is found as follows

$$\underline{\mathbf{T}}^{(\text{ABC})} = \underline{\boldsymbol{\sigma}} \cdot \hat{\underline{\mathbf{n}}}_{\text{ABC}} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \cdot \begin{Bmatrix} 3/7 \\ 6/7 \\ 2/7 \end{Bmatrix} = \begin{Bmatrix} 360/7 \\ 300/7 \\ -280/7 \end{Bmatrix} \text{ MPa}$$

$$= \frac{360}{7}\,\hat{\underline{\mathbf{i}}} + \frac{300}{7}\,\hat{\underline{\mathbf{j}}} - 40\,\hat{\underline{\mathbf{k}}} \text{ MPa}$$

End Example □

2.1.5 Principal Stresses and Principal Planes

The knowledge of principal stresses[2] help us find plane(s) on which the normal stress has the largest possible value or plane(s) on which the largest possible shear stress value. A principal plane is a plane such that the stress vector acting on that plane has no component which is tangent to the plane (i.e., there are no shear stresses acting on the plane):

$$\underline{\mathbf{T}}^{(\text{n})} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.19}$$

In order words, the stress vector has the same direction as the unit normal that describes the plane. The magnitude of the normal stress is known as **principal stress**. Now we proceed to derive the solution procedure.

[2]The principal state of stress represents the critical value that stresses can have in their normal planes at a point, in the absence of shear stresses, for any plane cutting through it. These values may be either zero, negative (meaning in compression), or positive (meaning in tension).

So far we have used the Cauchy's relationship to find the stress vector acting on a face whose unit normal and stress tensor is known. Now, let us assume that the principal stress vector is known and the unit normal is not known. Let λ be the magnitude of the stress vector acting on the principal plane. Using Cauchy's formula we can define the principal stress vector as

$$\underline{\mathbf{T}}^{(\mathrm{n})} = \lambda\,\underline{\hat{\mathbf{n}}} = \lambda \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.20}$$

Also, using Cauchy's relationship, the principal stress vector can be expressed in terms of the nine stress components as follows

$$\underline{\mathbf{T}}^{(\mathrm{n})} = \begin{Bmatrix} T_{\mathrm{x}}^{(\mathrm{n})} \\ T_{\mathrm{y}}^{(\mathrm{n})} \\ T_{\mathrm{z}}^{(\mathrm{n})} \end{Bmatrix} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{xy}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{xz}} & \tau_{\mathrm{yz}} & \sigma_{\mathrm{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.21}$$

Now using the knowledge of what the principal stress vector should be for a principal plane, Eq. (2.20), Cauchy's equation becomes

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{xy}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{xz}} & \tau_{\mathrm{yz}} & \sigma_{\mathrm{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.22}$$

Thus for a principal plane, we can write Cauchy's equations in matrix form as follows

$$\begin{bmatrix} \sigma_{\mathrm{xx}} - \lambda & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} - \lambda & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} - \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.23}$$

These equations have the trivial solution $n_x = n_y = n_z = 0$. However, this solution is not allowed because n_x, n_y, and n_z are the components of a unit vector, satisfying

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{2.24}$$

and at least one component must be nonzero (i.e., one). Hence, equations in (2.23) possess a nontrivial solution if the three equations are not independent of each other. In other words, the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\det \begin{bmatrix} \sigma_{\mathrm{xx}} - \lambda & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} - \lambda & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} - \lambda \end{bmatrix} = \begin{vmatrix} \sigma_{\mathrm{xx}} - \lambda & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{yx}} & \sigma_{\mathrm{yy}} - \lambda & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{zx}} & \tau_{\mathrm{zy}} & \sigma_{\mathrm{zz}} - \lambda \end{vmatrix} = 0 \tag{2.25}$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda - I_{\sigma_3} = 0 \tag{2.26}$$

where I_{σ_i}'s are the stress invariants. Using the definition of stress invariants:

$$I_{\sigma_1} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{2.27}$$

$$\begin{aligned} I_{\sigma_2} &= \det\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} + \det\begin{bmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{bmatrix} + \det\begin{bmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{bmatrix} \\ &= \sigma_{xx}\,\sigma_{yy} + \sigma_{zz}\,\sigma_{xx} + \sigma_{yy}\,\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \end{aligned} \tag{2.28}$$

$$\begin{aligned} I_{\sigma_3} &= \det\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \\ &= \sigma_{xx}\,\sigma_{yy}\,\sigma_{zz} + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_{xx}\,\tau_{yz}^2 - \sigma_{yy}\,\tau_{zx}^2 - \sigma_{zz}\,\tau_{xy}^2 \end{aligned} \tag{2.29}$$

The three roots of the characteristic equation, Eq. (2.26), are the principal stresses and may be obtained analytically[3]

$$\lambda_1 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3}\right) \tag{2.30}$$

$$\lambda_2 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) \tag{2.31}$$

$$\lambda_3 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) \tag{2.32}$$

$$\beta = \cos^{-1}\left[\frac{2I_{\sigma_1}^3 - 9I_{\sigma_1}I_{\sigma_2} + 27I_{\sigma_3}}{2\sqrt{\left(I_{\sigma_1}^2 - 3I_{\sigma_2}\right)^3}}\right] \qquad \text{(keep in radians)} \tag{2.33}$$

For each of these three solutions, the matrix of the system of equations defined by Eq. (2.29) has a zero determinant, and a non trivial solution exists for the directions on which the shear stresses vanish. Such direction is called a principal stress plane or simply principal planes. Since we are solving homogeneous equations, the solution will include an arbitrary constant which can be determined by enforcing the condition,

$$n_x^2 + n_y^2 + n_z^2 = 1$$

associated with the fact that vector $\underline{\hat{\mathbf{n}}}$ must be a unit vector. We will have three principal stress directions because we have three principal stresses. Furthermore, it can be shown that these three directions are mutually orthogonal.

The fact that the stress tensor is symmetric, the three principal stress, roots of Eq. (2.26), will be

[3]These can also be obtained using any computer program.

real-valued. The principal stresses are chosen as:

$$\begin{aligned}\sigma_1 &= \max[\lambda_1, \lambda_2, \lambda_3]\\ \sigma_3 &= \min[\lambda_1, \lambda_2, \lambda_3]\\ \sigma_2 &= \text{The remaining } \lambda\end{aligned} \tag{2.34}$$

Thus the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

It turns out that a state of stress not only has three extreme values of normal stress, but also three extreme values of shear stress, which are related to the three principal stresses as follows:

$$\tau_{12} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \qquad \tau_{13} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| \qquad \tau_{23} = \left|\frac{\sigma_2 - \sigma_3}{2}\right| \tag{2.35}$$

Observe that the absolute maximum shear stress at a point equals one-half the difference between the largest and the smallest principal stress, or:

$$\sigma_{\max} = \max[\sigma_1, \sigma_2, \sigma_3] \tag{2.36}$$

$$\sigma_{\min} = \min[\sigma_1, \sigma_2, \sigma_3] \tag{2.37}$$

$$\tau_{\max} = \left|\frac{\sigma_{\max} - \sigma_{\min}}{2}\right| \tag{2.38}$$

Finally, it must be pointed out that whereas the shear stress, by definition, vanishes on planes of principal stress, the normal stress is generally not zero on planes where the shear stress acquires its extreme values.

We are not only interested in the principal stresses but also the planes upon which they act on, as shown in Fig. 2.7. Hence, we now proceed to determine these plane.

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(1)}$

To find $\underline{\hat{\mathbf{n}}}^{(1)}$, the principal direction of σ_1, we substitute $\lambda = \sigma_1$ into Eq. (2.23) and use any two of the three equations, but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(1)}$ ($n_x^{(1)}$, $n_y^{(1)}$, and $n_z^{(1)}$) and the last component is obtained with

$$\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1$$

The above condition ensures that $\underline{\hat{\mathbf{n}}}^{(1)}$ is indeed a unit vector. Hence,

$$\underline{\hat{\mathbf{n}}}^{(1)} = \begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \\ n_z^{(1)} \end{Bmatrix} \tag{2.39}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(2)}$

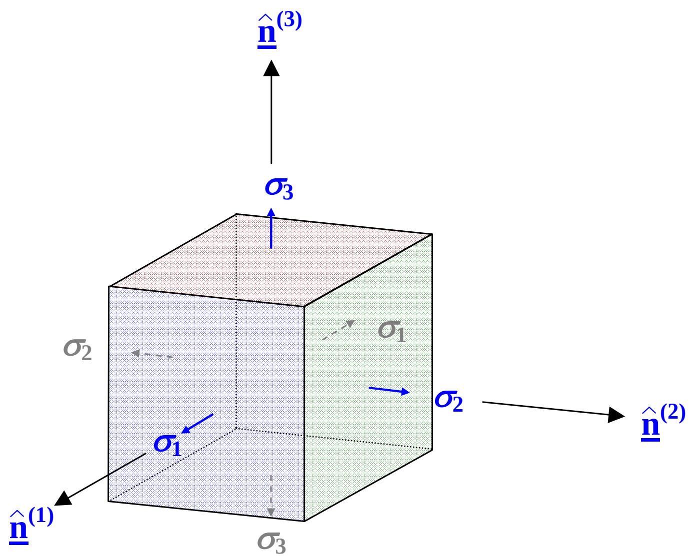


Figure 2.7: Principal state of stress

To find $\underline{\hat{\mathbf{n}}}^{(2)}$, the principal direction of σ_2, we substitute $\lambda = \sigma_2$ into Eq. (2.23) and use any two of the three equations, but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(2)}$ ($n_x^{(2)}$, $n_y^{(2)}$, and $n_z^{(2)}$) and the last component is obtained with

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1$$

The above condition ensures that $\underline{\hat{\mathbf{n}}}^{(2)}$ is indeed a unit vector. Hence,

$$\underline{\hat{\mathbf{n}}}^{(2)} = \left\{ \begin{array}{c} n_x^{(2)} \\ n_y^{(2)} \\ n_z^{(2)} \end{array} \right\} \tag{2.40}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(3)}$

To find $\underline{\hat{\mathbf{n}}}^{(3)}$, the principal direction of σ_3, we substitute $\lambda = \sigma_3$ into Eq. (2.23) and use any two of the three equations, but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(3)}$ ($n_x^{(3)}$, $n_y^{(3)}$, and $n_z^{(3)}$) and the last component is obtained with

$$\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1$$

The above condition ensures that $\underline{\hat{\mathbf{n}}}^{(3)}$ is indeed a unit vector. Hence,

$$\underline{\hat{\mathbf{n}}}^{(3)} = \left\{ \begin{array}{c} n_x^{(3)} \\ n_y^{(3)} \\ n_z^{(3)} \end{array} \right\} \tag{2.41}$$

Note that the principal planes are orthogonal to each other. In other words, the three principal normals are perpendicular to one another and thus

$$\underline{\hat{\mathbf{n}}}^{(3)} = \underline{\hat{\mathbf{n}}}^{(1)} \times \underline{\hat{\mathbf{n}}}^{(2)} \tag{2.42}$$

Example 2.3.

Determine the three principal stresses and corresponding principal planes for the state of stress given in Example 2.2. Also determine the extreme shear stresses.

$$\begin{bmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{bmatrix} \text{ MPa}$$

For a principal plane, Cauchy's equations can be written in matrix form as follows

$$\begin{bmatrix} \sigma_{xx}-\lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & -60 \\ 0 & -60 & 40-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\det \begin{bmatrix} \sigma_{xx}-\lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\lambda \end{bmatrix} = \begin{vmatrix} 40-\lambda & 40 & 0 \\ 40 & 50-\lambda & -60 \\ 0 & -60 & 40-\lambda \end{vmatrix} = 0$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1}\lambda^2 + I_{\sigma_2}\lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants.

$$\begin{aligned} I_{\sigma_1} &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ &= (40 + 50 + 40) \text{ MPa} = 130 \text{ MPa} \\ I_{\sigma_2} &= \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} + \det \begin{bmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{bmatrix} + \det \begin{bmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{bmatrix} \\ &= \begin{vmatrix} 40 & 40 \\ 40 & 50 \end{vmatrix} + \begin{vmatrix} 40 & 0 \\ 0 & 40 \end{vmatrix} + \begin{vmatrix} 50 & -60 \\ -60 & 40 \end{vmatrix} \\ &= \{(40)(50) - (40)(40)\} + \{(40)(40)\} + \{(50)(40) - (-60)(-60)\} = 400 \text{ MPa}^2 \end{aligned}$$

$$I_{\sigma_3} = \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{vmatrix} 40 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 40 \end{vmatrix}$$

$$= (40)\begin{vmatrix} 50 & -60 \\ -60 & 40 \end{vmatrix} - (40)\begin{vmatrix} 40 & -60 \\ 0 & 40 \end{vmatrix} + (0)\begin{vmatrix} 40 & 50 \\ 0 & -60 \end{vmatrix} = -128000 \text{ MPa}^3$$

Therefore, the characteristic equation can be written as

$$\lambda^3 - 130\,\lambda^2 + 400\,\lambda + 128000 = 0$$

The three roots of the characteristic equation are the principal stresses and are obtained analytically as follows:

$$\beta = \cos^{-1}\left[\frac{2I_{\sigma_1}^3 - 9I_{\sigma_1}I_{\sigma_2} + 27I_{\sigma_3}}{2\sqrt{\left(I_{\sigma_1}^2 - 3I_{\sigma_2}\right)^3}}\right] = \cos^{-1}\left[\frac{235}{157\sqrt{157}}\right] = 1.45105\,\text{rads}$$

$$\lambda_1 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3}\right) = 117.284 \text{ MPa}$$

$$\lambda_2 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) = -27.2842 \text{ MPa}$$

$$\lambda_3 = \frac{I_{\sigma_1}}{3} + \frac{2}{3}\sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}}\ \cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) = 40.00 \text{ MPa}$$

and the principal stresses are

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 117.284 \text{ MPa}$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -27.2842 \text{ MPa}$$
$$\sigma_2 = 40.00 \text{ MPa}$$

As we can see the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

The three extreme values of shear stress are:

$$\tau_{12} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| = 38.6421 \text{ MPa}$$
$$\tau_{13} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 72.2842 \text{ MPa}$$
$$\tau_{23} = \left|\frac{\sigma_2 - \sigma_3}{2}\right| = 33.6421 \text{ MPa}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(1)}$

To find $\underline{\hat{\mathbf{n}}}^{(1)}$, the principal direction of $\sigma_1 = 117.284$ MPa, we substitute $\lambda = \sigma_1$ into Eq. (2.23) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(1)}$ ($n_x^{(1)}$, $n_y^{(1)}$, and $n_z^{(1)}$) and the last component is obtained with

$$\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1 \tag{2.43}$$

Therefore,

$$\begin{bmatrix} -77.2842 & 40 & 0 \\ 40 & -67.2842 & -60 \\ 0 & -60 & -77.2842 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} -77.2842\, n_x^{(1)} &+ 40\, n_y^{(1)} &+ 0 &= 0 \\ 40\, n_x^{(1)} &+ -67.2842\, n_y^{(1)} &+ -60\, n_z^{(1)} &= 0 \\ 0 &+ -60\, n_y^{(1)} &+ -77.2842\, n_z^{(1)} &= 0 \end{aligned}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(1)}$ (we could have solved in terms of any other component). From the first equation:

$$\begin{aligned} -77.2842\, n_x^{(1)} + 40\, n_y^{(1)} + 0 &= 0 \\ 77.2842\, n_x^{(1)} &= 40\, n_y^{(1)} \\ n_x^{(1)} &= 0.51757\, n_y^{(1)} \end{aligned} \tag{2.44}$$

From the second equation:

$$\begin{aligned} 40\, n_x^{(1)} + -67.2842\, n_y^{(1)} + -60\, n_z^{(1)} &= 0 \\ 40\left(0.51757\, n_y^{(1)}\right) + -67.2842\, n_y^{(1)} + -60\, n_z^{(1)} &= 0 \\ n_y^{(1)} &= -1.28807\, n_z^{(1)} \end{aligned} \tag{2.45}$$

Substituting Eq. (2.45) into Eq. (2.44) we get:

$$n_x^{(1)} = 0.51757\, n_y^{(1)} = 0.51757\left(-1.28807\, n_z^{(1)}\right) = -0.666667\, n_z^{(1)}$$

Thus

$$n_x^{(1)} = -0.666667\, n_z^{(1)} \qquad n_y^{(1)} = -1.28807\, n_z^{(1)} \tag{2.46}$$

Now obtain $n_z^{(1)}$ using Eq. (2.43)

$$\begin{aligned}
\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 &= 1 \\
\left(-0.666667\, n_z^{(1)}\right)^2 + \left(-1.28807\, n_z^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 &= 1 \\
3.10357 \left(n_z^{(1)}\right)^2 &= 1
\end{aligned}$$

Then

$$n_z^{(1)} = \pm 0.567635$$

Note that the signs represent that the stress vector acts on opposite ends of the same plane. This makes sense as it ensures equilibrium. Now taking the positive sign (arbitrarily) of $n_z^{(1)}$ and substituting into Eq. (2.46)

$$n_z^{(1)} = 0.567635 \qquad n_x^{(1)} = -0.378424 \qquad n_y^{(1)} = -0.731154$$

Thus, the principal stress $\sigma_1 = 117.284$ MPa acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(1)} = \left\{ \begin{array}{c} -0.378424 \\ -0.731154 \\ 0.567635 \end{array} \right\}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(2)}$

To find $\underline{\hat{\mathbf{n}}}^{(2)}$, the principal direction of $\sigma_2 = 40.00$ MPa, we substitute $\lambda = \sigma_2$ into Eq. (2.23) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(2)}$ ($n_x^{(2)}$, $n_y^{(2)}$, and $n_z^{(2)}$) and the last component is obtained with

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1 \tag{2.47}$$

Therefore,

$$\left[\begin{array}{rrr} 0 & 40 & 0 \\ 40 & 10 & -60 \\ 0 & -60 & 0 \end{array} \right] \left\{ \begin{array}{c} n_x \\ n_y \\ n_z \end{array} \right\}^{(2)} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}$$

$$\begin{array}{rcrcrcl}
0\, n_x^{(2)} & + & 40\, n_y^{(2)} & + & 0 & = & 0 \\
40\, n_x^{(2)} & + & 10\, n_y^{(2)} & + & -60\, n_z^{(2)} & = & 0 \\
0 & + & -60\, n_y^{(2)} & + & 0 & = & 0
\end{array}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(2)}$ (we could have solved in terms of any other component)

$$n_x^{(2)} = 1.5\, n_z^{(2)} \qquad n_y^{(2)} = 0\, n_z^{(2)} = 0$$

Now obtain $n_z^{(2)}$ using Eq. (2.47)

$$\begin{aligned}
\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 &= 1 \\
\left(1.5\, n_z^{(2)}\right)^2 + \left(0\, n_z^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 &= 1 \\
3.25 \left(n_z^{(2)}\right)^2 &= 1
\end{aligned}$$

Then

$$n_z^{(2)} = \pm 0.5547$$

Note that the signs represent that the stress vector acts on opposite ends of the same plane. This makes sense as it ensures equilibrium. Now taking the positive sign (arbitrarily) of $n_z^{(2)}$:

$$n_z^{(2)} = 0.5547 \qquad n_x^{(2)} = 0.83205 \qquad n_y^{(2)} = 0$$

Thus, the principal stress $\sigma_2 = 40.00$ MPa acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(2)} = \begin{Bmatrix} 0.83205 \\ 0.0 \\ 0.5547 \end{Bmatrix}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(3)}$

To find $\underline{\hat{\mathbf{n}}}^{(3)}$, the principal direction of $\sigma_3 = -27.2842$ MPa, we substitute $\lambda = \sigma_3$ into Eq. (2.23) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(3)}$ ($n_x^{(3)}$, $n_y^{(3)}$, and $n_z^{(3)}$) and the last component is obtained with

$$\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1 \tag{2.48}$$

Therefore,

$$\begin{bmatrix} 67.2842 & 40 & 0 \\ 40 & 77.2842 & -60 \\ 0 & -60 & 67.2842 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned}
67.2842\, n_x^{(3)} + 40\, n_y^{(3)} + 0 &= 0 \\
40\, n_x^{(3)} + 77.2842\, n_y^{(3)} + -60\, n_z^{(3)} &= 0 \\
0 + -60\, n_y^{(3)} + 67.2842\, n_z^{(3)} &= 0
\end{aligned}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(3)}$ (we could have solved in terms of any other component)

$$n_x^{(3)} = -0.666667\, n_z^{(3)} \qquad n_y^{(3)} = 1.1214\, n_z^{(3)}$$

Now obtain $n_z^{(3)}$ using Eq. (2.48)

$$
\begin{aligned}
\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 &= 1 \\
\left(-0.666667\, n_z^{(3)}\right)^2 + \left(1.1214\, n_z^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 &= 1 \\
2.70199 \left(n_z^{(3)}\right)^2 &= 1
\end{aligned}
$$

Then

$$n_z^{(3)} = \pm 0.608357$$

Note that the signs represent that the stress vector acts on opposite ends of the same plane. This makes sense as it ensures equilibrium. Now taking the positive sign (arbitrarily) of $n_z^{(3)}$:

$$n_z^{(3)} = 0.608357 \qquad n_x^{(3)} = -0.405571 \qquad n_y^{(3)} = 0.682213$$

Thus, the principal stress $\sigma_3 = -27.2842$ MPa acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(3)} = \begin{Bmatrix} -0.405571 \\ 0.682213 \\ 0.608357 \end{Bmatrix}$$

Also, we could have obtained this by using Eq. (2.42):

$$
\begin{aligned}
\underline{\hat{\mathbf{n}}}^{(3)} = \underline{\hat{\mathbf{n}}}^{(1)} \times \underline{\hat{\mathbf{n}}}^{(2)} &= \begin{vmatrix} \underline{\hat{\mathbf{i}}} & \underline{\hat{\mathbf{j}}} & \underline{\hat{\mathbf{k}}} \\ -0.378424 & -0.731154 & 0.567635 \\ 0.83205 & 0.0 & 0.5547 \end{vmatrix} \\
&= \underline{\hat{\mathbf{i}}} \begin{vmatrix} -0.731154 & 0.567635 \\ 0.0 & 0.5547 \end{vmatrix} - \underline{\hat{\mathbf{j}}} \begin{vmatrix} -0.378424 & 0.567635 \\ 0.83205 & 0.5547 \end{vmatrix} \\
&\quad + \underline{\hat{\mathbf{k}}} \begin{vmatrix} -0.378424 & -0.731154 \\ 0.83205 & 0.0 \end{vmatrix} \\
\underline{\hat{\mathbf{n}}}^{(3)} &= -0.405571\, \underline{\hat{\mathbf{i}}} + 0.682213\, \underline{\hat{\mathbf{j}}} + 0.608357\, \underline{\hat{\mathbf{k}}}
\end{aligned}
$$

End Example □

2.2 State of Plane Stress

A particular state of stress of great practical importance is the state of plane stress. The state of stress at a point is given by the stress tensor

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{2.49}$$

For applications for which a material is formed into thin sheets and plates of uniform thickness, it is often appropriate to assume that the stress components are confined to a plane, say x-y plane. In other words,

$$\tau_{xz} = \tau_{yz} = 0 \tag{2.50}$$

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \tag{2.51}$$

Now for many problems, such as in aerospace and mechanical engineering applications,

$$\sigma_{zz} \ll \sigma_{xx} \qquad \sigma_{zz} \ll \sigma_{yy} \tag{2.52}$$

If this is the case, then we can take

$$\sigma_{zz} \approx 0 \tag{2.53}$$

This type of problems are known as plane stress problems and the three dimensional state of stress reduces to three independent components,

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.54}$$

and it is shown in Fig. 2.8. In short, plane stress assumption is acceptable when the thickness is far smaller than (at least) other dimension.

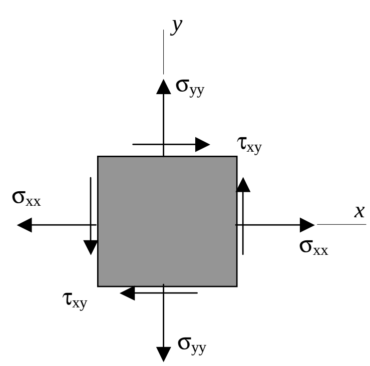


Figure 2.8: Positive stresses on a two dimensional element.

2.2.1 Principal stresses for Plane State of Stress

Recall that the knowledge of principal stresses help us find plane(s) on which the normal stress has the largest possible value or plane(s) on which the largest possible shear stress value. Next, we will review three different methods used to obtain the principal stresses and maximum shear stresses for a plane state of stress.

2.2.2 Principal stresses: Eigenvalue Approach

A principal plane is a plane such that the stress vector acting on that plane has no component which is tangent to the plane:

$$\underline{\mathbf{T}}^{(n)} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.55}$$

For a plane stress problem, at a principal plane Cauchy's equations can be written in matrix form as follows

$$\begin{bmatrix} \sigma_{xx} - \lambda & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} - \lambda & 0 \\ 0 & 0 & 0 - \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.56}$$

The above posses a nontrivial solution if the three equations are not independent of each other. In other words, the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\det \begin{bmatrix} \sigma_{xx} - \lambda & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} - \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{vmatrix} \sigma_{xx} - \lambda & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \tag{2.57}$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1}\lambda^2 + I_{\sigma_2}\lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants. Using the definition of stress invariants:

$$\begin{aligned} I_{\sigma_1} &= \sigma_{xx} + \sigma_{yy} \\ I_{\sigma_2} &= \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} + \det \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix} + \det \begin{bmatrix} \sigma_{yy} & 0 \\ 0 & 0 \end{bmatrix} = \sigma_{xx}\sigma_{yy} - \tau_{xy}^2 \\ I_{\sigma_3} &= \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

Thus the characteristic equation becomes

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = 0 \quad \rightarrow \quad \lambda\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) \;= 0 \tag{2.58}$$

The three roots of the characteristic equation, Eq. (2.58), are the principal stresses and can be obtained analytically:

$$\begin{aligned}
\lambda_1 &= \frac{I_{\sigma_1}}{2} + \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} \\
\lambda_2 &= \frac{I_{\sigma_1}}{2} - \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} \\
\lambda_3 &= 0
\end{aligned}$$

The principal stresses are chosen as:

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3]$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3]$$

Thus the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

2.2.3 Principal stresses: Transformation Equations Approach

Plane stress transformation matrix for a rotation about the z-axis is

$$\underline{\mathbf{a}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.59}$$

Therefore the plane stresses can be transformed as follows

$$\underline{\overline{\boldsymbol{\sigma}}} = \underline{\mathbf{a}}\,\underline{\boldsymbol{\sigma}}\,\underline{\mathbf{a}}^T \tag{2.60}$$

This yields to the plane stress transformation formulas

$$\overline{\sigma}_{\text{xx}} = \sigma_{\text{ave}} + \sigma_{\text{diff}}\,\cos 2\,\theta + \tau_{\text{xy}}\,\sin 2\,\theta \tag{2.61a}$$

$$\overline{\sigma}_{\text{yy}} = \sigma_{\text{ave}} - \sigma_{\text{diff}}\,\cos 2\,\theta - \tau_{\text{xy}}\,\sin 2\,\theta \tag{2.61b}$$

$$\overline{\tau}_{\text{xy}} = -\sigma_{\text{diff}}\,\sin 2\,\theta + \tau_{\text{xy}}\,\cos 2\,\theta \tag{2.61c}$$

where

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{xx}} + \sigma_{\text{yy}}}{2} \qquad\qquad \sigma_{\text{diff}} = \frac{\sigma_{\text{xx}} - \sigma_{\text{yy}}}{2}$$

2.2.4 Principal stresses: Mohr's Circle Approach

The transformation equations for plane stress can be represented in graphical form by a plot known as Mohr's circle. The Mohr's circle, although not so popular, is a useful tool for stress analysis at a material point. Numerical techniques, such as eigenvalue problem, have substituted this technique. However, the Mohr's circle helps the understanding of the physical meaning of some specific problems. Thus, we will apply the two-dimensional representation of the three-dimensional state of stress at a point[4].

Example 2.4.

Mohr's Stress Circle
At a point on the surface of a turbine engine the stresses are

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} -50 & 20 & 0 \\ 20 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

Using Mohr's circle and only considering in-plane stresses, determine the following quantities: a)stresses acting on an element inclined at an angle $\alpha = 40°$; b) principal stresses; c) maximum in-plane shear stresses.

2.4a) Determine the Mohr circle radius and center
The average stress acting on the differential element will be:

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{xx}} + \sigma_{\text{yy}}}{2} = \frac{(-50) + (-20)}{2} \text{ MPa} = -35 \text{ MPa}$$

The difference in stresses acting on the differential element will be:

$$\sigma_{\text{diff}} = \frac{\sigma_{\text{xx}} - \sigma_{\text{yy}}}{2} = \frac{(-50) - (-20)}{2} \text{ MPa} = -15 \text{ MPa}$$

The radius of the in-plane state of stress is:

$$R = \sqrt{\tau_{\text{xy}}^2 + \sigma_{\text{diff}}^2} = \sqrt{(20)^2 + (-15)^2} \text{ MPa} = 25 \text{ MPa}$$

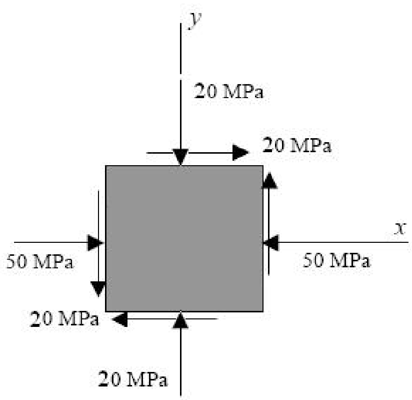
[4]A full description and derivation of the Mohr's circle is found in Appendix B.

The center of the circle is:

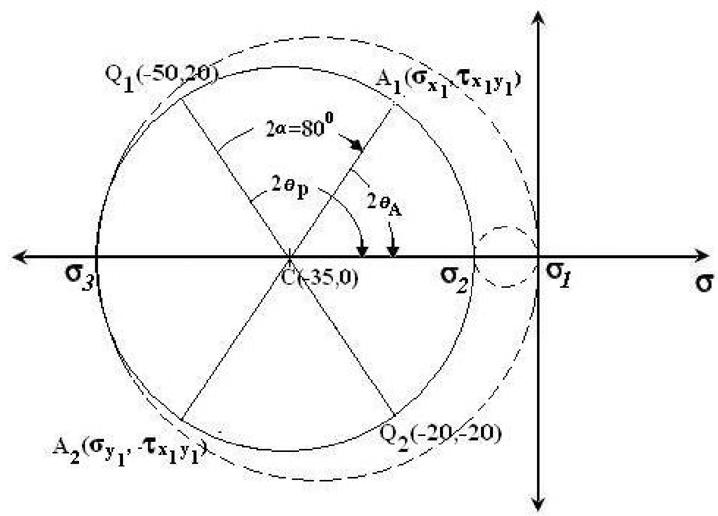
$$C = C(\sigma_{\text{ave}}, 0) = C(-35 \text{ MPa}, 0)$$

2.4b) Locate the two points and draw the circle:

$$Q_1 = Q_1(-50, 20) \qquad Q_2 = Q_2(-20, -20) \qquad C = C(-35, 0)$$



(a) stresses on a two dimensional element



(b) Mohr's circle for plane stress in the x-y plane

2.4c) Calculate angles:

Principal stresses act on an element inclined at an angle θ_p

$$2\,\theta'_p = \tan^{-1}\left[\frac{\tau_{\text{xy}}}{\sigma_{\text{diff}}}\right] = \tan^{-1}\left[\frac{(20)}{(-15)}\right] = -53.130^\circ$$

$$2\,\theta_p = 2\,\theta'_p - 180^\circ = 126.853^\circ \qquad \rightarrow \qquad \theta_p = 63.426^\circ$$

Note that we in CASE B. because $2\,\theta_p$ is measured from $\overline{Q_1C}$ to positive σ-axis. Minimum and maximum in-plane shear stresses act on an element inclined at an angle θ_s

$$2\,\theta_s = 2\,\theta_p \pm 90^\circ = 126.87^\circ \pm 90^\circ$$

$$\theta_s = \theta_p \pm 45^\circ = 63.435^\circ \pm 45^\circ$$

Transformed stresses act on an element inclined at an angle $\alpha = 40^\circ$

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha = 126.87^\circ - 80^\circ = 46.87^\circ$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element.

2.4d) Determine the normal and shear stresses on the inclined plane(s)

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} = \sigma_{\text{ave}} + R\cos(2\theta_A) = (-35) + (25)\cos(46.87°) = -17.91 \text{ MPa}$$

$$\sigma_{y_1} = \sigma_{\text{ave}} - R\cos(2\theta_A) = (-35) - (25)\cos(46.87°) = -52.09 \text{ MPa}$$

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1y_1} = R\sin(2\theta_A) = (25)\sin(46.87°) = 18.2451 \text{ MPa}$$

2.4e) Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

Note that when calculating principal stresses $2\alpha = 2\theta_p \rightarrow 2\theta_A = 0°$, therefore the principal stresses are

$$\begin{aligned}\lambda_1 &= \sigma_{\text{ave}} + R = (-35) + (25) = -10 \text{ MPa}\\ \lambda_2 &= \sigma_{\text{ave}} - R = (-35) - (25) = -60 \text{ MPa}\\ \lambda_3 &= 0 \text{ MPa}\end{aligned}$$

The principal stresses are chosen as:

$$\begin{aligned}&\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 0 \text{ MPa}\\ &\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -60 \text{ MPa}\\ &\sigma_2 = -10\end{aligned}$$

Note $\sigma_1 > \sigma_2 > \sigma_3$.

The maximum and minimum normal stresses acting on an element inclined at an angle θ_p are

$$\begin{aligned}\sigma_{\max} = \sigma_1 = 0 \text{ MPa}\\ \sigma_{\min} = \sigma_3 = -60 \text{ MPa}\end{aligned}$$

The in-plane maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{\max}\Big|_{\text{in-plane}} = R = \frac{\sigma_2 - \sigma_3}{2} = 25 \text{ MPa}$$

The maximum in-plane shear stresses will be:

$$\begin{aligned}\tau_{12} &= \frac{\sigma_1 - \sigma_2}{2} = 5 \text{ MPa}\\ \tau_{13} &= \frac{\sigma_1 - \sigma_3}{2} = 25 \text{ MPa}\\ \tau_{23} &= \frac{\sigma_2 - \sigma_3}{2} = 30 \text{ MPa}\end{aligned}$$

The overall maximum shear stress acting on an element inclined at an angle θ_s is

$$\tau_{\max} = \left| \frac{\sigma_{\max} - \sigma_{\min}}{2} \right| = 30 \text{ MPa}$$

2.4f) Show all results on sketches of properly oriented elements

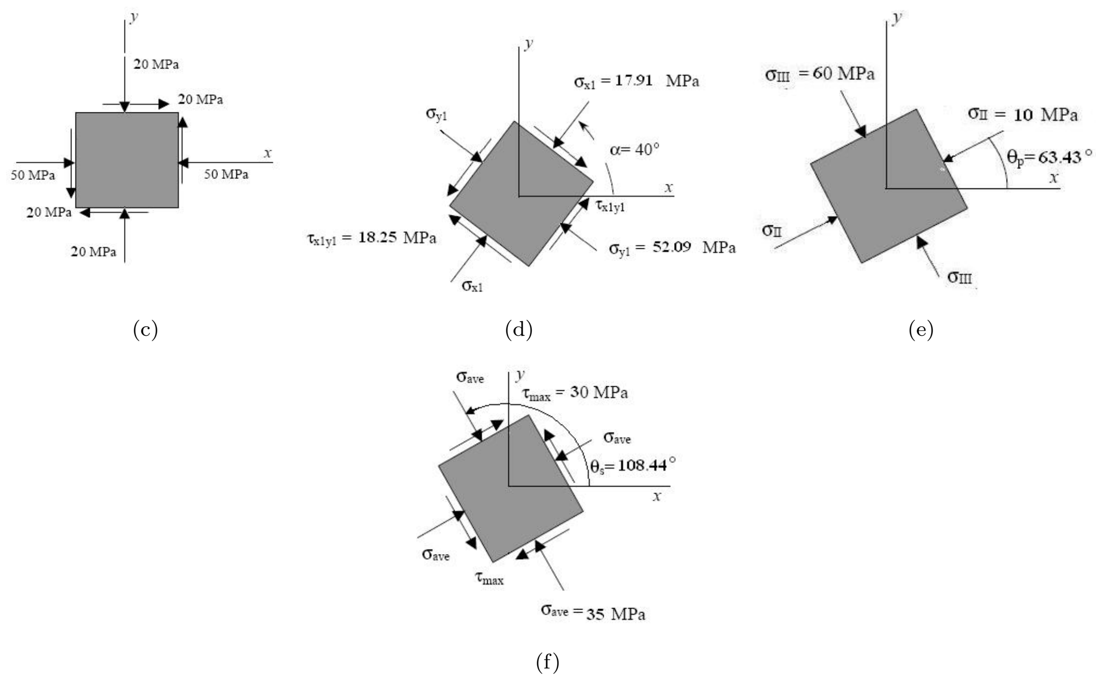


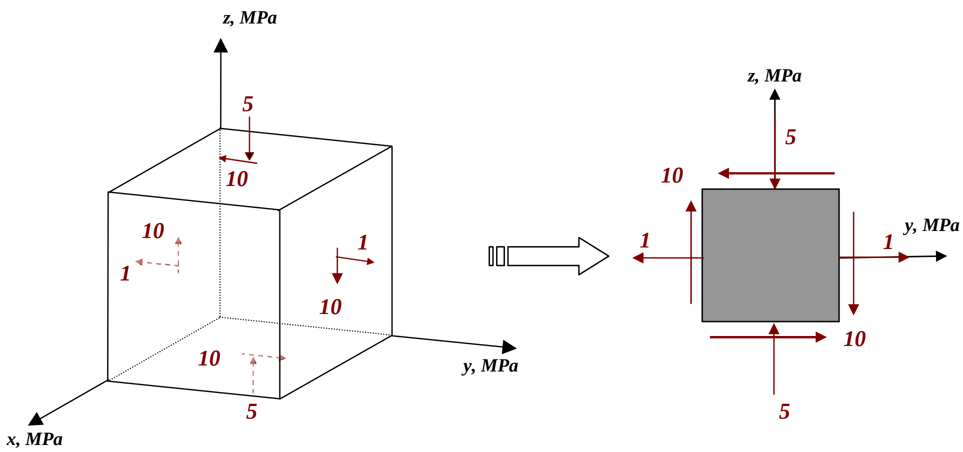
Figure 2.9: a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum in-plane shear stresses.

End Example □

Example 2.5.

An element in plane stress at the lateral surface of a wing panel is subjected to the following stresses

$$\underline{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -10 \\ 0 & -10 & -5 \end{bmatrix} \text{ MPa}$$



Considering only the in-plane stresses and using Mohr Circle determine:

1. Stresses acting on a element inclined at an angle $\theta = 45°$.
2. Principal stresses and maximum shear stresses.

2.5a) Calculate the radius and center of the Mohr's circle
The average stress acting on the differential element will be:

$$\sigma_{\text{ave}} = \frac{\sigma_{yy} + \sigma_{zz}}{2} = \frac{1 + (-5)}{2} = -2 \text{ MPa} \tag{2.62}$$

$$\tag{2.63}$$

The difference in stresses acting on the differential element will be:

$$\sigma_{\text{diff}} = \frac{\sigma_{yy} - \sigma_{zz}}{2} = \frac{1 - (-5)}{2} = 3 \text{ MPa} \tag{2.64}$$

$$\tag{2.65}$$

The radius of the inplane state of stress is:

$$R = \sqrt{\tau_{\rm yz}^2 + \sigma_{\rm diff}^2} = \sqrt{(-10)^2 + (3)^2} = 10.44033 \text{ MPa} \tag{2.66}$$

$$\tag{2.67}$$

The center of the circle is:

$$C = C(-2, 0)3 \text{ MPa} \tag{2.68}$$

2.5b) Draw the circle and locate all points

$$Q_1 = Q_1(\sigma_{\rm yy}, \tau_{\rm yz}) = Q_1(1, -10) \text{ MPa}$$
$$Q_2 = Q_2(\sigma_{\rm zz}, -\tau_{\rm yz}) = Q_1(-5, 10) \text{ MPa}$$
$$C = C(\sigma_{\rm ave}, 0) = C(-2, 0) \text{ MPa}$$

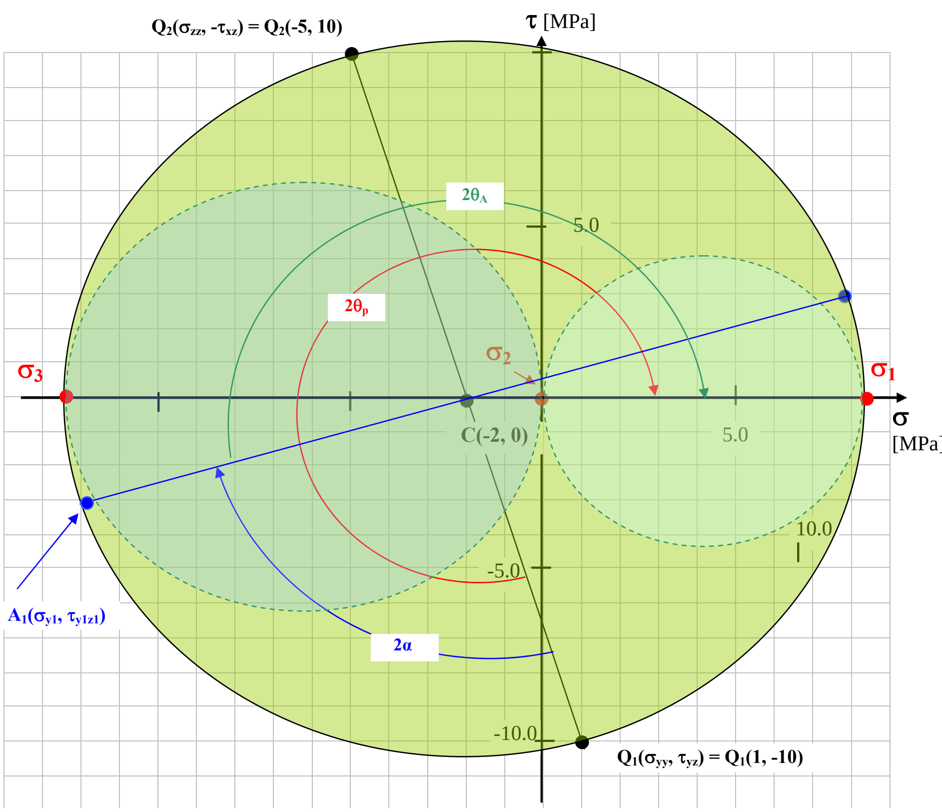


Figure 2.10: Mohr's circle for plane stress in the y-z plane.

2.5c) Calculate angles:

(All measured positive clockwise from $\overline{Q_1C}$)

First calculate $2\,\theta_p'$,

$$\tan 2\,\theta_p' = \frac{\tau_{\text{yz}}}{\sigma_{\text{diff}}} \qquad 2\,\theta_p' = \tan^{-1}\left\{\frac{\tau_{\text{yz}}}{\sigma_{\text{diff}}}\right\} = -73.3008^\circ$$

Now consider the location of Q_1:

CASE D: $Q_1 \rightarrow$ fourth quadrant $(\sigma_{\text{yy}} > 0, \quad \tau_{\text{yz}} < 0)$ $\qquad 2\,\theta_p = 360^\circ - |2\,\theta_p'|$

Thus

$$2\,\theta_p = 360^\circ - |2\,\theta_p'| = 360^\circ - 73.3008^\circ = 286.699^\circ$$

Principal stresses act on an element inclined at an angle θ_p are

$$\theta_p = \frac{1}{2}\,(2\,\theta_p) = 143.35^\circ$$

Minimum/maximum in-plane shear stresses act on an element inclined at an angle θ_s

$$2\,\theta_s = 2\,\theta_p \pm 90^\circ$$

Note that at a rotation of $2\,\theta_s = 2\,\theta_p + 90^\circ$ from $\overline{Q_1C}$ the value of the in-plane shear stress is negative thus it gives the minimum shear stress, the maximum is obtained by taking $2\,\theta_s = 2\,\theta_p - 90^\circ$. In short,

Maximum Shear Stress: $2\,\theta_s = 2\,\theta_p - 90^\circ = 163.301^\circ \rightarrow \tau_{\text{max}}$

Minimum Shear Stress: $2\,\theta_s = 2\,\theta_p + 90^\circ = -16.6992^\circ \rightarrow \tau_{\text{min}}$

Note we used the fact that $2\,\theta_s > 360^\circ$ thus

$$2\,\theta_s = (360^\circ - 2\,\theta_p) \pm 90^\circ$$

Transformed stresses act on an element inclined at an angle α

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha = 196.699^\circ$$

Note: When working in the y–z plane, all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element. Also, note that $2\,\theta_p$ is measured from $\overline{Q_1C}$ to positive σ-axis.

2.5d) Determine the normal and shear stresses on the inclined plane(s)

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{y_1} = \sigma_{\text{ave}} + R\cos(2\,\theta_A) = -12.0 \text{ MPa}$$

$$\sigma_{z_1} = \sigma_{\text{ave}} + R\cos(2\,\theta_A + 180^\circ) = \sigma_{\text{ave}} - R\cos(2\,\theta_A) = 8.0 \text{ MPa}$$

The shear stresses acting on an element inclined at an angle α are

$$\tau_{y_1 z_1} = R\,\sin(2\,\theta_A) = -3 \text{ MPa}$$

2.5e) Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0°$, therefore the principal normal stresses are found as follows

$$\begin{aligned}
\lambda_1 &= \sigma_{\text{ave}} + R = 8.4403 \text{ MPa} \\
\lambda_2 &= \sigma_{\text{ave}} - R = -12.4403 \text{ MPa} \\
\lambda_3 &= 0
\end{aligned}$$

The principal stresses are chosen as:

$$\begin{aligned}
\sigma_1 &= \max[\lambda_1, \lambda_2, \lambda_3] = 8.4403 \text{ MPa} \\
\sigma_3 &= \min[\lambda_1, \lambda_2, \lambda_3] = -12.4403 \text{ MPa}
\end{aligned}$$

Thus the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

The maximum and minimum normal stresses acting on an element inclined at an angle θ_p are

$$\begin{aligned}
\sigma_{\text{max}} &= \sigma_1 = 8.4403 \text{ MPa} \\
\sigma_{\text{min}} &= \sigma_3 = -12.4403 \text{ MPa}
\end{aligned}$$

The in-plane maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{\text{max}}\Big|_{\text{in-plane}} = R = \frac{\sigma_1 - \sigma_2}{2} = 10.4403 \text{ MPa}$$

The maximum in-plane shear stresses will be:

$$\begin{aligned}
\tau_{12} &= \frac{\sigma_1 - \sigma_2}{2} = 4.220 \text{ MPa} \\
\tau_{13} &= \frac{\sigma_1 - \sigma_3}{2} = 6.220 \text{ MPa} \\
\tau_{23} &= \frac{\sigma_2 - \sigma_3}{2} = 10.440 \text{ MPa}
\end{aligned}$$

The overall maximum shear stress acting on an element inclined at an angle θ_s is

$$\tau_{\text{max}} = \left|\frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2}\right| = 10.4403 \text{ MPa}$$

2.5f) Show all results on sketches of properly oriented elements

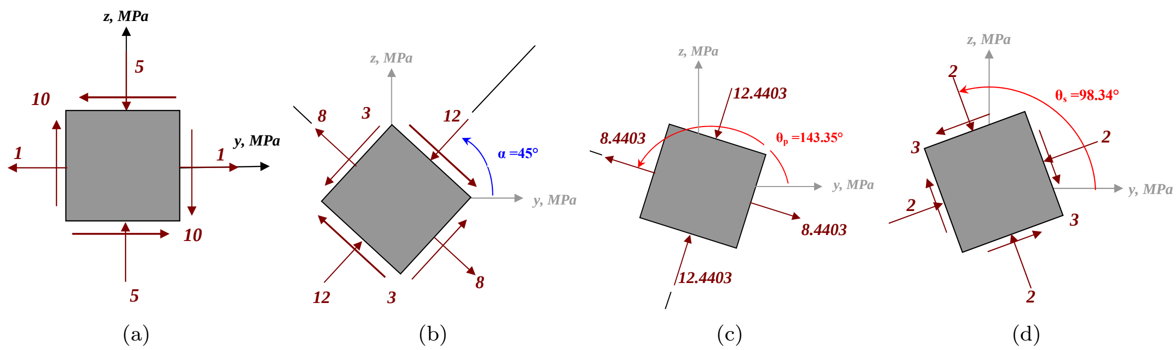


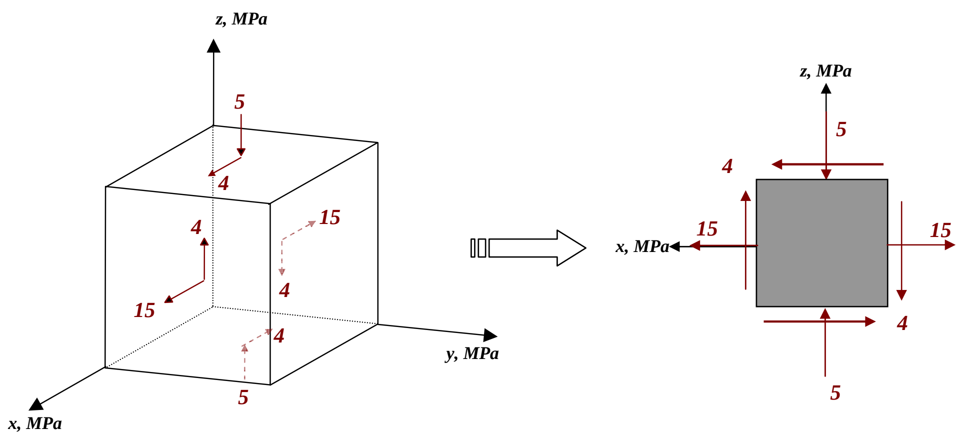
Figure 2.11: a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum in-plane shear stresses.

End Example □

Example 2.6.

An element in plane stress at the lateral surface of a wing panel is subjected to the following stresses

$$\underline{\sigma} = \begin{bmatrix} 15 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -5 \end{bmatrix} \text{ MPa}$$



Considering only the in-plane stresses and using Mohr's Circle determine:

1. Stresses acting on a element inclined at an angle $\theta = 45°$.
2. Principal stresses and maximum shear stresses.

2.6a) Calculate the radius and center of the Mohr's circle
The average stress acting on the differential element will be:

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{xx}} + \sigma_{\text{zz}}}{2} = \frac{(15) + (-5)}{2} \text{ MPa} = 5 \text{ MPa}$$

The difference in stresses acting on the differential element will be:

$$\sigma_{\text{diff}} = \frac{\sigma_{\text{xx}} - \sigma_{\text{zz}}}{2} = \frac{(15) - (-5)}{2} \text{ MPa} = 10 \text{ MPa}$$

The radius of the in-plane state of stress is:

$$R = \sqrt{\tau_{\text{xz}}^2 + \sigma_{\text{diff}}^2} = \sqrt{(4)^2 + (10)^2} \text{ MPa} = 10.7703 \text{ MPa}$$

The center of the circle is:

$$C \quad = C(\sigma_{\text{ave}}, 0) = C(5, 0) \text{ MPa}$$

2.6b) Draw the circle and locate all points

$$Q_1 = Q_1(\sigma_{\text{xx}}, \tau_{\text{xz}}) = Q_1(15, 4) \qquad Q_2 = Q_2(\sigma_{\text{zz}}, -\tau_{\text{xz}}) = Q_2(-5, -4) \qquad C = C(\sigma_{\text{ave}}, 0) = C(5, 0)$$

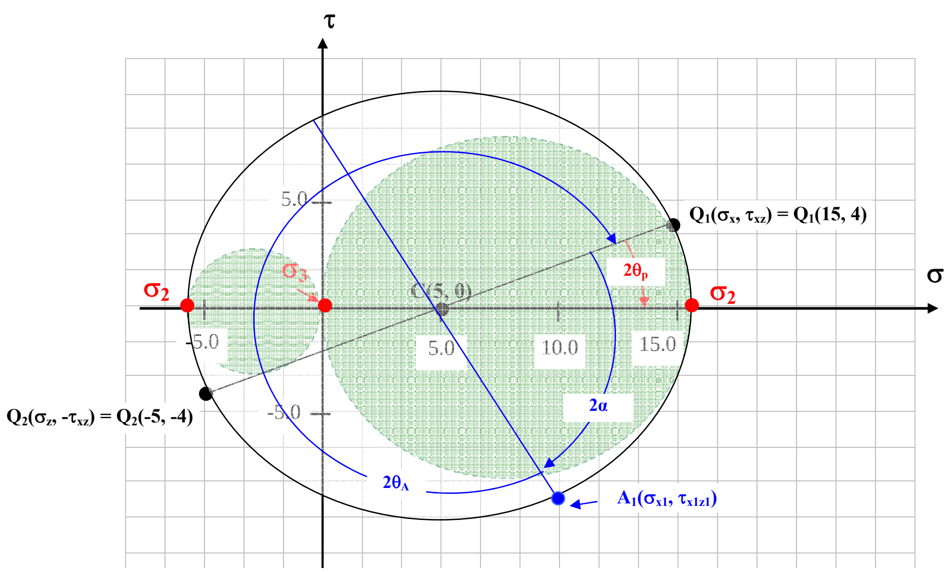


Figure 2.12: Mohr's circle for plane stress in the x-z plane.

2.6c) Calculate angles:

(All measured positive clockwise from $\overline{Q_1C}$)

First calculate $2\,\theta_p'$,

$$\tan 2\,\theta_p' = \frac{\tau_{\text{xz}}}{\sigma_{\text{diff}}} \qquad 2\,\theta_p' = \tan^{-1}\left\{\frac{\tau_{\text{xz}}}{\sigma_{\text{diff}}}\right\} = \tan^{-1}\left\{\frac{4}{10}\right\} = 21.8014^\circ = 0.380506 \text{ rads}$$

Now consider the location of Q_1:

$$\text{CASE A:} \quad Q_1 \to \text{first quadrant} \quad (\sigma_{\text{xx}} > 0, \quad \tau_{\text{xy}} > 0)$$

Thus

$$2\,\theta_p = 2\,\theta_p' = 21.8014^\circ = 0.380506 \text{ rads}$$

Principal stresses act on an element inclined at an angle θ_p are

$$\theta_p \quad = \quad \frac{1}{2}\,(2\,\theta_p) = 10.9007^\circ = 0.190253 \text{ rads}$$

Minimum/maximum in-plane shear stresses act on an element inclined at an angle θ_s

$$2\,\theta_s \quad = \quad 2\,\theta_p \pm 90^\circ$$

Note that at a rotation of $2\,\theta_s = 2\,\theta_p + 90^\circ$ from $\overline{Q_1C}$ the value of the in-plane shear stress is negative thus it gives the minimum shear stress, the maximum is obtained by taking $2\,\theta_s = 2\,\theta_p - 90^\circ$. In short,

$$\text{Maximum Shear Stress: } \tau_{\max}: \quad 2\,\theta_s \quad = 2\,\theta_p - 90^\circ = -68.1986^\circ \quad = -1.19029 \text{ rads}$$

$$\text{Minimum Shear Stress: } \tau_{\min}: \quad 2\,\theta_s \quad = 2\,\theta_p + 90^\circ = 111.801^\circ \quad = 1.9513 \text{ rads}$$

Transformed stresses act on an element inclined at an angle α

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha = -68.1986^\circ = -1.19029 \text{ rads} \qquad (\alpha = 45^\circ)$$

or

$$2\,\theta_A + 360^\circ = 291.801^\circ = 5.0929 \text{ rads} \qquad \text{(to measure clockwise)}$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element. Also, note that $2\,\theta_p$ is measured from $\overline{Q_1C}$ to positive σ-axis.

2.6d) Determine the normal and shear stresses on the inclined plane(s)

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} \quad = \quad \sigma_{\text{ave}} + R\cos(2\,\theta_A) = 9 \text{ MPa}$$

$$\sigma_{z_1} \quad = \quad \sigma_{\text{ave}} - R\cos(2\,\theta_A) = 1 \text{ MPa}$$

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1 z_1} \quad = \quad R\,\sin(2\,\theta_A) = -10 \text{ MPa}$$

2.6e) Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0^\circ$, therefore the principal normal stresses are found as follows

$$\begin{aligned} \lambda_1 &= \sigma_{\text{ave}} + R = (5) + (10.7703) = 15.77033 \text{ MPa} \\ \lambda_2 &= \sigma_{\text{ave}} - R = (5) - (10.7703) = -5.77033 \text{ MPa} \\ \lambda_3 &= 0 \end{aligned}$$

The principal stresses are chosen as:

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 15.77033 \text{ MPa}$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -5.770333 \text{ MPa}$$

Thus the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

The maximum and minimum normal stresses acting on an element inclined at an angle θ_p are

$$\sigma_{\max} = \sigma_1 = 15.77033 \text{ MPa}$$
$$\sigma_{\min} = \sigma_3 = -5.770333 \text{ MPa}$$

The in-plane maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{\max}\Big|_{\text{in-plane}} = R = \frac{\sigma_1 - \sigma_2}{2} = 10.77033 \text{ MPa}$$

The maximum in-plane shear stresses will be:

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = 7.885 \text{ MPa}$$
$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = 2.885 \text{ MPa}$$
$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = 10.770 \text{ MPa}$$

The overall maximum shear stress acting on an element inclined at an angle θ_s is

$$\tau_{\max} = \left|\frac{\sigma_{\max} - \sigma_{\min}}{2}\right| = 10.77033 \text{ MPa}$$

2.6f) Show all results on sketches of properly oriented elements

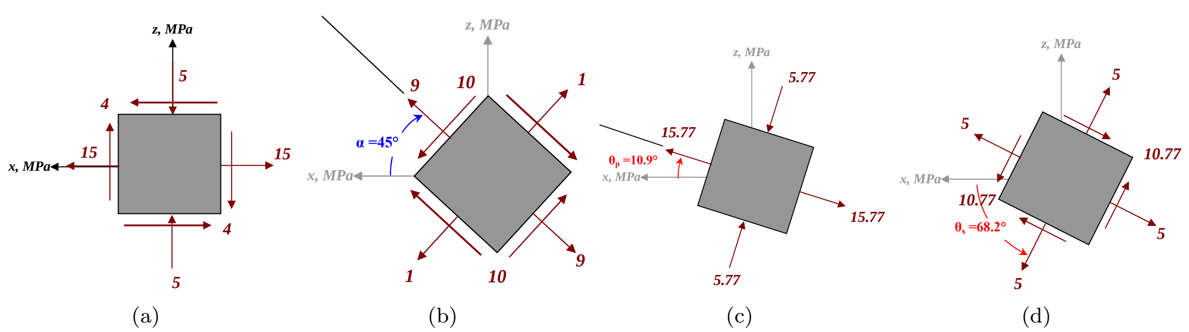


Figure 2.13: a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum in-plane shear stresses.

End Example □

Case 2.1.

Uniaxial Tension

Consider the state of stress at a given point as

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_\text{o} & \sigma_\text{o} & 0 \\ \sigma_\text{o} & \sigma_\text{o} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where σ_o is a constant stress, determine the principal stresses and plot the Mohr's circles.

The principal stresses are determined by finding the eigenvalues of the stress tensor:

$$\det \begin{bmatrix} \sigma_\text{xx} - \lambda & \tau_\text{xy} & \tau_\text{xz} \\ \tau_\text{yx} & \sigma_\text{yy} - \lambda & \tau_\text{yz} \\ \tau_\text{zx} & \tau_\text{zy} & \sigma_\text{zz} - \lambda \end{bmatrix} = \begin{vmatrix} \sigma_\text{o} - \lambda & \sigma_\text{o} & 0 \\ \sigma_\text{o} & \sigma_\text{o} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

which leads to the characteristic equation that can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1} \lambda^2 + I_{\sigma_2} \lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants.

$$\begin{aligned} I_{\sigma_1} &= \sigma_\text{xx} + \sigma_\text{yy} + \sigma_\text{zz} = 2\,\sigma_\text{o} \\ I_{\sigma_2} &= \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xy} \\ \tau_\text{yx} & \sigma_\text{yy} \end{bmatrix} + \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xz} \\ \tau_\text{zx} & \sigma_\text{zz} \end{bmatrix} + \det \begin{bmatrix} \sigma_\text{yy} & \tau_\text{yz} \\ \tau_\text{zy} & \sigma_\text{zz} \end{bmatrix} \\ &= \begin{vmatrix} \sigma_\text{o} & \sigma_\text{o} \\ \sigma_\text{o} & \sigma_\text{o} \end{vmatrix} + \begin{vmatrix} \sigma_\text{o} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \sigma_\text{o} & 0 \\ 0 & 0 \end{vmatrix} = 0 \\ I_{\sigma_3} &= \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xy} & \tau_\text{xz} \\ \tau_\text{yx} & \sigma_\text{yy} & \tau_\text{yz} \\ \tau_\text{zx} & \tau_\text{zy} & \sigma_\text{zz} \end{bmatrix} = \begin{vmatrix} \sigma_\text{o} & \sigma_\text{o} & 0 \\ \sigma_\text{o} & \sigma_\text{o} & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \end{aligned}$$

Thus, the characteristic equation becomes

$$\lambda^3 - 2\,\sigma_\text{o}\,\lambda^2 = \lambda^2\,(\lambda - 2\,\sigma_\text{o}) = 0$$

The three roots of the characteristic equation are

$$\lambda_1 = 2\,\sigma_\text{o} \quad \lambda_2 = 0 \quad \lambda_3 = 0$$

and the principal stresses are

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 2\,\sigma_\circ$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = 0$$
$$\sigma_2 = 0$$

The Mohr's circle is shown in Fig. 2.14. Here because of the double-zero root, one of the three Mohr's circles degenerated into a point (origin) and the other two circles coincide. Also, note that physically this is simply equivalent to a one-dimensional tension in the principal plane $\underline{\hat{\mathbf{n}}}^{(1)}$.

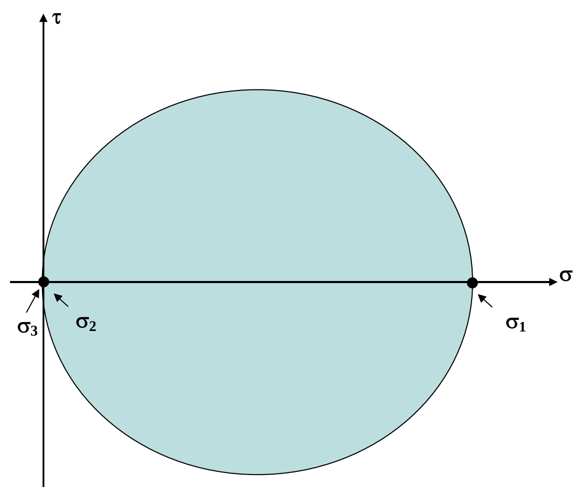


Figure 2.14: Mohr's circle case for uniaxial state of stress.

End Case □

Case 2.2.

Similar to Uniaxial Tension

Consider the state of stress at a given point as

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 2\,\sigma_\text{o} & 0 & 0 \\ 0 & \sigma_\text{o} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where σ_o is a constant stress, determine the principal stresses and plot the Mohr's circles.

The principal stresses are determined by finding the eigenvalues of the stress tensor:

$$\det \begin{bmatrix} \sigma_\text{xx} - \lambda & \tau_\text{xy} & \tau_\text{xz} \\ \tau_\text{yx} & \sigma_\text{yy} - \lambda & \tau_\text{yz} \\ \tau_\text{zx} & \tau_\text{zy} & \sigma_\text{zz} - \lambda \end{bmatrix} = \begin{vmatrix} 2\,\sigma_\text{o} - \lambda & 0 & 0 \\ 0 & \sigma_\text{o} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

which leads to the characteristic equation that can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants.

$$I_{\sigma_1} = \sigma_\text{xx} + \sigma_\text{yy} + \sigma_\text{zz} = 3\,\sigma_\text{o}$$

$$I_{\sigma_2} = \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xy} \\ \tau_\text{yx} & \sigma_\text{yy} \end{bmatrix} + \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xz} \\ \tau_\text{zx} & \sigma_\text{zz} \end{bmatrix} + \det \begin{bmatrix} \sigma_\text{yy} & \tau_\text{yz} \\ \tau_\text{zy} & \sigma_\text{zz} \end{bmatrix}$$

$$= \begin{vmatrix} 2\,\sigma_\text{o} & 0 \\ 0 & \sigma_\text{o} \end{vmatrix} + \begin{vmatrix} 2\,\sigma_\text{o} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \sigma_\text{o} & 0 \\ 0 & 0 \end{vmatrix} = 2\,\sigma_\text{o}^2$$

$$I_{\sigma_3} = \det \begin{bmatrix} \sigma_\text{xx} & \tau_\text{xy} & \tau_\text{xz} \\ \tau_\text{yx} & \sigma_\text{yy} & \tau_\text{yz} \\ \tau_\text{zx} & \tau_\text{zy} & \sigma_\text{zz} \end{bmatrix} = \begin{vmatrix} 2\,\sigma_\text{o} & 0 & 0 \\ 0 & \sigma_\text{o} & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Thus, the characteristic equation becomes

$$\lambda^3 - 3\,\sigma_\text{o}\,\lambda^2 + 2\,\sigma_\text{o}^2\,\lambda = \lambda\,\left(\lambda - 3\,\sigma_\text{o}\,\lambda + 2\,\sigma_\text{o}^2\right) = 0$$

The three roots of the characteristic equation are

$$\lambda_1 = 2\,\sigma_\text{o} \quad \lambda_2 = \sigma_\text{o} \quad \lambda_3 = 0$$

and the principal stresses are

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 2\,\sigma_o$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = 0$$
$$\sigma_2 = \sigma_o$$

The Mohr's circle is shown in Fig. 2.15. Here one could try to infer it is a one-dimensional case but this is not correct. The reason is that there is a Mohr's circle between σ_2 and σ_3.

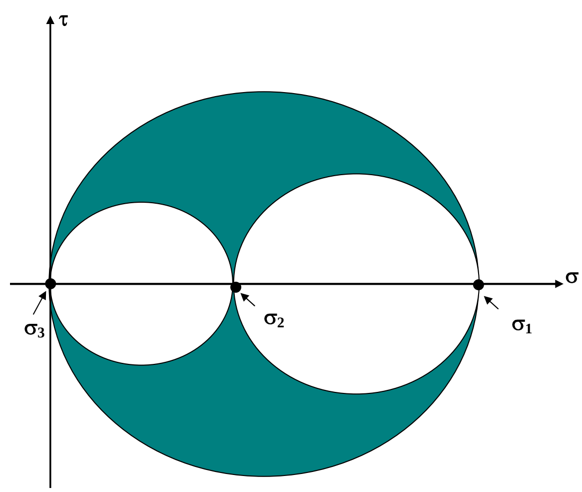


Figure 2.15: Mohr's circle case for triaxial state of stress.

End Case □

Case 2.3.

Hydrostatic State of Stress

Consider the state of stress at a given point as

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_o & 0 & 0 \\ 0 & \sigma_o & 0 \\ 0 & 0 & \sigma_o \end{bmatrix}$$

where σ_o is a constant stress, determine the principal stresses and plot the Mohr's circles.

The principal stresses are determined by finding the eigenvalues of the stress tensor:

$$\det \begin{bmatrix} \sigma_{xx} - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \lambda & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \lambda \end{bmatrix} = \begin{vmatrix} \sigma_o - \lambda & 0 & 0 \\ 0 & \sigma_o - \lambda & 0 \\ 0 & 0 & \sigma_o - \lambda \end{vmatrix} = 0$$

which leads to the characteristic equation that can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1} \lambda^2 + I_{\sigma_2} \lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants.

$$I_{\sigma_1} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 3\,\sigma_o$$

$$I_{\sigma_2} = \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} + \det \begin{bmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{bmatrix} + \det \begin{bmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$= \begin{vmatrix} \sigma_o & 0 \\ 0 & \sigma_o \end{vmatrix} + \begin{vmatrix} \sigma_o & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \sigma_o & 0 \\ 0 & 0 \end{vmatrix} = \sigma_o^2$$

$$I_{\sigma_3} = \det \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{vmatrix} \sigma_o & 0 & 0 \\ 0 & \sigma_o & 0 \\ 0 & 0 & \sigma_o \end{vmatrix} = \sigma_o^3$$

Thus, the characteristic equation becomes

$$\lambda^3 - 3\,\sigma_o\,\lambda^2 + \sigma_o^2\,\lambda - \sigma_o^3 = (\lambda - \sigma_o)\,(\lambda - \sigma_o)\,(\lambda - \sigma_o) = 0$$

The three roots of the characteristic equation are

$$\lambda_1 = \sigma_o \quad \lambda_2 = \sigma_o \quad \lambda_3 = \sigma_o$$

and the principal stresses are

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = \sigma_\circ$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = \sigma_\circ$$
$$\sigma_2 = \sigma_\circ$$

The Mohr's circle is shown in Fig. 2.16. Here because of the triple-zero root, all of the three Mohr's circles degenerated into a point. The classical physical example of this is the state of stress in a fluid at rest which is known as hydrostatic stress, and for which $\sigma_\circ = -p$, the static pressure.

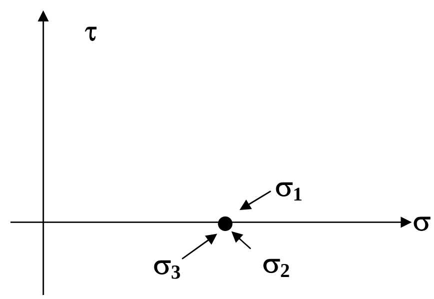


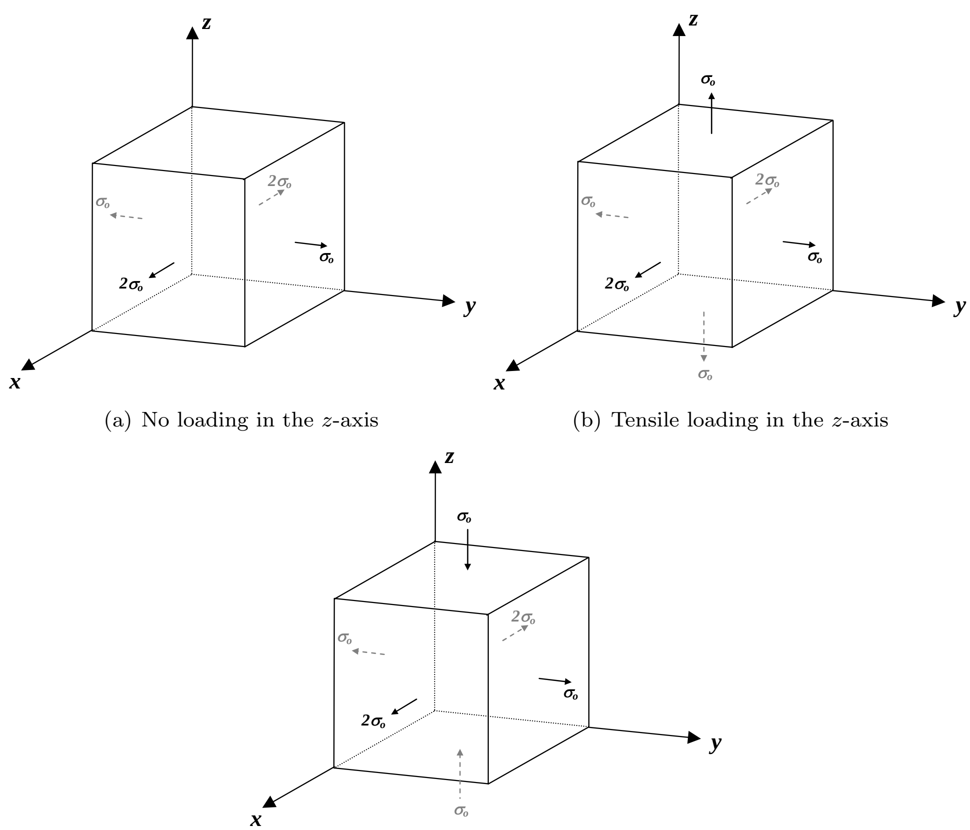
Figure 2.16: Mohr's circle case for hydrostatic state of stress.

End Case □

Case 2.4.

Consequences on the Overall Maximum Shear Stress

Consider three different state of stresses with the same loading in the x and y direction but different loadings in the z-direction:



(a) No loading in the z-axis

(b) Tensile loading in the z-axis

(c) Compressive loading in the z-axis

For each of the state of stresses determine the consequences on the overall maximum shear stress:

$$\underline{\boldsymbol{\sigma}}_{\mathrm{A}} = \begin{bmatrix} 2\,\sigma_{\mathrm{o}} & 0 & 0 \\ 0 & \sigma_{\mathrm{o}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.69a}$$

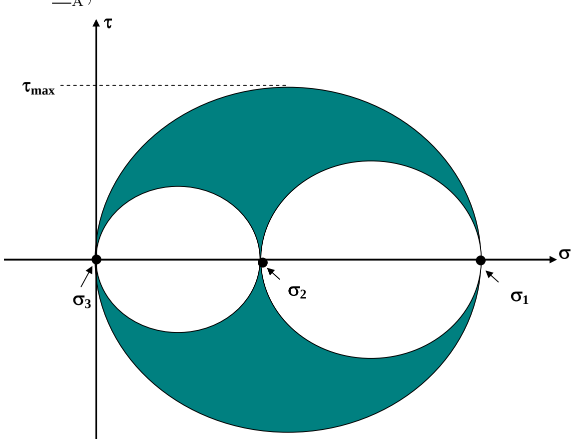
$$\underline{\boldsymbol{\sigma}}_{\mathrm{B}} = \begin{bmatrix} 2\,\sigma_{\mathrm{o}} & 0 & 0 \\ 0 & \sigma_{\mathrm{o}} & 0 \\ 0 & 0 & \sigma_{\mathrm{o}} \end{bmatrix} \tag{2.69b}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{C}} = \begin{bmatrix} 2\,\sigma_{\mathrm{o}} & 0 & 0 \\ 0 & \sigma_{\mathrm{o}} & 0 \\ 0 & 0 & -\sigma_{\mathrm{o}} \end{bmatrix} \tag{2.69c}$$

where σ_{o} is a constant stress.

First we evaluate the principal stresses for each case and then find the overall maximum shear stress.

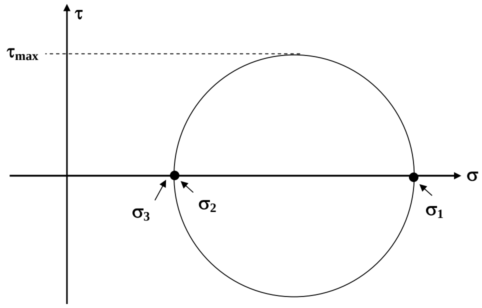
For $\underline{\boldsymbol{\sigma}}_{\mathrm{A}}$,



$$\underline{\boldsymbol{\sigma}}_{\mathrm{A}} = \begin{bmatrix} 2\,\sigma_{\mathrm{o}} & 0 & 0 \\ 0 & \sigma_{\mathrm{o}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \sigma_1 = 2\,\sigma_{\mathrm{o}} \quad \sigma_2 = \sigma_{\mathrm{o}} \quad \sigma_3 = 0$$

$$\tau_{\max} = \left| \frac{\sigma_{\max} - \sigma_{\min}}{2} \right| = \sigma_{\mathrm{o}}$$

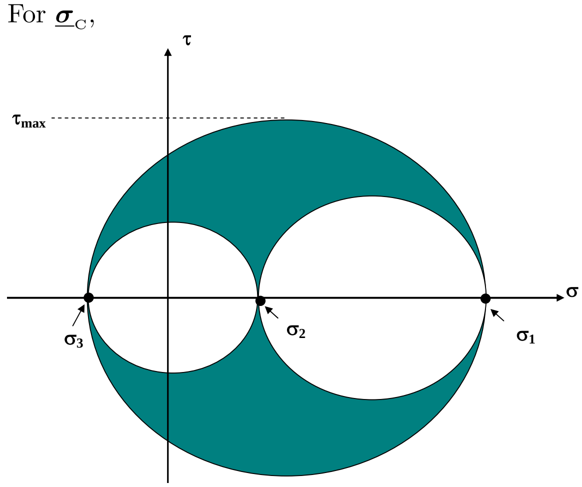
For $\underline{\boldsymbol{\sigma}}_{\mathrm{B}}$,



$$\underline{\boldsymbol{\sigma}}_{\rm B} = \begin{bmatrix} 2\,\sigma_{\rm o} & 0 & 0 \\ 0 & \sigma_{\rm o} & 0 \\ 0 & 0 & \sigma_{\rm o} \end{bmatrix} \qquad \rightarrow \quad \sigma_1 = 2\,\sigma_{\rm o} \quad \sigma_2 = \sigma_3 = \sigma_{\rm o}$$

$$\tau_{\rm max} = \left| \frac{\sigma_{\rm max} - \sigma_{\rm min}}{2} \right| = \frac{\sigma_{\rm o}}{2}$$

For $\underline{\boldsymbol{\sigma}}_{\rm C}$,



$$\underline{\boldsymbol{\sigma}}_{\rm C} = \begin{bmatrix} 2\,\sigma_{\rm o} & 0 & 0 \\ 0 & \sigma_{\rm o} & 0 \\ 0 & 0 & -\sigma_{\rm o} \end{bmatrix} \qquad \rightarrow \quad \sigma_1 = 2\,\sigma_{\rm o} \quad \sigma_2 = \sigma_{\rm o} \quad \sigma_3 = -\sigma_{\rm o}$$

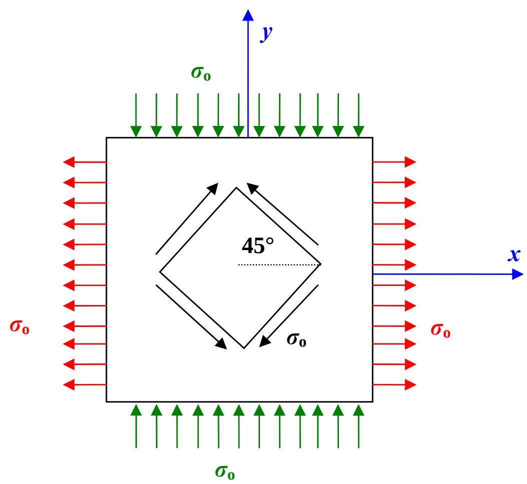
$$\tau_{\rm max} = \left| \frac{\sigma_{\rm max} - \sigma_{\rm min}}{2} \right| = \frac{3}{2}\,\sigma_{\rm o}$$

From these very interesting cases we can observe that when adding a compressive traction to the point of stress, it improves the shear capacity; and when adding a tensile traction, it worsens the shear capacity. This observations became handy when studying steady-load theories of failure.

End Case □

Case 2.5.

Pure Shear



A stress state of great practical importance is the state of pure shear characterized by principal stresses of equal magnitude but opposite signs:

$$\underline{\boldsymbol{\sigma}}_{\rm A} \quad = \quad \begin{bmatrix} \sigma_{\rm o} & 0 & 0 \\ 0 & -\sigma_{\rm o} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.70a)$$

First we evaluate the principal stresses for each case and then find the overall maximum shear stress.

$$\underline{\boldsymbol{\sigma}}_{\rm A} = \begin{bmatrix} \sigma_{\rm o} & 0 & 0 \\ 0 & -\sigma_{\rm o} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \sigma_1 = \sigma_{\rm o} \quad \sigma_2 = 0 \quad \sigma_3 = -\sigma_{\rm o}$$

$$\tau_{\rm max} = \left| \frac{\sigma_{\rm max} - \sigma_{\rm min}}{2} \right| = \sigma_{\rm o}$$

On faces oriented at 45° angles with respect to the principal stress directions, the direct stresses vanish and the shear has a maximum value, equal in magnitude to the common magnitudes of the two principal stresses.

End Case □

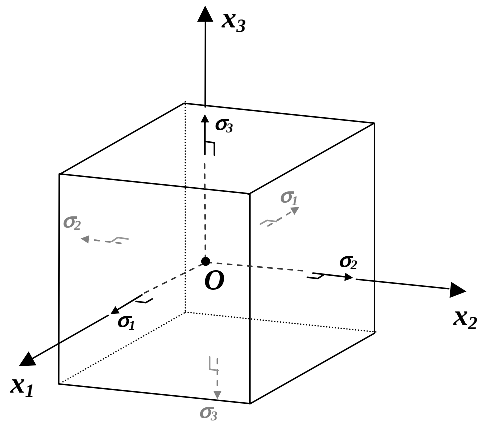


Figure 2.17: General state of stress for stresses acting on octahedral planes.

2.3 Important Stresses

2.3.1 Octahedral Stresses

Sometimes it is advantageous to represent the stresses on an octahedral stress element rather than on a conventional cubic element of principal stresses. Figures 2.17 and 2.18 and show the general state of stress for stresses acting on octahedral planes. These figures show the orientation of the eight octahedral planes that are associated with a given stress state. Each octahedral plane cuts across a corner of a principal element, so that the eight planes together form an octahedron. The following characteristics of the stresses on an octahedral plane should be noted:

1. Identical normal stresses act on all eight planes. Thus, the normal stresses tend to compress or enlarge the octahedron but not to distort it.

2. Identical shear stresses act on all eight planes. Thus, the shear stresses tend to distort the octahedron without changing its volume.

The fact that the normal and shear stresses are of equal magnitude for the eight planes is a powerful tool in failure analysis. Furthermore, the octahedral normal and shear stress can be expressed in terms of the principal normal stresses. Hence, our goal is to determine the octahedral normal stress, σ_{oct}, and the octahedral shear stress, τ_{oct}. The octahedral stresses are also known as hydrostatic stresses and will are denoted as $\underline{\mathbf{T}}^{(\text{oct})}$.

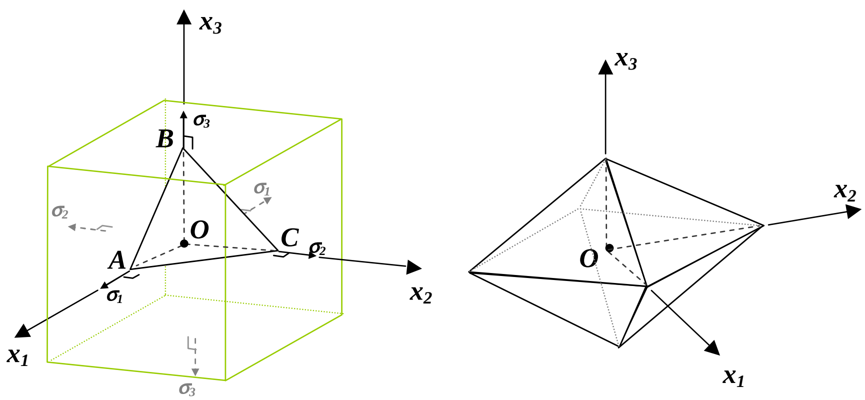


Figure 2.18: Tetrahedron element at **O**.

In order to derive these stresses we use Cauchy's relationship. We start the analysis with the principal stress element at point **O**. The general state of stress at **O** has only principal stresses

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{2.71}$$

and they are acting on the principal axes x_1-x_2-x_3, as shown in Fig. 2.18. All three axes are mutually orthogonal: x_1 is the principal plane $\hat{\underline{\mathbf{n}}}^{(1)}$, x_2 is the principal plane $\hat{\underline{\mathbf{n}}}^{(2)}$, and x_3 is the principal plane $\hat{\underline{\mathbf{n}}}^{(3)}$. Figure 2.18 shows the equilibrium of an infinitesimal tetrahedron at **O**.

We need to find the direction of the unit vector normal to the oblique face of the tetrahedron, $\hat{\underline{\mathbf{n}}}$:

$$\hat{\underline{\mathbf{n}}}_{\text{ACB}} = \frac{\underline{\mathbf{A}}_{\text{ACB}}}{\|\underline{\mathbf{A}}_{\text{ACB}}\|} \tag{2.72}$$

where $\underline{\mathbf{A}}_{\text{ACB}}$ is the area of the tetrahedron's oblique face. Note that for a tetrahedron $OA = OB = OC$. Let us take $OA = OB = OC = \Delta$. Let us proceed to obtain the area of **ACB**:

$$\underline{\mathbf{A}}_{\text{ACB}} = \frac{1}{2}\underline{\mathbf{r}}_{\text{(AC)}} \times \underline{\mathbf{r}}_{\text{(AB)}}$$

where

$$\underline{\mathbf{r}}_{\text{(AC)}} = \underline{\mathbf{C}} - \underline{\mathbf{A}} = \begin{Bmatrix} 0 \\ \Delta \\ 0 \end{Bmatrix} - \begin{Bmatrix} \Delta \\ 0 \\ 0 \end{Bmatrix} = -\Delta\,\hat{\underline{\mathbf{i}}} + \Delta\,\hat{\underline{\mathbf{j}}}$$

$$\underline{\mathbf{r}}_{\text{(AB)}} = \underline{\mathbf{B}} - \underline{\mathbf{A}} = \begin{Bmatrix} 0 \\ 0 \\ \Delta \end{Bmatrix} - \begin{Bmatrix} \Delta \\ 0 \\ 0 \end{Bmatrix} = -\Delta\,\hat{\underline{\mathbf{i}}} + \Delta\,\hat{\underline{\mathbf{j}}}$$

Hence, the area and its magnitude are

$$\underline{\mathbf{A}}_{\text{ACB}} = \frac{1}{2}\,\underline{\mathbf{r}}_{(\text{AC})} \times \underline{\mathbf{r}}_{(\text{AB})} = \frac{1}{2}\,\Delta^2\,\hat{\underline{\mathbf{i}}} + \frac{1}{2}\,\Delta^2\,\hat{\underline{\mathbf{j}}} + \frac{1}{2}\,\Delta^2\,\hat{\underline{\mathbf{k}}}$$

$$\|\underline{\mathbf{A}}_{\text{ACB}}\| = \sqrt{\underline{\mathbf{A}}_{\text{ABC}} \cdot \underline{\mathbf{A}}_{\text{ABC}}} = \sqrt{A_x^2 + A_y^2 + A_z^2} = \frac{\sqrt{3}}{2}\,\Delta^2$$

Thus, the unit vector on face **ACB** is

$$\hat{\underline{\mathbf{n}}}_{\text{ACB}} = \frac{1}{\sqrt{3}}\,\hat{\underline{\mathbf{i}}} + \frac{1}{\sqrt{3}}\,\hat{\underline{\mathbf{j}}} + \frac{1}{\sqrt{3}}\,\hat{\underline{\mathbf{k}}} \tag{2.73}$$

Now we find the stress vector on face **ACB** using Cauchy's formula

$$\underline{\mathbf{T}}^{(\text{ACB})} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \begin{Bmatrix} \frac{\sigma_1}{\sqrt{3}} \\ \frac{\sigma_2}{\sqrt{3}} \\ \frac{\sigma_3}{\sqrt{3}} \end{Bmatrix} = \frac{\sigma_1}{\sqrt{3}}\,\hat{\underline{\mathbf{i}}} + \frac{\sigma_2}{\sqrt{3}}\,\hat{\underline{\mathbf{j}}} + \frac{\sigma_3}{\sqrt{3}}\,\hat{\underline{\mathbf{k}}}$$

Since the octahedral stress vector is the stress vector acting on face **ACB**:

$$\underline{\mathbf{T}}^{(\text{oct})} = \underline{\mathbf{T}}^{(\text{ACB})} = \frac{\sigma_1}{\sqrt{3}}\,\hat{\underline{\mathbf{i}}} + \frac{\sigma_2}{\sqrt{3}}\,\hat{\underline{\mathbf{j}}} + \frac{\sigma_3}{\sqrt{3}}\,\hat{\underline{\mathbf{k}}} \tag{2.74}$$

The magnitude of the normal octahedral stress can be obtained using Eq. (2.5):

$$\sigma_{\text{oct}} = \underline{\mathbf{T}}^{(\text{oct})} \cdot \hat{\underline{\mathbf{n}}}_{\text{oct}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{2.75}$$

where $\hat{\underline{\mathbf{n}}}_{\text{oct}} = \hat{\underline{\mathbf{n}}}_{\text{ACB}}$. The magnitude of the shear octahedral stress can be obtained using Eq. (2.7):

$$\tau_{\text{oct}} = \sqrt{\left\|\underline{\mathbf{T}}^{(\text{oct})}\right\|^2 - \sigma_{\text{oct}}^2} = \frac{\sqrt{2}}{3}\,\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\,\sigma_2 - \sigma_1\,\sigma_3 - \sigma_2\,\sigma_3} \tag{2.76}$$

We can rearrange the above equation as follows

$$\begin{aligned} \tau_{\text{oct}} &= \frac{1}{3}\,\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\ &= \frac{1}{3}\,\sqrt{(\sigma_{\text{xx}} - \sigma_{\text{yy}})^2 + (\sigma_{\text{yy}} - \sigma_{\text{zz}})^2 + (\sigma_{\text{zz}} - \sigma_{\text{xx}})^2 + 6\,\left(\tau_{\text{xy}}^2 + \tau_{\text{yz}}^2 + \tau_{\text{xz}}^2\right)} \end{aligned} \tag{2.77}$$

The octahedral stresses can also be expressed in terms of the stress invariants as follows

$$\begin{bmatrix} \sigma_1 - \lambda & 0 & 0 \\ 0 & \sigma_2 - \lambda & 0 \\ 0 & 0 & \sigma_3 - \lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The nontrivial solution is given when the determinant of the matrix of coefficients of n_x, n_y, and n_z vanish:

$$\begin{vmatrix} \sigma_1-\lambda & 0 & 0 \\ 0 & \sigma_2-\lambda & 0 \\ 0 & 0 & \sigma_3-\lambda \end{vmatrix} = 0$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda - I_{\sigma_3} = 0$$

where I_{σ_i}'s are the stress invariants. Using the definition of stress invariants:

$$I_{\sigma_1} = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_{\sigma_2} = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_3 \end{vmatrix} + \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{vmatrix} = \sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_1\,\sigma_3$$

$$I_{\sigma_3} = \begin{vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{vmatrix} = \sigma_1\,\sigma_2\,\sigma_3$$

We can express the normal octahedral (hydrostatic) stress in terms of the stress invariants as follows:

$$\sigma_{\text{oct}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_{\sigma_1}}{3} \tag{2.78}$$

and the shear octahedral stress in terms of the stress invariants as follows:

$$\tau_{\text{oct}} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} = \frac{1}{3}\sqrt{2\,I_{\sigma_1}^2 - 6\,I_{\sigma_2}} \tag{2.79}$$

2.3.2 Von Mises Stress

The triaxial state of stress can be expressed in terms of the following equation

$$\tau_{\text{oct}} = \frac{\sqrt{2}}{3}\,\sigma_{\text{eq}} \tag{2.80}$$

where σ_{eq} is known as the von Misses stress, effective stress, or equivalent uniaxial stress. The derivation for the above equation is left for the discussion of steady-load failure criterions. However, here we define the von Mises stress using the expression for τ_{oct}:

$$\begin{aligned}
\sigma_{\text{eq}} &= \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} \\
&= \sqrt{\frac{(\sigma_{\text{xx}}-\sigma_{\text{yy}})^2+(\sigma_{\text{yy}}-\sigma_{\text{zz}})^2+(\sigma_{\text{zz}}-\sigma_{\text{xx}})^2+6\,\left(\tau_{\text{xy}}^2+\tau_{\text{yz}}^2+\tau_{\text{xz}}^2\right)}{2}} \\
&= \sqrt{I_{\sigma_1}^2-3\,I_{\sigma_2}}
\end{aligned} \tag{2.81}$$

Note that the von Mises stress is invariant as it only depends on stress invariants.

For a uniaxial state of stress

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\text{xx}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\sigma_2 = \sigma_3 = 0$,

$$\sigma_{\text{eq}} = \sigma_1 = \sigma_{\text{xx}}$$

For a state of plane stress (say the x-y plane)

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & 0 \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\sigma_3 = 0$,

$$\sigma_{\text{eq}} = \sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\,\sigma_2} = \sqrt{\sigma_{\text{xx}}^2+\sigma_{\text{yy}}^2-\sigma_{\text{yy}}\,\sigma_{\text{xx}}+3\,\tau_{\text{xy}}^2}$$

Reconsider the particular case of combined loading, where $\sigma_{\text{yy}} = \sigma_{\text{zz}} = \tau_{\text{xz}} = \tau_{\text{yz}} = 0$. Then the above expression becomes

$$\sigma_{\text{eq}} = \sqrt{\sigma_{\text{xx}}^2+3\,\tau_{\text{xy}}^2}$$

2.4 Theory of Strains

A structure deforms as we apply loads. The deformation may produce change in the dimensions of the body, change in its shape. Deformations induce strains throughout the structure. Thus, strain may defined as a measure of the relative distortion of the material in the vicinity of a given point. Normal strains have units of [Length]/[Length] and shear strains have units of [radians]; and they are directionally dependent.

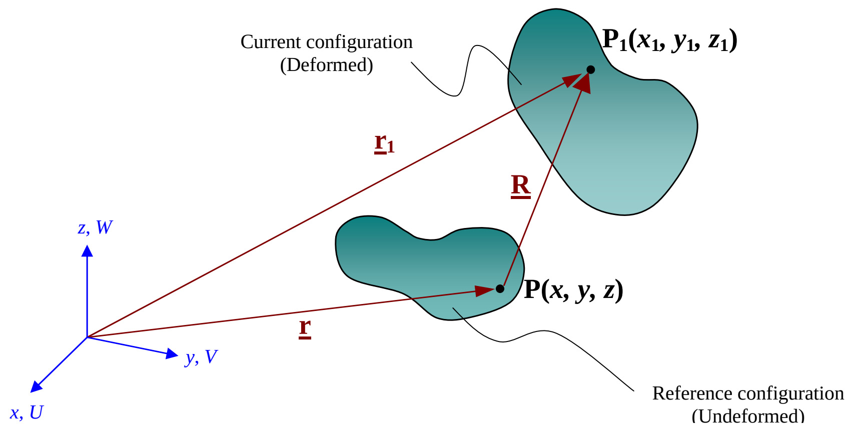


Figure 2.19: Deformation of a solid body from the initial configuration, $\mathcal{C}^0$, to the current configuration, $\mathcal{C}^1$

The formulation of strains is more complex that that of stresses. One reason is due to the non-linear terms. As is was in the case of stresses, we are interested in deriving the state of strain in the neighborhood of a given point. The state of strain may be defined as the deformation of a solid in the neighborhood of a given point, say point **P** of position vector

$$\underline{\mathbf{r}} = x\,\hat{\underline{\mathbf{i}}} + y\,\hat{\underline{\mathbf{j}}} + z\,\hat{\underline{\mathbf{k}}} \tag{2.82}$$

as shown in Fig. 2.19.

To better understand the concept of strain let us start by defining the deformed and undeformed states. The reference configuration is the configuration of the solid in its undeformed state. Under the effect of applied loads, the body deforms and assumes a new configuration, called the deformed configuration. Figure 2.19 shows an arbitrary body in its initial and deformed configurations. Let the body in its undeformed configuration have a volume designated Γ, external surface area Ω, mass density ρ, and reference material points of the body to cartesian coordinates x, y, z. Denote the deformed configuration with a volume Γ_1, external surface area Ω_1, mass density ρ_1, and reference material points of the body to cartesian coordinates x_1, y_1, z_1. Furthermore, let the coordinate system of the reference (undeformed) and current (deformed) configuration coincide.

The initial position of a point, **P**, with coordinates (x, y, z) is given by the position vector $\underline{\mathbf{r}}$, and the current position of the same point, $\mathbf{P}_1$, with coordinates (x_1, y_1, z_1) is given by the position vector $\underline{\mathbf{r}}_1$. Let us define the position vector $\underline{\mathbf{r}}_1$ as

$$\underline{\mathbf{r}}_1 = \underline{\mathbf{r}} + \underline{\mathbf{R}}$$

where[5]

$$\underline{\mathbf{R}} = \left\{ \begin{array}{c} U(x,y,z) \\ V(x,y,z) \\ W(x,y,z) \end{array} \right\} \qquad \underline{\mathbf{r}} = \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} \qquad \underline{\mathbf{r}}_1 = \left\{ \begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right\}$$

When loads are applied to an initially unstressed body, each unconstrained material particle undergoes a small displacement, moving from its initial location to a new location a small distance away, see Fig. 2.19.

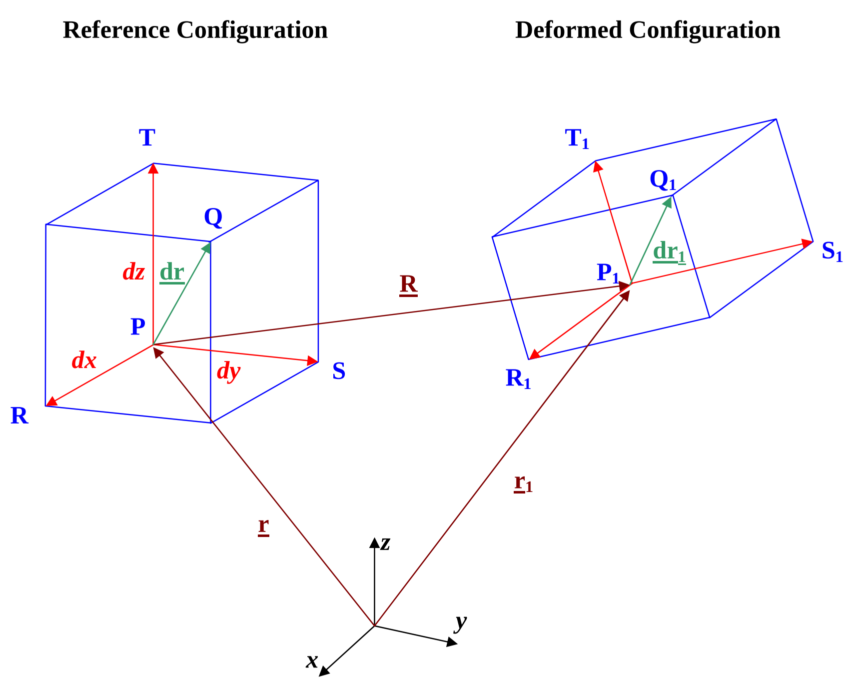


Figure 2.20: The neighborhood of point $\mathbf{P}$ in the reference and deformed configurations..

In order to visualize the deformed configuration, consider a small rectangular parallelepiped **PQRST** of differential volume $d\Gamma = dx\,dy\,dz$. Let us cut the parallelepiped in the neighborhood of point $\mathbf{P}$, as shown in Fig. 2.20. All the material particle that form the rectangular parallelepiped **PQRST** in the reference configuration now form the parallelepiped $\mathbf{P}_1\mathbf{Q}_1\mathbf{R}_1\mathbf{S}_1\mathbf{T}_1$ in the deformed configuration. The state of strain at a point characterizes the deformation of the parallelepiped without any consideration for the loads that created the deformation.

The displacement vector is a measure of how much a material point moves from the reference to the deformed configuration. The components of the displacement vector in the cartesian coordinates are

$$\underline{\mathbf{R}} = U(x,y,z)\,\hat{\underline{\mathbf{i}}} + V(x,y,z)\,\hat{\underline{\mathbf{j}}} + W(x,y,z)\,\hat{\underline{\mathbf{k}}} \tag{2.83}$$

[5]They could be time-dependent.

This displacement field describes the displacement of any point within the solid and consists of two parts: a rigid body motion and a deformation or straining of the solid. The rigid body motion itself consists of two parts: a rigid body translation and rigid body rotation. By definition, a rigid body motion does not strain the body. Strain measures extract from the displacement field the part that deforms the body.

Consider a differential line element $d\underline{\mathbf{r}}$ joining an arbitrary material point **P** to a neighboring material point **Q**, as shown in Fig. 2.20. The components of $d\underline{\mathbf{r}}$ are:

$$d\underline{\mathbf{r}} = dx\,\hat{\underline{\mathbf{i}}} + dy\,\hat{\underline{\mathbf{j}}} + dz\,\hat{\underline{\mathbf{k}}}$$

and the magnitude is

$$ds = \|d\underline{\mathbf{r}}\| = \sqrt{dx^2 + dy^2 + dz^2}$$

Thus the unit vector un the direction of $d\underline{\mathbf{r}}$ is

$$\hat{\underline{\mathbf{n}}} = \frac{d\underline{\mathbf{r}}}{ds} = \frac{dx}{ds}\,\hat{\underline{\mathbf{i}}} + \frac{dy}{ds}\,\hat{\underline{\mathbf{j}}} + \frac{dz}{ds}\,\hat{\underline{\mathbf{k}}}$$

During deformation, point **P** undergoes a displacement $\underline{\mathbf{R}}$, carrying it to $\mathbf{P}_1$. The displacement vector joins a point in the deformed body to its new location in the deformed state. The coordinates of $\mathbf{P}_1$ therefore are

$$x_1 = x + U(x,y,z) \qquad y_1 = y + V(x,y,z) \qquad z_1 = z + W(x,y,z) \tag{2.84}$$

Thus, after deformation, the vector joining $\mathbf{P}_1$ to $\mathbf{Q}_1$ is $d\underline{\mathbf{r}}_1$:

$$d\underline{\mathbf{r}}_1 = dx_1\,\hat{\underline{\mathbf{i}}} + dy_1\,\hat{\underline{\mathbf{j}}} + dz_1\,\hat{\underline{\mathbf{k}}}$$

where

$$dx_1 = dx + dU \qquad dy_1 = dy + dV \qquad dz_1 = dz + dW \tag{2.85}$$

From differential calculus, the total derivatives in Eq. (2.85) are:

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz$$

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz$$

$$dW = \frac{\partial W}{\partial x}\,dx + \frac{\partial W}{\partial y}\,dy + \frac{\partial W}{\partial z}\,dz$$

Thus, Eq. (2.85) may be written as:

$$dx_1 = dx + \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz$$

$$dy_1 = dy + \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz$$

$$dz_1 = dz + \frac{\partial W}{\partial x}\,dx + \frac{\partial W}{\partial y}\,dy + \frac{\partial W}{\partial z}\,dz$$

2.4.1 State of Strain

A material line is an ensemble of material particles that are in a straight line in the reference configuration of the body. For instance, segments **PR**, **PS** and **PT** of the reference configuration are material lines. Due to the deformation of the body, all the material particles forming material line **PR** will move to segment $\mathbf{P}_1\mathbf{R}_1$ in the deformed configuration. Due to the differential nature of this segment, it can be assumed to remain straight in the deformed configuration.

When comparing segment **PR** in the undeformed and deformed configurations, the motion consists of two parts: a change in orientation and a change in length. Clearly, the orientation change is a rigid body motion, whereas the change in length is a deformation or stretching of the material line. Similarly, segments **PR** and **PS** form a rectangle in the reference configuration, but a parallelogram in the deformed configuration. Here again, the change in orientation of the rectangle is a rigid body rotation, but the angular distortion of the rectangle into a parallelogram represents a deformation of the body. Stretching of the material lines and angular distortion between two material lines will be selected as measures of the state of strain at a point.

The stretching or relative elongations of materials lines **PR**, **PS** and **PT** will be denoted $e_{\rm xx}$, $e_{\rm yy}$ and $e_{\rm zz}$, respectively. The angular distortions between segments **PS** and **PT**, **PR** and **PT**, and **PR** and **PS** will be denoted $\gamma_{\rm yz}$, $\gamma_{\rm xz}$, and $\gamma_{\rm xy}$, respectively.

Extensional Strains

The extensional strain of the line element **PQ** in Fig. 2.20, is the ratio of the change of its length to its original length:

$$e_{\rm n} = \frac{ds_1 - ds}{ds} = \frac{ds_1}{ds} - 1$$

The extensional strain is obviously a dimensionless quantity and a typical order of magnitude is 10^{-3}. It is written often as 1000μ and is read as "1000 microstrain", where $\mu = 10^{-6}$ length units/ per length unit. In order to derive these strains let us only look at the material line **PR**. This will give us the total strain in the x-direction. The relative elongation, $e_{\rm xx}$, of the material line **PR** is defined as:

$$e_{\rm xx} = \frac{ds_1}{ds} - 1 = \frac{\mathbf{P}_1\mathbf{R}_1}{\mathbf{PR}} - 1$$

The length of the material lines in the undeformed and deformed configurations are

$$\mathbf{PR} = \left\| dx\,\underline{\hat{\mathbf{i}}} \right\| = dx$$

$$\mathbf{P}_1\mathbf{R}_1 = \left\| (dx + dU)\,\underline{\hat{\mathbf{i}}} \right\| = \sqrt{1 + 2\,\frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2}\,dx$$

The extensional, or relative, elongation becomes

$$e_{xx} = \frac{\mathbf{P}_1\mathbf{R}_1}{\mathbf{PR}} - 1$$

$$= \frac{\sqrt{1 + 2\frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2}\, dx}{dx} - 1$$

$$= \sqrt{1 + 2\frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2} - 1$$

We can replace the squared operator by using the following expansion

$$(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 + \cdots$$

and for a small ϵ

$$(1+\epsilon)^{1/2} \approx 1 + \frac{1}{2}\epsilon \tag{2.86}$$

Thus,

$$e_{xx} = \frac{\partial U}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right\}$$

Similarly,

$$e_{yy} = \frac{\partial V}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right\}$$

$$e_{zz} = \frac{\partial W}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right\}$$

In most aerospace engineering materials, strains on the order of 1% or more may cause damage, which is unacceptable. The fact that in most applications are indeed quite small when compared to 1, justifies a fundamental assumption of linear elasticity which states that all displacement components remain very small so that all second order terms can be neglected. As a consequence:

$$|U| \ll 1 \qquad |V| \ll 1 \qquad |W| \ll 1 \qquad \left|\frac{\partial U}{\partial x}\right| \ll 1 \qquad \left|\frac{\partial U}{\partial y}\right| \ll 1 \qquad \left|\frac{\partial U}{\partial z}\right| \ll 1$$

$$\left|\frac{\partial V}{\partial x}\right| \ll 1 \qquad \left|\frac{\partial V}{\partial y}\right| \ll 1 \qquad \left|\frac{\partial V}{\partial z}\right| \ll 1 \qquad \left|\frac{\partial W}{\partial x}\right| \ll 1 \qquad \left|\frac{\partial W}{\partial y}\right| \ll 1 \qquad \left|\frac{\partial W}{\partial z}\right| \ll 1$$

With these assumptions, the expressions for the relative elongation are reduced to:

$$e_{xx} = \frac{\partial U}{\partial x} \qquad e_{yy} = \frac{\partial V}{\partial y} \qquad e_{zz} = \frac{\partial W}{\partial z}$$

But we should treat this assumption with care as for some applications, such as helicopter blades, nonlinear terms are extremely important.

Shear Strain

Now let us obtain the appropriate measure of the change in shape of the solid body. To better understand the derivation, let us limit to the y-z plane, see Fig. 2.21.

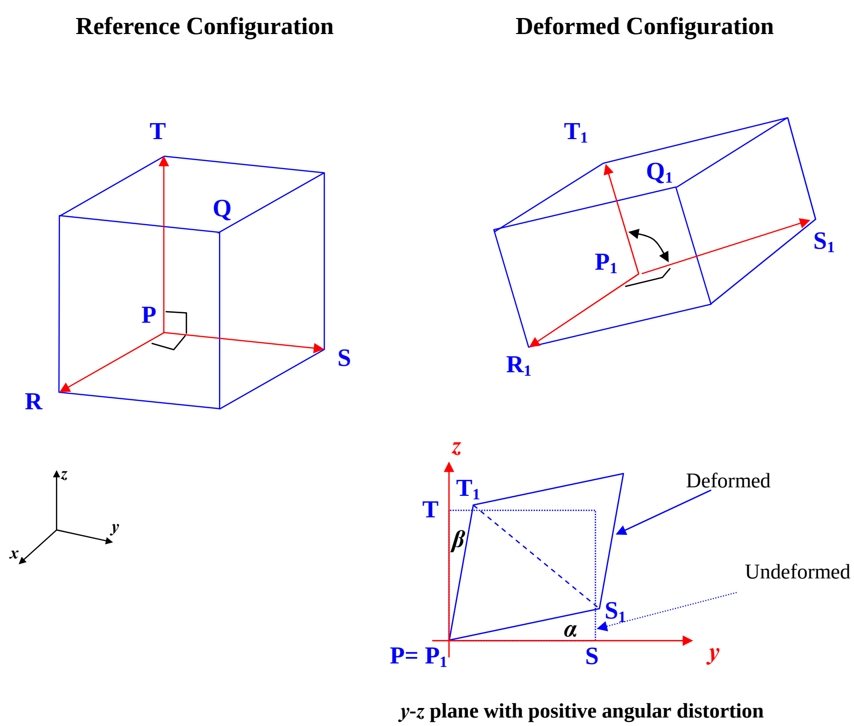


Figure 2.21: Shear deformation in the reference and deformed configurations..

The shear strain, or the angular distortion, γ_{yz}, between two material lines $\mathbf{PT}$ and $\mathbf{PS}$ is defined as the change of the initially right angle

$$\gamma_{yz} = \alpha + \beta = \measuredangle\mathbf{TPS} - \measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1 = \frac{\pi}{2} - \measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1$$

where $\measuredangle\mathbf{TPS}$ is is used to indicate the angle between segments $\mathbf{PT}$ and $\mathbf{PS}$. The shear strain, or the angular distortion, are also nondimensional quantities. To eliminate the difference between the two angles, the basic properties of the sine function are used: the sine of the angular distortion becomes

$$\sin\,\gamma_{yz} = \sin\left(\frac{\pi}{2} - \measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1\right) = \cos\measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1 \quad \rightarrow \quad \gamma_{yz} = \arcsin\left(\cos\measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1\right)$$

The cosine of the angle between the two material lines is computed from the following trigonometric identity, the law of cosines, applied to triangle $\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1$ in the deformed configuration

$$\|\mathbf{T}_1\mathbf{S}_1\|^2 = \|\mathbf{P}_1\mathbf{T}_1\|^2 + \|\mathbf{P}_1\mathbf{S}_1\|^2 - 2\,\|\mathbf{P}_1\mathbf{T}_1\|\,\|\mathbf{P}_1\mathbf{S}_1\|\,\cos\measuredangle\mathbf{T}_1\mathbf{P}_1\mathbf{S}_1$$

solving in terms of the cosine of the angle $\measuredangle \mathbf{T}_1\mathbf{P}_1\mathbf{S}_1$

$$\cos \measuredangle \mathbf{T}_1\mathbf{P}_1\mathbf{S}_1 = \frac{\|\mathbf{T}_1\mathbf{S}_1\|^2 - \|\mathbf{P}_1\mathbf{T}_1\|^2 - \|\mathbf{P}_1\mathbf{S}_1\|^2}{2\,\|\mathbf{P}_1\mathbf{T}_1\|\,\|\mathbf{P}_1\mathbf{S}_1\|}$$

Thus, the shear strain is

$$\gamma_{\text{yz}} = \arcsin\left(\cos \measuredangle \mathbf{T}_1\mathbf{P}_1\mathbf{S}_1\right) = \gamma_{\text{yz}} = \arcsin\left\{\frac{\|\mathbf{T}_1\mathbf{S}_1\|^2 - \|\mathbf{P}_1\mathbf{T}_1\|^2 - \|\mathbf{P}_1\mathbf{S}_1\|^2}{2\,\|\mathbf{P}_1\mathbf{T}_1\|\,\|\mathbf{P}_1\mathbf{S}_1\|}\right\}$$

The same procedure as used above in determining e_{xx} is used to compute $\|\mathbf{P}_1\mathbf{T}_1\|$ and $\|\mathbf{P}_1\mathbf{S}_1\|$:

$$\underline{\mathbf{P}_1\mathbf{T}_1} = \left(dy\,\hat{\underline{\mathbf{j}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial y}\,dy\right) = \left(\hat{\underline{\mathbf{j}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial y}\right)dy$$

$$\underline{\mathbf{P}_1\mathbf{S}_1} = \left(dz\,\hat{\underline{\mathbf{k}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial z}\,dz\right) = \left(\hat{\underline{\mathbf{k}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial z}\right)dz$$

Hence,

$$\underline{\mathbf{T}_1\mathbf{S}_1} = \underline{\mathbf{P}_1\mathbf{S}_1} - \underline{\mathbf{P}_1\mathbf{T}_1}$$

With some mathematical manipulation, it can be shown that

$$\begin{aligned}
\text{numerator} &= \|\mathbf{T}_1\mathbf{S}_1\|^2 - \|\mathbf{P}_1\mathbf{T}_1\|^2 - \|\mathbf{P}_1\mathbf{S}_1\|^2 \\
&= 2\left(\hat{\underline{\mathbf{j}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial y}\right)\cdot\left(\hat{\underline{\mathbf{k}}} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial z}\right)dy\,dz \\
&= 2\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} + \frac{\partial \underline{\underline{\mathbf{R}}}}{\partial y}\cdot\frac{\partial \underline{\underline{\mathbf{R}}}}{\partial z}\right)dy\,dz \\
&= 2\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} + \frac{\partial U}{\partial y}\,\frac{\partial U}{\partial z} + \frac{\partial V}{\partial y}\,\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\,\frac{\partial W}{\partial z}\right)dy\,dz
\end{aligned}$$

The denominator is expressed in the same manner:

$$\begin{aligned}
\text{denominator} &= 2\,\|\mathbf{P}_1\mathbf{T}_1\|\,\|\mathbf{P}_1\mathbf{S}_1\| \\
&= 2\sqrt{1 + 2\frac{\partial U}{\partial y} + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2}\,dy\;\times \\
&\qquad \sqrt{1 + 2\frac{\partial W}{\partial z} + \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2}\,dz
\end{aligned}$$

Using Eq. (2.86), it can be shown that

$$\text{denominator} = 2\,\|\mathbf{P}_1\mathbf{T}_1\|\,\|\mathbf{P}_1\mathbf{S}_1\| \approx 1 + \frac{\partial U}{\partial y} + \frac{\partial W}{\partial z}$$

For moderately small rotations, the angular distortion, or the shear strain about the x-axis can be approximated as

$$\gamma_{\text{yz}} = \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} + \frac{\partial U}{\partial y}\frac{\partial U}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\frac{\partial W}{\partial z}$$

Likewise,

$$\gamma_{\text{xy}} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial V}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial W}{\partial x}$$

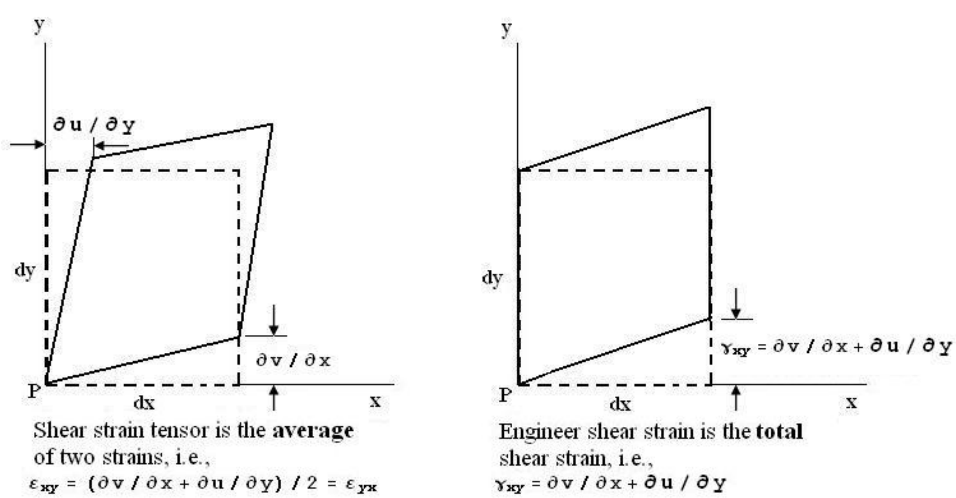
$$\gamma_{\text{xz}} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial U}{\partial z} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial z} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial z}$$

For small displacements,

$$\gamma_{\text{xy}} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \quad \gamma_{\text{xz}} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \quad \gamma_{\text{yz}} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial z} \tag{2.87}$$

Now recall that the engineering shear strain (γ_{shear}) is related to the true elasticity shear strain (e_{shear}) as

$$e_{\text{shear}} = \frac{\gamma_{\text{shear}}}{2} \tag{2.88}$$



Thus

$$e_{xy} = \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) \tag{2.89}$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)$$

These strains are often called the Green-Cauchy strains.

Green Tensor or Cauchy's Strain Tensor

The components os strain relative to any set of orthogonal axes are therefore known if we are given the state of strain (relative to a cartesian coordinate system),

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}$$

The above is known as the strain tensor. Further we can show that the strain tensor is a symmetric strain matrix,

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} \tag{2.90}$$

In terms of the engineering shear strains

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix} \tag{2.91}$$

Example 2.7.

The following displacement field describes the movement of a body under load:

$$\underline{\mathbf{R}} = 0.01\,\left(x^2+3\right)\,\underline{\hat{\mathbf{i}}} + 0.01\,\left(3\,y^2\,z\right)\,\underline{\hat{\mathbf{j}}} + 0.01\,\left(x+3\,z\right)\,\underline{\hat{\mathbf{k}}} \qquad \text{m}$$

(2.7a) Determine the strain tensor.

The strain tensor is

$$\underline{e} = \begin{bmatrix} e_{xx} & \frac{1}{2}\,\gamma_{xy} & \frac{1}{2}\,\gamma_{xz} \\ \frac{1}{2}\,\gamma_{xy} & e_{yy} & \frac{1}{2}\,\gamma_{yz} \\ \frac{1}{2}\,\gamma_{xz} & \frac{1}{2}\,\gamma_{yz} & e_{zz} \end{bmatrix}$$

and from the given problem,

$$U(x,y,z) = 0.01\,\left(x^2+3\right)$$

$$V(x,y,z) = 0.01\,\left(3\,y^2\,z\right)$$

$$W(x,y,z) = 0.01\,\left(x+3\,z\right)$$

Now, evaluating all strains we get

$$e_{xx} = \frac{\partial U}{\partial x} = 2\,(0.01)\,x \qquad e_{yy} = \frac{\partial V}{\partial y} = 6\,(0.01)\,y\,z$$

$$e_{zz} = \frac{\partial W}{\partial z} = 0.03 \qquad \gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0$$

$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = 0.010 \qquad \gamma_{yz} = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = 3\,(0.01)\,y^2$$

Thus, the strain tensor is

$$\underline{e} = \begin{bmatrix} e_{xx} & \frac{1}{2}\,\gamma_{xy} & \frac{1}{2}\,\gamma_{xz} \\ \frac{1}{2}\,\gamma_{xy} & e_{yy} & \frac{1}{2}\,\gamma_{yz} \\ \frac{1}{2}\,\gamma_{xz} & \frac{1}{2}\,\gamma_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} 2\,x & 0 & 0.5 \\ 0 & 6\,y\,z & 1.5\,y^2 \\ 0.5 & 1.5\,y^2 & 3 \end{bmatrix} \times 10^{-2}$$

(2.7b) Determine the state of strain at (0,2,3).

At the given point: $x = 0$, $y = 2$, $z = 3$. Thus the state of strain is

$$\underline{\boldsymbol{e}} = \begin{bmatrix} 0.000 & 0.000 & 0.005 \\ 0.000 & 0.360 & 0.060 \\ 0.005 & 0.060 & 0.030 \end{bmatrix}$$

End Example □

2.4.2 Strain compatibility equations

We defined the displacement field, $\underline{\mathbf{R}}$, to describe the motion and deformation of a solid. In order to accomplish this, the displacement field must be a single-valued, continuous function. If it is not single-valued, it means that certain points can have more than one displacement at a time, which is physically impossible. A a discontinuous displacement field means that originally infinitesimally close points would be separated by a finite amount in the deformed geometry, which takes place in presence of imperfections or cracks. However, this is beyond the scope of this text.

The state of strain also has a means of describing deformation of a solid for purposes of relating simply to the state of stress. Hence, we must ensure properly association with a physically deformation of a solid, free from cracks and fissures. To better understand this suppose a solid is before deformation consists of a system of infinitesimal contiguous cubes. Then, we must choose strain component functions that ensure that

1. each element is continuous.

2. no element partially or fully occupies the same space at the same time.

This is known as ensuring strain compatibility. One way to ensure strain compatible is by starting with a single-valued, continuous displacement field and developing the strains from this field in accordance with the strain-displacement equations discussed earlier. However, for some circumstances we may only be able to assume the strains functions instead of the displacement field. In order to ensure that the strains are indeed compatible, we can derive certain equations known as the *compatibility equations*. Compatibility equations guarantee for certain classes of bodies that the strains have the proper functions.

Consider the following derivatives of the shear strain components

$$\frac{\partial^2 \gamma_{\mathrm{yz}}}{\partial y\,\partial z} = \frac{\partial^2}{\partial y\,\partial z}\left\{\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right\} = \frac{\partial^3 V}{\partial y\,\partial^2 z} + \frac{\partial^3 W}{\partial^2 y\,\partial z} = \frac{\partial^2 e_{\mathrm{yy}}}{\partial z^2} + \frac{\partial^2 e_{\mathrm{zz}}}{\partial y^2}$$

or

$$2\,\frac{\partial^2 e_{\mathrm{yz}}}{\partial y\,\partial z} = \frac{\partial^2 e_{\mathrm{yy}}}{\partial z^2} + \frac{\partial^2 e_{\mathrm{zz}}}{\partial y^2}$$

This clearly implies that the shear and axial strain components are not independent. Consider now a different set of derivatives

$$\frac{\partial^2 e_{\mathrm{xx}}}{\partial y\,\partial z} = \frac{\partial^3 U}{\partial x\,\partial y\,\partial z} \qquad \frac{\partial \gamma_{\mathrm{yz}}}{\partial x} = \frac{\partial^2 V}{\partial x\,\partial z} + \frac{\partial^2 W}{\partial x\,\partial y}$$

$$\frac{\partial \gamma_{\mathrm{xz}}}{\partial y} = \frac{\partial^2 U}{\partial y\,\partial z} + \frac{\partial^2 W}{\partial x\,\partial y} \qquad \frac{\partial \gamma_{\mathrm{xy}}}{\partial z} = \frac{\partial^2 U}{\partial y\,\partial z} + \frac{\partial^2 V}{\partial x\,\partial z}$$

It can be shown that all the above four equations can be written into one:

$$2\,\frac{\partial^2 e_{\mathrm{xx}}}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left\{-\frac{\partial \gamma_{\mathrm{yz}}}{\partial x} + \frac{\partial \gamma_{\mathrm{xz}}}{\partial y} + \frac{\partial \gamma_{\mathrm{xy}}}{\partial z}\right\}$$

or

$$\frac{\partial^2 e_{xx}}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left\{-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right\}$$

This is another relationship between the shear and axial strain components. Similar relationship can be obtained through cyclical permutations of the indices to yield the six strain compatibility equations

$$\begin{aligned}
\frac{\partial^2 e_{xy}}{\partial x\,\partial y} &= \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} \\
\frac{\partial^2 e_{yz}}{\partial y\,\partial z} &= \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} \\
\frac{\partial^2 e_{zx}}{\partial z\,\partial x} &= \frac{\partial^2 e_{xx}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial x^2} \\
\frac{\partial^2 e_{xx}}{\partial y\,\partial z} &= \frac{\partial}{\partial x}\left\{-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right\} \\
\frac{\partial^2 e_{yy}}{\partial x\,\partial z} &= \frac{\partial}{\partial y}\left\{-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{yx}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right\} \\
\frac{\partial^2 e_{zz}}{\partial x\,\partial y} &= \frac{\partial}{\partial z}\left\{-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{zy}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right\}
\end{aligned} \tag{2.92}$$

In short, note that at the beginning of this chapter, we found that to satisfy equilibrium, stresses had to vary with position in such a way as to satisfy the equilibrium equations. Similarly, strains must vary with position so as to satisfy the compatibility equations in order to represent physically realizable deformations.

2.4.3 Cauchy's relationship for Strains

Cauchy's relation can be extended for strains. Recall that to find the stress vector at a surface with a unit normal $\underline{\hat{\mathbf{n}}}_{(\mathrm{s})}$, we had

$$\begin{Bmatrix} T_{\mathrm{x}} \\ T_{\mathrm{y}} \\ T_{\mathrm{z}} \end{Bmatrix} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \tau_{\mathrm{xy}} & \tau_{\mathrm{xz}} \\ \tau_{\mathrm{xy}} & \sigma_{\mathrm{yy}} & \tau_{\mathrm{yz}} \\ \tau_{\mathrm{xz}} & \tau_{\mathrm{yz}} & \sigma_{\mathrm{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

$$\underline{\mathbf{T}}^{(\mathrm{s})} = \underline{\boldsymbol{\sigma}} \cdot \underline{\hat{\mathbf{n}}}_{(\mathrm{s})}$$

Analog to the above formulation, the relationship may be extended to strains as follows:

$$\begin{Bmatrix} E_{\mathrm{x}} \\ E_{\mathrm{y}} \\ E_{\mathrm{z}} \end{Bmatrix} = \begin{bmatrix} e_{\mathrm{xx}} & \frac{1}{2}\gamma_{\mathrm{xy}} & \frac{1}{2}\gamma_{\mathrm{xz}} \\ \frac{1}{2}\gamma_{\mathrm{xy}} & e_{\mathrm{yy}} & \frac{1}{2}\gamma_{\mathrm{yz}} \\ \frac{1}{2}\gamma_{\mathrm{xz}} & \frac{1}{2}\gamma_{\mathrm{yz}} & e_{\mathrm{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.93}$$

$$\underline{\mathbf{E}}^{(\mathrm{s})} = \underline{\mathbf{e}} \cdot \underline{\hat{\mathbf{n}}}_{(\mathrm{s})}$$

Example 2.8.

The following displacement field describes the movement of a body under load:

$$\underline{\mathbf{R}} = 0.01\,\left(x^2 + y^2\right)\,\hat{\underline{\mathbf{i}}} + 0.01\,(3 + x\,z)\,\hat{\underline{\mathbf{j}}} - \left(0.006\,z^2\right)\,\hat{\underline{\mathbf{k}}} \qquad \text{ft}$$

(2.8a) Determine the strain tensor.

The strain tensor is

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix}$$

From the problem,

$$U(x, y, z) = 0.01\,\left(x^2 + y^2\right)$$

$$V(x, y, z) = 0.01\,(3 + x\,z)$$

$$W(x, y, z) = -\left(0.006\,z^2\right)$$

Now, evaluating all strains we get

$$e_{xx} = \frac{\partial U}{\partial x} = 2\,(0.01)\,x$$

$$e_{yy} = \frac{\partial V}{\partial y} = 0$$

$$e_{zz} = \frac{\partial W}{\partial z} = -.012\,z$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 2\,(0.01)\,y + (0.01)\,z$$

$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = (0.01)\,x$$

Thus, the strain tensor is

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} 2\,x & y+\frac{z}{2} & 0 \\ y+\frac{z}{2} & 0 & \frac{x}{2} \\ 0 & \frac{x}{2} & -1.2\,z \end{bmatrix} \times 10^{-2}$$

(2.8b) Determine the state of strain at (0,1,3).

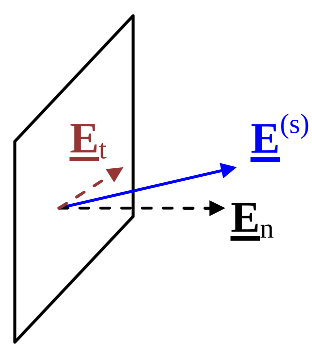
At the given point $x = 0, \quad y = 1, \quad z = 3$. Thus the state of strain is

$$\underline{\boldsymbol{e}} = \begin{bmatrix} 0 & 0.025 & 0 \\ 0.025 & 0 & 0 \\ 0 & 0 & -0.036 \end{bmatrix}$$

(2.8c) Compute the normal and tangential strain at (0,1,3) in the direction of

$$\underline{\hat{\mathbf{n}}}_{(s)} = 0.6\,\underline{\hat{\mathbf{i}}} + 0.8\,\underline{\hat{\mathbf{j}}} = \begin{Bmatrix} 0.6 \\ 0.8 \\ 0.0 \end{Bmatrix}$$

$$\underline{\mathbf{E}}^{(s)} = \begin{bmatrix} 0 & 0.025 & 0 \\ 0.025 & 0 & 0 \\ 0 & 0 & -0.036 \end{bmatrix} \begin{Bmatrix} 0.6 \\ 0.8 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.0200 \\ 0.015 \\ 0.000 \end{Bmatrix}$$



The normal component of the strain is

$$e_{nn} = \left[\underline{\mathbf{E}}^{(s)}\right]^{T} \cdot \underline{\hat{\mathbf{n}}}_{(s)} = \begin{Bmatrix} 0.0200 & 0.015 & 0.000 \end{Bmatrix} \cdot \begin{Bmatrix} 0.6 \\ 0.8 \\ 0.0 \end{Bmatrix} = 0.0240$$

$$\underline{\mathbf{E}}_n = e_{\text{nn}}\, \underline{\hat{\mathbf{n}}}_{(\text{s})} = 0.0240 \begin{Bmatrix} 0.6 \\ 0.8 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.0144 \\ 0.0192 \\ 0.0000 \end{Bmatrix}$$

The tangential component of the strain is:

$$\underline{\mathbf{E}}_t = \underline{\mathbf{E}}^{(\text{s})} - \underline{\mathbf{E}}_n = \begin{Bmatrix} 0.0056 \\ -0.0042 \\ 0.0000 \end{Bmatrix}$$

and the magnitude is

$$e_{\text{tt}} = \sqrt{(0.0056)^2 + (-0.0042)^2 + (0.0)^2} = 0.007$$

End Example □

2.4.4 Principal Strains and Principal Planes

Similar to principal stresses, the knowledge of principal strains help us find plane(s) on which the normal strains has the largest possible value or plane(s) on which the largest possible shear strain value. A principal plane is a plane such that the strain vector acting on that plane has no component which is tangent to the plane (i.e., there are no shear strains acting on the plane):

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \tag{2.94}$$

In order words, the strain vector has the same direction as the unit normal that describes the plane. The magnitude of the normal strain is known as **principal strain**. The procedure is similar to that used to obtain principal stresses for a given three-dimensional state of stress[6]

The derivation of the eigenvalue problem is similar to that for the stress analysis. If ϕ is the magnitude of the strain vector acting on the principal plane the strain vector is defined as

$$\phi\,\underline{\hat{\mathbf{n}}} = \phi \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \phi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{bmatrix} \phi & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{2.95}$$

Now using the knowledge of what the strain vector should be for a principal plane, Eq. (2.95),

$$\begin{bmatrix} \phi & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{bmatrix} e_{\text{xx}} & \frac{1}{2}\gamma_{\text{xy}} & \frac{1}{2}\gamma_{\text{xz}} \\ \frac{1}{2}\gamma_{\text{xy}} & e_{\text{yy}} & \frac{1}{2}\gamma_{\text{yz}} \\ \frac{1}{2}\gamma_{\text{xz}} & \frac{1}{2}\gamma_{\text{yz}} & e_{\text{zz}} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

Thus for a principal plane,

$$\begin{bmatrix} e_{\text{xx}} - \phi & \frac{1}{2}\gamma_{\text{xy}} & \frac{1}{2}\gamma_{\text{xz}} \\ \frac{1}{2}\gamma_{\text{xy}} & e_{\text{yy}} - \phi & \frac{1}{2}\gamma_{\text{yz}} \\ \frac{1}{2}\gamma_{\text{xz}} & \frac{1}{2}\gamma_{\text{yz}} & e_{\text{zz}} - \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.96}$$

These equations have the trivial solution $n_x = n_y = n_z = 0$, which is not allowed, since n_x, n_y, and n_z

[6] All stress equations apply but take $\tau \rightarrow \dfrac{\gamma}{2}$ and $\sigma \rightarrow e$ for the case of strains.

are the components of a unit vector, satisfying

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{2.97}$$

Equations in (2.96) possess a nontrivial solution if the three equations are not independent of each other. In other words, the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\begin{vmatrix} e_{xx} - \phi & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} - \phi & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} - \phi \end{vmatrix} = 0$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the strain invariants as follows

$$\phi^3 - I_{\epsilon_1}\phi^2 + I_{\epsilon_2}\phi - I_{\epsilon_3} = 0 \tag{2.98}$$

where I_{ϵ_i}'s are[7]:

$$I_{\epsilon_1} = e_{xx} + e_{yy} + e_{zz} \tag{2.99}$$

$$\begin{aligned} I_{\epsilon_2} &= \begin{vmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & e_{yy} \end{vmatrix} + \begin{vmatrix} e_{xx} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xz} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{yz} & e_{zz} \end{vmatrix} \\ &= e_{xx}\,e_{yy} + e_{zz}\,e_{xx} + e_{yy}\,e_{zz} - \frac{1}{4}\left(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2\right) \end{aligned} \tag{2.100}$$

$$\begin{aligned} I_{\epsilon_3} &= \det \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix} \\ &= e_{xx}\,e_{yy}\,e_{zz} + \frac{1}{4}\left(\gamma_{xy}\,\gamma_{yz}\,\gamma_{xz} - e_{xx}\,\gamma_{yz}^2 - e_{yy}\,\gamma_{xz}^2 - e_{zz}\,\gamma_{xy}^2\right) \end{aligned} \tag{2.101}$$

The three roots of the characteristic equation, Eq. (2.98), are the principal strains and can be obtained

[7]When working with Cauchy's strains, substitute γ for $2\,e$, i.e., $\gamma_{xy} \rightarrow 2\,e_{xy}$.

analytically:

$$e_1 = \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\,I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3}\right) \tag{2.102}$$

$$e_2 = \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\,I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) \tag{2.103}$$

$$e_3 = \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\,I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) \tag{2.104}$$

$$\beta = \cos^{-1}\left[\frac{2I_{\epsilon_1}^3 - 9I_{\epsilon_1}I_{\epsilon_2} + 27I_{\epsilon_3}}{2\sqrt{\left(I_{\epsilon_1}^2 - 3I_{\epsilon_2}\right)^3}}\right] \qquad \text{(keep in radians)} \tag{2.105}$$

Since the strain tensor is symmetric, the principal strains, roots of Eq. (2.98), will be three real-valued solutions. For each of these three solutions, the matrix of the system of equations defined by Eq. (2.96) will have a zero determinant and a non trivial solution for the directions cosines that now define the direction for which the shear strains vanish. Such direction is called a principal strain direction. Since the equations to be solved are homogeneous, their solution will include an arbitrary constant which can be determined by enforcing the condition

$$n_x^2 + n_y^2 + n_z^2 = 1$$

associated with the fact that vector $\underline{\hat{\mathbf{n}}}$ must be a unit vector. Since there exist three principal strains, three principal strain directions will exist. It can be shown that these three directions are mutually orthogonal.

It turns out that a state of strain not only has three extreme values of normal strain, but also three extreme values of shear strain, which are related to the three principal strains as follows:

$$e_{12} = \left|\frac{e_1 - e_2}{2}\right| \qquad e_{13} = \left|\frac{e_1 - e_3}{2}\right| \qquad e_{23} = \left|\frac{e_2 - e_3}{2}\right| \tag{2.106}$$

or

$$\gamma_{12} = |e_1 - e_2| \qquad \gamma_{13} = |e_1 - e_3| \qquad \gamma_{23} = |e_2 - e_3| \tag{2.107}$$

Observe that the absolute maximum shear strain at a point equals one-half the difference between the largest and the smallest principal strain, or:

$$e_{\max} = \max[e_1, e_2, e_3] \tag{2.108}$$

$$e_{\min} = \min[e_1, e_2, e_3] \tag{2.109}$$

$$\gamma_{\max} = |e_{\max} - e_{\min}| \tag{2.110}$$

Note that whereas the shear strain vanishes on planes of principal strain, the normal strain is generally nonzero on planes where the shear strain acquires its extreme values.

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(1)}$

To find $\underline{\hat{\mathbf{n}}}^{(1)}$, the principal direction of e_1, we substitute $\phi = e_1$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(1)}$ ($n_x^{(1)}$, $n_y^{(1)}$, and $n_z^{(1)}$) and the last component is obtained with

$$\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1$$

thus

$$\underline{\hat{\mathbf{n}}}^{(1)} = \begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \\ n_z^{(1)} \end{Bmatrix}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(2)}$

To find $\underline{\hat{\mathbf{n}}}^{(2)}$, the principal direction of e_2, we substitute $\phi = e_2$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(2)}$ ($n_x^{(2)}$, $n_y^{(2)}$, and $n_z^{(2)}$) and the last component is obtained with

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1$$

thus

$$\underline{\hat{\mathbf{n}}}^{(2)} = \begin{Bmatrix} n_x^{(2)} \\ n_y^{(2)} \\ n_z^{(2)} \end{Bmatrix}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(3)}$

To find $\underline{\hat{\mathbf{n}}}^{(3)}$, the principal direction of e_3, we substitute $\phi = e_3$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(3)}$ ($n_x^{(3)}$, $n_y^{(3)}$, and $n_z^{(3)}$) and the last component is obtained with

$$\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1$$

thus

$$\underline{\hat{\mathbf{n}}}^{(3)} = \begin{Bmatrix} n_x^{(3)} \\ n_y^{(3)} \\ n_z^{(3)} \end{Bmatrix}$$

Note that the principal planes are orthogonal to each other. In other words, the three principal normals are perpendicular to one another and thus

$$\underline{\hat{\mathbf{n}}}^{(3)} = \underline{\hat{\mathbf{n}}}^{(1)} \times \underline{\hat{\mathbf{n}}}^{(2)}$$

Example 2.9.

Determine the three principal strains and the corresponding principal planes for the state of strain given in Example 2.7.

From the problem we found that the state of strain at (0,2,3) was

$$\underline{\boldsymbol{e}} = \begin{bmatrix} 0.000 & 0.000 & 0.005 \\ 0.000 & 0.360 & 0.060 \\ 0.005 & 0.060 & 0.030 \end{bmatrix}$$

For a principal plane, Cauchy's equations can be written in matrix form as follows

$$\begin{bmatrix} e_{\mathrm{xx}} - \phi & e_{\mathrm{xy}} & e_{\mathrm{xz}} \\ e_{\mathrm{yx}} & e_{\mathrm{yy}} - \phi & e_{\mathrm{yz}} \\ e_{\mathrm{zx}} & e_{\mathrm{zy}} & e_{\mathrm{zz}} - \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0.000 - \phi & 0.000 & 0.005 \\ 0.000 & 0.360 - \phi & 0.060 \\ 0.005 & 0.060 & 0.030 - \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\det \begin{bmatrix} e_{\mathrm{xx}} - \phi & e_{\mathrm{xy}} & e_{\mathrm{xz}} \\ e_{\mathrm{yx}} & e_{\mathrm{yy}} - \phi & e_{\mathrm{yz}} \\ e_{\mathrm{zx}} & e_{\mathrm{zy}} & e_{\mathrm{zz}} - \phi \end{bmatrix} = \begin{vmatrix} 0.000 - \phi & 0.000 & 0.005 \\ 0.000 & 0.360 - \phi & 0.060 \\ 0.005 & 0.060 & 0.030 - \phi \end{vmatrix} = 0$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the stress invariants as follows

$$\phi^3 - I_{\epsilon_1} \phi^2 + I_{\epsilon_2} \phi - I_{\epsilon_3} = 0$$

where I_{ϵ_i}'s are the strain invariants.

$$
\begin{aligned}
I_{\epsilon_1} &= e_{xx} + e_{yy} + e_{zz} = 0.390 \\
I_{\epsilon_2} &= \begin{vmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{vmatrix} + \begin{vmatrix} e_{xx} & e_{xz} \\ e_{xz} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{yy} & e_{yz} \\ e_{yz} & e_{zz} \end{vmatrix} \\
&= e_{xx}\, e_{yy} + e_{zz}\, e_{xx} + e_{yy}\, e_{zz} - e_{xy}^2 + e_{yz}^2 + e_{xz}^2 = 0.007175 \\
I_{\epsilon_3} &= \det \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} \\
&= e_{xx}\, e_{yy}\, e_{zz} + \left(2\, e_{xy}\, e_{yz}\, e_{xz} - e_{xx}\, e_{yz}^2 - e_{yy}\, e_{xz}^2 - e_{zz}\, e_{xy}^2\right) = -0.000009
\end{aligned}
$$

Therefore, the characteristic equation can be written as

$$\lambda^3 - 0.390\, \lambda^2 + 0.007175\, \lambda + 0.000009 = 0$$

The three roots of the characteristic equation are the principal strains and are obtained analytically as follows:

$$\beta = \cos^{-1}\left[\frac{2\, I_{\epsilon_1}^3 - 9\, I_{\epsilon_1} I_{\epsilon_2} + 27\, I_{\epsilon_3}}{2\sqrt{\left(I_{\epsilon_1}^2 - 3\, I_{\epsilon_2}\right)^3}}\right] = 0.11567\, \text{rads}$$

$$
\begin{aligned}
\phi_1 &= \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\, I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3}\right) = 0.370573 \\
\phi_2 &= \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\, I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) = -0.001178 \\
\phi_3 &= \frac{I_{\epsilon_1}}{3} + \frac{2}{3}\sqrt{I_{\epsilon_1}^2 - 3\, I_{\epsilon_2}}\ \cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) = 0.02061
\end{aligned}
$$

and the principal strains are

$$
\begin{aligned}
& e_1 = \max[\phi_1, \phi_2, \phi_3] = 0.370573 \\
& e_3 = \min[\phi_1, \phi_2, \phi_3] = -0.001178 \\
& e_2 = 0.02061
\end{aligned}
$$

As we can see the principal strains are given as follows

$$e_1 > e_2 > e_3$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(1)}$

To find $\underline{\hat{\mathbf{n}}}^{(1)}$, the principal direction of $e_1 = 0.370573$, we substitute $\phi = e_1$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(1)}$ ($n_x^{(1)}$, $n_y^{(1)}$, and $n_z^{(1)}$) and the last component is obtained with

$$\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1 \tag{2.111}$$

Therefore,

$$\begin{bmatrix} -0.370573 & 0.0 & 0.005 \\ 0.0 & -0.010573 & 0.06 \\ 0.005 & 0.06 & -0.340573 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} -0.370573\, n_x^{(1)} &+ 0 + 0.005\, n_z^{(1)} = 0 \\ 0 &+ -0.010573\, n_y^{(1)} + 0.06\, n_z^{(1)} = 0 \\ 0.005\, n_x^{(1)} &+ 0.06\, n_y^{(1)} + -0.340573\, n_z^{(1)} = 0 \end{aligned}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(1)}$ (we could have solved in terms of any other component). Using the first two equations:

$$n_x^{(1)} = 0.01349\, n_z^{(1)} \qquad n_y^{(1)} = 5.6748\, n_z^{(1)} \tag{2.112}$$

Now obtain $n_z^{(1)}$ using Eq. (2.111)

$$\begin{aligned} \left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 &= 1 \\ \left(0.01349\, n_z^{(1)}\right)^2 + \left(5.6748\, n_z^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 &= 1 \end{aligned}$$

Then

$$n_z^{(1)} = \pm 0.173535$$

Now taking the positive sign (arbitrarily) of $n_z^{(1)}$ and substituting into Eq. (2.112)

$$n_z^{(1)} = 0.173535 \qquad n_x^{(1)} = 0.00234144 \qquad n_y^{(1)} = 0.984825$$

Thus, the principal strain $e_1 = 0.370573$ acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(1)} = \begin{Bmatrix} 0.00234144 \\ 0.984825 \\ 0.173535 \end{Bmatrix}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(2)}$

To find $\underline{\hat{\mathbf{n}}}^{(2)}$, the principal direction of $e_2 = 0.0206061$, we substitute $\phi = e_2$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(2)}$ ($n_x^{(2)}$, $n_y^{(2)}$, and $n_z^{(2)}$) and the last component is obtained with

$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1 \tag{2.113}$$

Therefore,

$$\begin{bmatrix} -0.0206061 & 0.0 & 0.005 \\ 0.0 & 0.33939 & 0.06 \\ 0.005 & 0.06 & 0.00939 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} -0.0206061\, n_x^{(2)} \quad + \quad 0 \quad + \quad 0.005\, n_z^{(2)} &= 0 \\ 0 \quad + \quad 0.33939\, n_y^{(2)} \quad + \quad 0.06\, n_z^{(2)} &= 0 \\ 0.005\, n_x^{(2)} \quad + \quad 0.06\, n_y^{(2)} \quad + \quad 0.00939\, n_z^{(2)} &= 0 \end{aligned}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(2)}$ (we could have solved in terms of any other component). Using the first two equations:

$$n_x^{(2)} = 0.2426\, n_z^{(2)} \qquad n_y^{(2)} = -0.17678\, n_z^{(2)} \tag{2.114}$$

Now obtain $n_z^{(2)}$ using Eq. (2.113)

$$\begin{aligned} \left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 &= 1 \\ \left(0.2426\, n_z^{(2)}\right)^2 + \left(-0.17678\, n_z^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 &= 1 \end{aligned}$$

Then

$$n_z^{(2)} = \pm 0.957769$$

Now taking the positive sign (arbitrarily) of $n_z^{(2)}$ and substituting into Eq. (2.114)

$$n_z^{(2)} = 0.957769 \qquad n_x^{(2)} = 0.232399 \qquad n_y^{(2)} = -0.16932$$

Thus, the principal strain $e_2 = 0.00206061$ acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(2)} = \begin{Bmatrix} 0.232399 \\ -0.16932 \\ 0.957769 \end{Bmatrix}$$

Principal Plane: $\underline{\hat{\mathbf{n}}}^{(3)}$

To find $\underline{\hat{\mathbf{n}}}^{(3)}$, the principal direction of $e_3 = -0.00117862$, we substitute $\phi = e_1$ into Eq. (2.96) and use only two equations but not all three. This will give two of the three components of $\underline{\hat{\mathbf{n}}}^{(3)}$ ($n_x^{(3)}$, $n_y^{(3)}$, and $n_z^{(3)}$) and the last component is obtained with

$$\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1 \tag{2.115}$$

Therefore,

$$\begin{bmatrix} 0.00117862 & 0.0 & 0.005 \\ 0.0 & 0.361178 & 0.06 \\ 0.005 & 0.06 & 0.031178 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} 0.00117862\, n_x^{(3)} + 0 + 0.005\, n_z^{(3)} &= 0 \\ 0 + 0.361178\, n_y^{(3)} + 0.06\, n_z^{(3)} &= 0 \\ 0.005\, n_x^{(3)} + 0.06\, n_y^{(3)} + 0.031178\, n_z^{(3)} &= 0 \end{aligned}$$

Using the first two equations (we could have used any two equations) and solving all variables in terms of $n_z^{(3)}$ (we could have solved in terms of any other component). Using the first two equations:

$$n_x^{(3)} = -4.24225\, n_z^{(3)} \qquad n_y^{(3)} = -0.166123\, n_z^{(3)} \tag{2.116}$$

Now obtain $n_z^{(3)}$ using Eq. (2.115)

$$\begin{aligned} \left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 &= 1 \\ \left(-4.24225\, n_z^{(3)}\right)^2 + \left(-0.166123\, n_z^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 &= 1 \end{aligned}$$

Then

$$n_z^{(3)} = \pm 0.229269$$

Now taking the positive sign (arbitrarily) of $n_z^{(3)}$ and substituting into Eq. (2.116)

$$n_z^{(3)} = 0.229269 \qquad n_x^{(3)} = -0.972618 \qquad n_y^{(3)} = -0.0380868$$

Thus, the principal strain $e_3 = -0.0405854$ acts on a plane with the unit normal

$$\underline{\hat{\mathbf{n}}}^{(3)} = \begin{Bmatrix} -0.972618 \\ -0.0380868 \\ 0.229269 \end{Bmatrix}$$

Also, we could have obtained this by using Eq. (2.42):

$$\underline{\hat{\mathbf{n}}}^{(3)} = \underline{\hat{\mathbf{n}}}^{(1)} \times \underline{\hat{\mathbf{n}}}^{(2)} = \begin{vmatrix} \underline{\hat{\mathbf{i}}} & \underline{\hat{\mathbf{j}}} & \underline{\hat{\mathbf{k}}} \\ 0.00234144 & 0.984825 & 0.173535 \\ 0.232399 & -0.16932 & 0.957769 \end{vmatrix}$$

$$= -0.972618\,\underline{\hat{\mathbf{i}}} - 0.0380868\,\underline{\hat{\mathbf{j}}} + 0.229269\,\underline{\hat{\mathbf{k}}}$$

End Example □

2.5 State of Plane Strain

The state of strain in the neighborhood of a point is given by the strain tensor

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix} \tag{2.117}$$

The plane state of strain is of great practical importance in aerospace engineering. In this case, the displacement component along the direction of z-axis is assumed to vanish, or to be negligible compared to the displacement components in the other two directions.

For applications for which a material is formed into thick sheets and plates of uniform thickness, it is often appropriate to assume that the strain components are confined to a plane, say x-y plane. In other words,

$$\gamma_{xz} = \gamma_{yz} = 0 \tag{2.118}$$

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & e_{yy} & 0 \\ 0 & 0 & e_{zz} \end{bmatrix} \tag{2.119}$$

Now for many problems, such as in aerospace applications,

$$e_{zz} \ll e_{xx} \qquad\qquad e_{zz} \ll e_{yy} \tag{2.120}$$

If this is the case, then we can take

$$e_{zz} \approx 0 \tag{2.121}$$

This type of problems are known as plane strain problems and the three dimensional state of strain reduces to three independent components,

$$\underline{\boldsymbol{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & e_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{xy} & 0 \\ e_{xy} & e_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.122}$$

2.5.1 Principal strains for State of Plane Strain

Now we will briefly discuss three different methods used to obtain the principal strains and maximum shear strains.

Principal strains: Eigenvalue Approach

A principal plane is a plane such that the strain vector acting on that plane has no component which is tangent to the plane (i.e., there are no shear strains acting on the plane):

$$\begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$$

For a plane strain problem,

$$\begin{bmatrix} e_{\mathrm{xx}} - \phi & e_{\mathrm{xy}} & 0 \\ e_{\mathrm{yx}} & e_{\mathrm{yy}} - \phi & 0 \\ 0 & 0 & 0 - \phi \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The above posses a nontrivial solution if the three equations are not independent of each other. In other words, the determinant of the matrix of coefficients of n_x, n_y, and n_z must vanish:

$$\begin{vmatrix} e_{\mathrm{xx}} - \phi & e_{\mathrm{xy}} & 0 \\ e_{\mathrm{yx}} & e_{\mathrm{yy}} - \phi & 0 \\ 0 & 0 & 0 - \phi \end{vmatrix} = 0$$

The characteristic equation obtained by expanding the determinant can be expressed in terms of the strain invariants as follows

$$\phi^3 - I_{\epsilon_1}\,\phi^2 + I_{\epsilon_2}\,\phi - I_{\epsilon_3} = 0$$

where I_{ϵ_i}'s are the strain invariants. Using the definition of strain invariants:

$$\begin{aligned} I_{\epsilon_1} &= e_{\mathrm{xx}} + e_{\mathrm{yy}} \\ I_{\epsilon_2} &= \begin{vmatrix} e_{\mathrm{xx}} & \frac{1}{2}\gamma_{\mathrm{xy}} \\ \frac{1}{2}\gamma_{\mathrm{xy}} & e_{\mathrm{yy}} \end{vmatrix} + \begin{vmatrix} e_{\mathrm{xx}} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} e_{\mathrm{yy}} & 0 \\ 0 & 0 \end{vmatrix} = e_{\mathrm{xx}}\,e_{\mathrm{yy}} - \frac{1}{4}\gamma_{\mathrm{xy}}^2 \\ I_{\epsilon_3} &= \begin{vmatrix} e_{\mathrm{xx}} & \frac{1}{2}\gamma_{\mathrm{xy}} & 0 \\ \frac{1}{2}\gamma_{\mathrm{xy}} & e_{\mathrm{yy}} & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \end{aligned}$$

Thus the characteristic equation becomes

$$\phi^3 - I_{\epsilon_1}\,\phi^2 + I_{\epsilon_2}\,\phi = 0 \quad \rightarrow \quad \phi\,\left(\phi^2 - I_{\epsilon_1}\,\phi + I_{\epsilon_2}\right) = 0 \tag{2.123}$$

The three roots of the characteristic equation, Eq. (2.123), are the principal strains and can be obtained analytically:

$$
\begin{aligned}
e_1 &= \frac{I_{\epsilon_1}}{2} + \frac{1}{2}\sqrt{I_{\epsilon_1}^2 - 4\, I_{\epsilon_2}} \\
e_2 &= \frac{I_{\epsilon_1}}{2} - \frac{1}{2}\sqrt{I_{\epsilon_1}^2 - 4\, I_{\epsilon_2}} \\
e_3 &= 0
\end{aligned}
$$

2.5.2 Strain Measurements

The whole idea of state of stress at a given point of an elastic body must be verified using through experimental data or measurements. So far we have no experimental equipment to directly measure stresses. However, we do have gauges that can measure strains, and thus the state of strain. The state of stress can be obtained using the constitutive laws discussed in chapter **??**.

Let us now discuss how we determine the state of strain. In general, we obtain the measurements on an external surface rather than at an interior point. As we have discussed in previous sections, a two-dimensional state of strain in the neighborhood of a point is characterized by two components: extensional and angular (shear). The measurement of the first one, extensional strains, is easy; but the measurement of the extremely small angular changes associated with shear strain is very difficult to accomplish. The relative elongation at the surface of a body can be measured with the help of what are called electrical resistance strain gauges, or more simply, strain gauges. The complete state of strain at the surface of the body is specified by three independent quantities, i.e., either two extensional and a shear strain, or two principal strains and a principal direction. These can be computed from the measurement of relative elongation in three distinct directions.

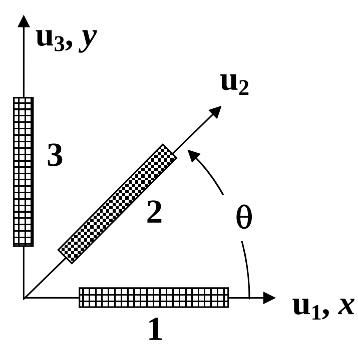


Figure 2.22: Three strain gauges at the surface of a solid: 3-gage rosette.

To better understand how this works, let ϵ_1, ϵ_2, and ϵ_3 be the experimentally measured relative elongations in those three directions. Note that a complete evaluation of the state of strain requires the knowledge of three strain components, and thus requires three independent measurements in three distinct directions. If the four gauges are properly working, the redundant information can be used to compensate for experimental errors.

In order to calculate the principal strains, we need the actual strains for the above measurements. These can be obtained by solving the following system of equations:

$$\epsilon_1 = e_{xx}\ \cos^2\theta_1 + e_{\rm yy}\ \sin^2\theta_1 + \gamma_{\rm xy}\ \sin\theta_1\ \cos\theta_1$$

$$\epsilon_2 = e_{\rm xx}\ \cos^2\theta_2 + e_{\rm yy}\ \sin^2\theta_2 + \gamma_{\rm xy}\ \sin\theta_2\ \cos\theta_2$$

$$\epsilon_3 = e_{\rm xx}\ \cos^2\theta_3 + e_{\rm yy}\ \sin^2\theta_3 + \gamma_{\rm xy}\ \sin\theta_3\ \cos\theta_3$$

where the θ_i's are measured counterclockwise from the x-axis. The above can also be written in matrix form,

$$\left\{ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\ \cos\theta_1 \\ \cos^2\theta_2 & \sin^2\theta_2 & \sin\theta_2\ \cos\theta_2 \\ \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\ \cos\theta_3 \end{array} \right] \left\{ \begin{array}{c} e_{\rm xx} \\ e_{\rm yy} \\ \gamma_{\rm xy} \end{array} \right\} \tag{2.124}$$

Example 2.10.

The strain gage measurements from a rosette are given as:

$$e_{\text{xx}} = \quad 2000\mu \qquad e_{\text{xx}+45^\circ} = \quad 1350\mu \qquad e_{\text{yy}} = \quad 950\mu$$

Determine the principal strains from the above components.

First we need to find all strains components in the x-y plane.

$$\theta_1 = 0^\circ \qquad \theta_2 = 45^\circ \qquad \theta_3 = 90^\circ$$

$$\epsilon_1 = e_{\text{xx}} = 2000\mu \qquad \epsilon_2 = e_{\text{xx}+45^\circ} = 1350\mu \qquad \epsilon_3 = e_{\text{yy}} = 950\mu$$

From Eq. (2.124):

$$\begin{Bmatrix} 2000 \\ 1350 \\ 950 \end{Bmatrix} \mu = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{Bmatrix} 2000 \\ 1350 \\ 950 \end{Bmatrix} \mu = \begin{Bmatrix} 2000 \\ 950 \\ -250 \end{Bmatrix} \mu$$

Note that for this rosette $\epsilon_1 = e_{\text{xx}}$ and $\epsilon_3 = e_{\text{yy}}$ and the only unknown is the shear strain γ_{xy}, which could have been directly calculated using the transformation relationship for ϵ_2:

$$\epsilon_2 = \epsilon(\theta) = e_{\text{xx}} \cos^2\theta + e_{\text{yy}} \sin^2\theta + \gamma_{\text{xy}} \sin\theta \cos\theta$$

$$= e_{\text{ave}} + e_{\text{dif}} \cos 2\theta + \frac{\gamma_{\text{xy}}}{2} \sin 2\theta$$

$$\epsilon(45^\circ) = e_{\text{ave}} + \frac{\gamma_{\text{xy}}}{2}$$

where,

$$e_{\text{ave}} = \frac{e_{\text{xx}} + e_{\text{yy}}}{2} = \frac{(2000) + (950)}{2}\, \mu = 1475\, \mu$$

$$e_{\text{dif}} = \frac{e_{\text{xx}} - e_{\text{yy}}}{2} = \frac{(2000) - (950)}{2}\, \mu = 525\, \mu$$

Now, half the shear strain, $\gamma_{\text{xy}}/2$, is

$$e_{\text{xy}} = \frac{\gamma_{\text{xy}}}{2} = \epsilon(45^\circ) - e_{\text{ave}} = (1350\, \mu) - (1475\, \mu) = -125\, \mu$$

Let's use the Mohr's circle to find the principal strains.

2.10a) Calculate the radius and center of the Mohr's circle

$$e_{\text{ave}} = \frac{e_{\text{xx}} + e_{\text{yy}}}{2} = \frac{(2000) + (950)}{2}\,\mu = 1475\,\mu$$

$$e_{\text{dif}} = \frac{e_{\text{xx}} - e_{\text{yy}}}{2} = \frac{(2000) - (950)}{2}\,\mu = 525\,\mu$$

$$R = \sqrt{\left(\frac{\gamma_{\text{xy}}}{2}\right)^2 + e_{\text{dif}}^2} = \sqrt{(-125)^2 + (525)^2}\,\mu = 539.676\,\mu$$

$$C = C(e_{\text{ave}}, 0) = C(1475\,\mu, 0)$$

2.10b) Draw the circle and locate all points

$$Q_1 = Q_1(e_{\text{xx}}, e_{\text{xy}}) = (2000, -125) \qquad Q_2 = Q_2(e_{\text{yy}}, -e_{\text{xy}}) = (950, 125) \qquad C = C(1475, 0)$$

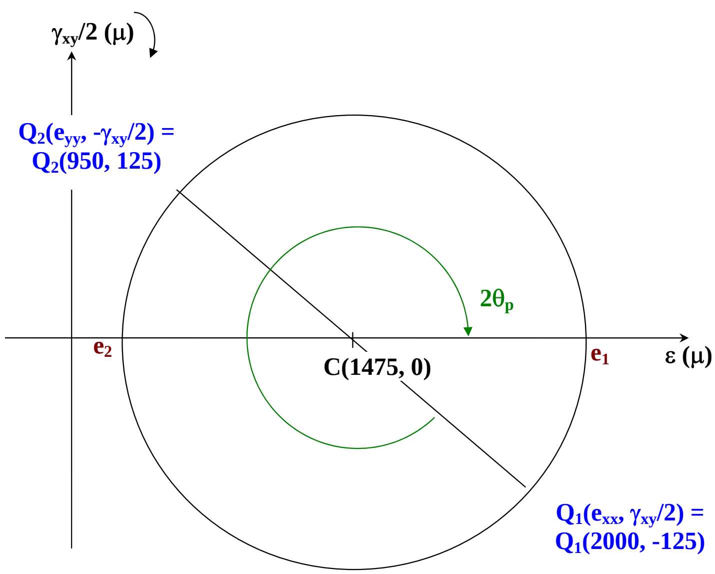


Figure 2.23: Mohr's circle for plane strain in the x-y plane.

2.10c) Calculate angles:

Principal stresses act on an element inclined at an angle θ_p

$$\begin{aligned} 2\theta_p' &= \tan^{-1}\left[\frac{\gamma_{\text{xy}}/2}{e_{\text{dif}}}\right] = \tan^{-1}\left[\frac{(-125)}{(525)}\right] = -13.3925^\circ \\ 2\theta_p &= 360^\circ - \left|2\theta_p'\right| = 346.6^\circ \end{aligned}$$

Note that we in CASE C because $2\,\theta_p$ is measured from $\overline{Q_1 C}$ to positive σ-axis. Recall Now consider the location of Q_1:

CASE A:	$Q_1 \rightarrow$ first quadrant	$(e_{xx} > 0, \quad \gamma_{xy}/2 > 0)$	$2\,\theta_p = 2\,\theta'_p$
CASE B:	$Q_1 \rightarrow$ second quadrant	$(e_{xx} < 0, \quad \gamma_{xy}/2 > 0)$	$2\,\theta_p = 180^\circ - \left|2\,\theta'_p\right|$
CASE C:	$Q_1 \rightarrow$ third quadrant	$(e_{xx} < 0, \quad \gamma_{xy}/2 < 0)$	$2\,\theta_p = 180^\circ + \left|2\,\theta'_p\right|$
CASE D:	$Q_1 \rightarrow$ fourth quadrant	$(e_{xx} > 0, \quad \gamma_{xy}/2 < 0)$	$2\,\theta_p = 360^\circ - \left|2\,\theta'_p\right|$

Minimum and maximum shear strain act on an element inclined at an angle θ_s

$$\begin{aligned} 2\,\theta_s &= 2\,\theta_p \pm 90^\circ = 346.6^\circ \pm 90^\circ \\ \theta_s &= \theta_p \pm 45^\circ = 173.3^\circ \pm 45^\circ \end{aligned}$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element.

2.10d) Determine the principal strains

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0^\circ$, therefore the principal stresses are

$$\begin{aligned} \phi_1 &= e_{ave} + R = (1475\,\mu) + (539.676\,\mu) = 2014.68\,\mu \\ \phi_2 &= e_{ave} - R = (1475\,\mu) - (539.676\,\mu) = 935.324\,\mu \\ \phi_3 &= 0 \end{aligned}$$

For the in-plane principal strain:

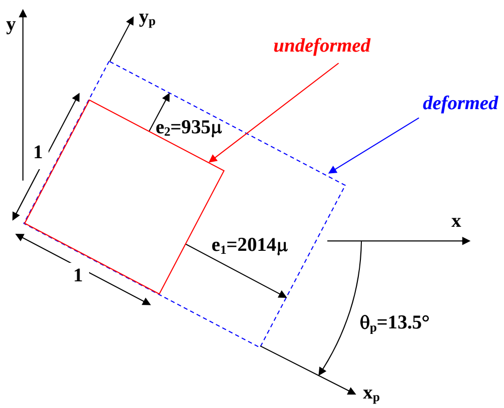
$$e_1 = 2014.68\,\mu \qquad e_2 = 935.324\,\mu$$

2.10e) Determine the maximum in-plane shear strain

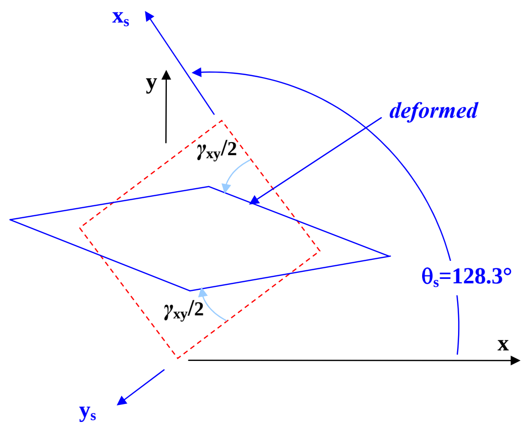
$$\left.\frac{\gamma_{max}}{2}\right|_{in-plane} = R = \frac{\epsilon_1 - \epsilon_2}{2} = 539.676\,\mu \tag{2.125}$$

$$\left.\gamma_{max}\right|_{in-plane} = 2\,R = \epsilon_1 - \epsilon_2 = 1079.35\,\mu \tag{2.126}$$

2.10f) Show all results on sketches of properly oriented elements



(a) Principal strain axes shown with a undeformed and deformed element



(b) Axes of maximum shearing with the deformed element

End Example □

2.6 Linear Elasticity for Structures

We describe the elastic field with fifteen unknowns (6 stresses, 6 strains, 3 displacements). In order to find the description of the elastic field, we would need fifteen equations.

1. **Displacement field (3 equations)**

   First, we begin with the assumption of a kinematically displacement field:

$$\begin{aligned} U(x,y,z) &= u(x,y,z) \\ V(x,y,z) &= v(x,y,z) \\ W(x,y,z) &= w(x,y,z) \end{aligned}$$

   This displacement approximation comes from the physical problem.

2. **Strain-displacement relationship (6 equations): Kinematics**

   The displacement gradients for our displacement field are:

$$\begin{aligned} g_1 &= \frac{\partial U(x,y,z)}{\partial x}, & g_2 &= \frac{\partial V(x,y,z)}{\partial x}, & g_3 &= \frac{\partial W(x,y,z)}{\partial x}, \\ g_4 &= \frac{\partial U(x,y,z)}{\partial y}, & g_5 &= \frac{\partial V(x,y,z)}{\partial y}, & g_6 &= \frac{\partial W(x,y,z)}{\partial y}, \\ g_7 &= \frac{\partial U(x,y,z)}{\partial z}, & g_8 &= \frac{\partial V(x,y,z)}{\partial z}, & g_9 &= \frac{\partial W(x,y,z)}{\partial z} \end{aligned}$$

   Using these displacements gradients, we obtain the strain-displacement relationship for small strains:

$$\begin{aligned} e_{xx} &= g_1 \\ e_{yy} &= g_5 \\ e_{zz} &= g_9 \\ 2\,e_{yz} &= g_6 + g_8 \\ 2\,e_{xz} &= g_3 + g_7 \\ 2\,e_{xy} &= g_2 + g_4 \end{aligned}$$

3. **Stress-Strain relationship (6 equations): Material Law**

The constitutive relationship for anisotropic materials is:

$$\underline{\mathbf{S}} = \underline{\mathbf{D}}\,\underline{\mathbf{E}}$$

$$\underline{\mathbf{S}} = \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2\,e_{yz} \\ 2\,e_{xz} \\ 2\,e_{xy} \end{Bmatrix}$$

4. **Equilibrium Equations (3 equations)**

   The three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

   Note that the body forces, b's, have units of [F]/[L^3].

5. **Finally, use the boundary conditions**

   Now, we find all the remaining unknown constants by using the conditions at the boundaries. For this we use Cauchy's relationship:

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

If we were to add the density as an unknown, then we would need to use the continuity equation. As we add more unknowns, we need the add that many equations to fully express the elastic field. Figure 2.24 shows the interaction/relationship between displacement, strains, stresses and loads. If we have a displacement, we should be able to find the strains and viceversa for any given problem. Given the strains, we should be able to find the stresses and viceversa for any given problem. Given the stresses, we should be able to determine the internal loads (stress resultants) and viceversa for any given problems. The external loads must always maintain the system in equilibrium with the internal ones.

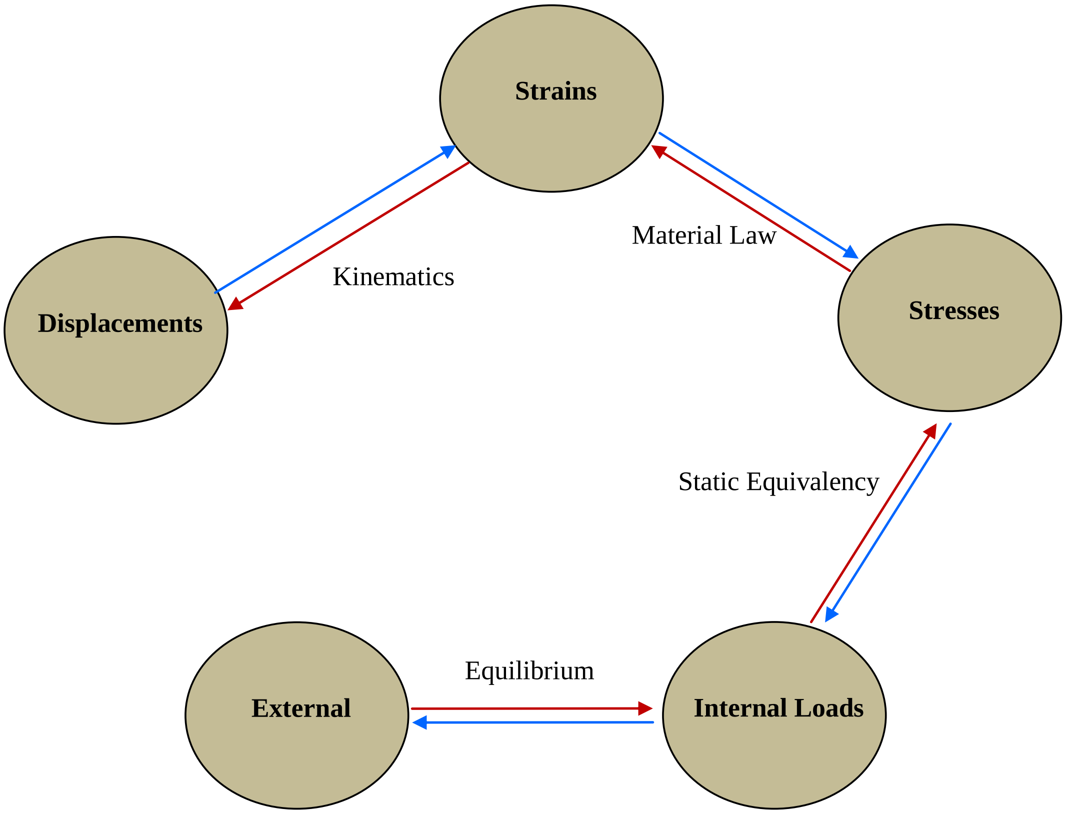


Figure 2.24: Relationship between displacement, strains, stresses and loads.

2.7 Alternative Stress and Strain Quantities

For a given problem different strains and stress measures can be used. However, it is important to consider the stresses and the strains as conjugate quantities in the sense that their product gives mechanical work. The present formulations are generally used for nonlinear analysis of structures.

2.7.1 Green-Lagrange strains

Consider the quantities defined in Fig. 2.19. Then derivatives of $\underline{\mathbf{r}}_1$ with respect to $\underline{\mathbf{r}}$ constitute the deformation gradient matrix, $\underline{\mathbf{F}}$, when arranged in Jacobian format:

$$\underline{\mathbf{F}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial z} \\ \dfrac{\partial y_1}{\partial x} & \dfrac{\partial y_1}{\partial y} & \dfrac{\partial y_1}{\partial z} \\ \dfrac{\partial z_1}{\partial x} & \dfrac{\partial z_1}{\partial y} & \dfrac{\partial z_1}{\partial z} \end{bmatrix} = \begin{bmatrix} 1+\dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & 1+\dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & 1+\dfrac{\partial W}{\partial z} \end{bmatrix} \tag{2.127}$$

The determinant of the deformation gradient matrix is known as the Jacobian determinant and is defined as

$$J = det[\underline{\mathbf{F}}] \tag{2.128}$$

The displacement gradients with respect to the reference configuration are defined as

$$\underline{\mathbf{G}} = \underline{\mathbf{F}} - \underline{\mathbf{I}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial x}-1 & \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial z} \\ \dfrac{\partial y_1}{\partial x} & \dfrac{\partial y_1}{\partial y}-1 & \dfrac{\partial y_1}{\partial z} \\ \dfrac{\partial z_1}{\partial x} & \dfrac{\partial z_1}{\partial y} & \dfrac{\partial z_1}{\partial z}-1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{bmatrix} = \begin{bmatrix} g_1 & g_4 & g_7 \\ g_2 & g_5 & g_8 \\ g_3 & g_6 & g_9 \end{bmatrix} \tag{2.129}$$

where $\underline{\mathbf{I}}$ is the identity matrix. Sometimes, it is convenient to arrange the displacements gradients in vector form as follows

$$\underline{\mathbf{g}}^{\mathrm{T}} = \left\{ \begin{array}{ccccccccc} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & g_9 \end{array} \right\} \tag{2.130}$$

Now, the displacement gradients for the displacement field are:

$$\begin{array}{lll} g_1 = \dfrac{\partial U(x,y,z)}{\partial x} & g_2 = \dfrac{\partial V(x,y,z)}{\partial x} & g_3 = \dfrac{\partial W(x,y,z)}{\partial x} \\ g_4 = \dfrac{\partial U(x,y,z)}{\partial y} & g_5 = \dfrac{\partial V(x,y,z)}{\partial y} & g_6 = \dfrac{\partial W(x,y,z)}{\partial y} \\ g_7 = \dfrac{\partial U(x,y,z)}{\partial z} & g_8 = \dfrac{\partial V(x,y,z)}{\partial z} & g_9 = \dfrac{\partial W(x,y,z)}{\partial z} \end{array} \tag{2.131}$$

The strains associated with the displacement field are computed using the Green-Lagrange strains. These strains can be expressed in terms of the displacement gradients as follows

$$\epsilon_1 = e_{xx} = g_1 + \frac{1}{2}\left(g_1^2 + g_2^2 + g_3^2\right) \tag{2.132a}$$

$$\epsilon_2 = e_{yy} = g_5 + \frac{1}{2}\left(g_4^2 + g_5^2 + g_6^2\right) \tag{2.132b}$$

$$\epsilon_3 = e_{zz} = g_9 + \frac{1}{2}\left(g_7^2 + g_8^2 + g_9^2\right) \tag{2.132c}$$

$$\epsilon_4 = 2\,e_{yz} = g_6 + g_8 + g_4\,g_7 + g_5\,g_8 + g_6\,g_9 \tag{2.132d}$$

$$\epsilon_5 = 2\,e_{xz} = g_3 + g_7 + g_1\,g_7 + g_2\,g_8 + g_3\,g_9 \tag{2.132e}$$

$$\epsilon_6 = 2\,e_{xy} = g_2 + g_4 + g_1\,g_4 + g_2\,g_5 + g_3\,g_6 \tag{2.132f}$$

The above may be rewritten in the quadratic form, as follows:

$$\epsilon_i = \underline{\mathbf{h}}_i^{\mathrm{T}}\,\underline{\mathbf{g}} + \frac{1}{2}\,\underline{\mathbf{g}}^{\mathrm{T}}\,\underline{\mathbf{H}}_i\,\underline{\mathbf{g}} \tag{2.133}$$

where the 9×1 vectors $\underline{\mathbf{h}}_i$'s and 9×9 matrices $\underline{\mathbf{H}}_i$'s are given in Appendix C. If we assume small displacements, small strains and rotations, these strains can be expressed in terms of the Green-Langrange strains as follows

$$E_1 = e_{xx} = g_1 \tag{2.134a}$$

$$E_2 = e_{yy} = g_5 \tag{2.134b}$$

$$E_3 = e_{zz} = g_9 \tag{2.134c}$$

$$E_4 = 2\,e_{yz} = g_6 + g_8 \tag{2.134d}$$

$$E_5 = 2\,e_{xz} = g_3 + g_7 \tag{2.134e}$$

$$E_6 = 2\,e_{xy} = g_2 + g_4 \tag{2.134f}$$

The above expressions are obtained by taking $\underline{\mathbf{H}}_i$ as the zero matrix. The Green-Lagrange strains are usually expressed in vectorial form as follows

$$\underline{\epsilon} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.135}$$

For arbitrary rigid-body motions (motions without deformations)

$$\underline{\mathbf{F}}^{\mathrm{T}}\,\underline{\mathbf{F}} = \underline{\mathbf{F}}\,\underline{\mathbf{F}}^{\mathrm{T}} = \underline{\mathbf{I}}$$

that is, $\underline{\mathbf{F}}$ is an orthogonal matrix. Displacement gradient matrices are connected by the relations

$$\underline{\mathbf{G}} = (\underline{\mathbf{I}} - \underline{\mathbf{J}})^{-1} - \underline{\mathbf{I}} \quad \text{and} \quad \underline{\mathbf{J}} = \underline{\mathbf{I}} - (\underline{\mathbf{I}} + \underline{\mathbf{G}})^{-1}$$

For small deformations,

$$\underline{\mathbf{G}} \approx \underline{\mathbf{J}}^{-1} \quad \text{and} \quad \underline{\mathbf{J}} \approx \underline{\mathbf{G}}^{-1}$$

Example 2.11.

Express the state of strain in Example 2.7 in terms of the Green-Lagrange linear and non-linear strains.

From Example 2.7, the displacement field was given as

$$U(x,y,z) = 0.01\left(x^2+3\right)$$

$$V(x,y,z) = 0.01\left(3\,y^2\,z\right)$$

$$W(x,y,z) = 0.01\,(x+3\,z)$$

and the state of strain at (0,2,3) is

$$\underline{\boldsymbol{e}} = \begin{bmatrix} 0.000 & 0.000 & 0.005 \\ 0.000 & 0.360 & 0.060 \\ 0.005 & 0.060 & 0.030 \end{bmatrix}$$

2.11a) Linear Green-Lagrange strains.

$$\epsilon_i = \underline{\mathbf{h}}_i^{\mathrm{T}}\,\underline{\mathbf{g}}$$

where $\underline{\mathbf{h}}_i$'s are given in Appendix C. The linear Green-Lagrange strains are:

$$\begin{aligned}
\epsilon_1 &= e_{xx} = g_1 \\
\epsilon_2 &= e_{\mathrm{yy}} = g_5 \\
\epsilon_3 &= e_{\mathrm{zz}} = g_9 \\
\epsilon_4 &= 2\,e_{\mathrm{yz}} = g_6 + g_8 \\
\epsilon_5 &= 2\,e_{\mathrm{xz}} = g_3 + g_7 \\
\epsilon_6 &= 2\,e_{\mathrm{xy}} = g_2 + g_4
\end{aligned}$$

where the displacement gradient is

$$g_1 = \frac{\partial U}{\partial x} = 0.02\,x = 0.000 \quad g_4 = \frac{\partial U}{\partial y} = 0.000 \quad g_7 = \frac{\partial U}{\partial z} = 0.000$$

$$g_2 = \frac{\partial V}{\partial x} = 0.000 \quad g_5 = \frac{\partial V}{\partial y} = 0.06yz \quad g_8 = \frac{\partial V}{\partial z} = 0.03y^2$$

$$g_3 = \frac{\partial W}{\partial x} = 0.01 \quad g_6 = \frac{\partial W}{\partial y} = 0.00 \quad g_9 = \frac{\partial W}{\partial z} = 0.03$$

The displacement gradient vector is

$$\underline{\mathbf{g}} = \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \\ g_9 \end{Bmatrix} = \begin{Bmatrix} 0.02\,x \\ 0.0 \\ 0.01 \\ 0.0 \\ 0.06\,y\,z \\ 0.0 \\ 0.03\,y^2 \\ 0.03 \end{Bmatrix}$$

and the displacement gradient matrix becomes:

$$\underline{\mathbf{G}} = \begin{bmatrix} 0.02\,x & 0.0 & 0.01 \\ 0.0 & 0.06\,y\,z & 0.03\,y^2 \\ 0.01 & 0.0 & 0.03 \end{bmatrix}_{(x,y,z)=(0,2,3)} = \begin{bmatrix} 0.0 & 0.0 & 0.01 \\ 0.0 & 0.36 & 0.48 \\ 0.01 & 0.0 & 0.03 \end{bmatrix}$$

The linear strains are

$$\underline{\epsilon} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} 0.02\,x \\ 0.06\,y\,z \\ 0.03 \\ 0.03\,y^2 \\ 0.01 \\ 0.00 \end{Bmatrix}_{(x,y,z)=(0,2,3)} = \begin{Bmatrix} 0.00 \\ 0.36 \\ 0.03 \\ 0.12 \\ 0.01 \\ 0.00 \end{Bmatrix}$$

2.11b) Full Green-Lagrange Strains.

$$\epsilon_i = \underline{\mathbf{h}}_i^{\mathrm{T}}\,\underline{\mathbf{g}} + \frac{1}{2}\,\underline{\mathbf{g}}^{\mathrm{T}}\,\underline{\mathbf{H}}_i\,\underline{\mathbf{g}}$$

where $\underline{\mathbf{h}}_i$'s and $\underline{\mathbf{H}}_i$'s are given in Appendix C.

$$\underline{\epsilon} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} 0.00005 + 0.02\,x + 0.0002\,x^2 \\ 0.06\,y\,z + 0.0018\,y^2\,z^2 \\ 0.03045 + 0.00045\,y^4 \\ 0.03\,y^2 + 0.0018\,y^3\,z \\ 0.0103 \\ 0.00 \end{Bmatrix}_{(x,y,z)=(0,2,3)} = \begin{Bmatrix} 0.00005 \\ 0.4248 \\ 0.0376 \\ 0.1632 \\ 0.0103 \\ 0.00 \end{Bmatrix}$$

2.11c) Comparing both solutions (Linear and Nonlinear Green-Lagrange Strains):

$$\text{percentage of error} = \begin{Bmatrix} e_{xx} \\ e_{xy} \\ e_{xz} \\ e_{yy} \\ e_{yz} \\ e_{zz} \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 2.91262 \\ 15.2542 \\ 26.4706 \\ 20.3187 \end{Bmatrix}$$

End Example □

2.7.2 Stress Measures

We introduced the concept of stress in the body through the Cauchy's formula at the beginning of the chapter. We used $\underline{\boldsymbol{\sigma}}$ to denote the Cauchy stress tensor, which is the true stress in the body. However, other stress measures may be defined as functions of Cauchy's stress tensor:

1. The first Piola-Kirchhoff stress tensor $\underline{\mathbf{P}}$

$$\underline{\mathbf{P}} = J\,\underline{\boldsymbol{\sigma}}\,\underline{\mathbf{F}}^{-\mathrm{T}}$$

   where J is the Jacobian determinant, and $\underline{\mathbf{F}}$ the deformation gradient matrix as defined by Eq. (2.127). The first Piola-Kirchhoff stress tensor is nonsymmetric.

2. The second Piola-Kirchhoff (PK2) stress tensor $\underline{\mathbf{S}}$

$$\underline{\mathbf{S}} = J\,\underline{\mathbf{F}}^{-1}\,\underline{\boldsymbol{\sigma}}\,\underline{\mathbf{F}}^{-\mathrm{T}}$$

   where J is the Jacobian determinant, $\underline{\boldsymbol{\sigma}}$ the Cauchy (true) stresses, and $\underline{\mathbf{F}}$ the deformation gradient matrix as defined by Eq. (2.127). The second Piola-Kirchhoff stress tensor is symmetric.

The stresses corresponding to the Green-Lagrange strains are the second Piola-Kirchhoff stresses. The three dimensional tensor in Cartesian coordinates is

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{\mathrm{xx}} & S_{\mathrm{xy}} & S_{\mathrm{xz}} \\ S_{\mathrm{yx}} & S_{\mathrm{yy}} & S_{\mathrm{yz}} \\ S_{\mathrm{zx}} & S_{\mathrm{zy}} & S_{\mathrm{zz}} \end{bmatrix} = \begin{bmatrix} S_{\mathrm{xx}} & S_{\mathrm{xy}} & S_{\mathrm{xz}} \\ S_{\mathrm{xy}} & S_{\mathrm{yy}} & S_{\mathrm{yz}} \\ S_{\mathrm{xz}} & S_{\mathrm{yz}} & S_{\mathrm{zz}} \end{bmatrix} \tag{2.136}$$

It can be shown that the PK2 stresses are linearly related to the Cauchy stresses as follows

$$\underline{\mathbf{S}} = \underline{\mathbf{S}}^0 + J\,\underline{\mathbf{F}}^{-1}\,\underline{\boldsymbol{\sigma}}\,\underline{\mathbf{F}}^{-\mathrm{T}} \tag{2.137}$$

where $\underline{\mathbf{S}}^0$ are the prestresses, J the Jacobian determinant, $\underline{\mathbf{F}}$ the deformation gradient matrix, $\underline{\mathbf{S}}$ the PK2 stresses, and $\underline{\boldsymbol{\sigma}}$ the Cauchy (true) stresses defined as

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\mathrm{xx}} & \sigma_{\mathrm{xy}} & \sigma_{\mathrm{xz}} \\ \sigma_{\mathrm{xy}} & \sigma_{\mathrm{yy}} & \sigma_{\mathrm{yz}} \\ \sigma_{\mathrm{xz}} & \sigma_{\mathrm{yz}} & \sigma_{\mathrm{zz}} \end{bmatrix} \tag{2.138}$$

From continuity equation, we know that the total mass of the entire body must be conserved:

$$\rho_1\, d\Gamma_1 = \rho\, d\Gamma \qquad \Rightarrow \qquad \rho_1\, det[\underline{\mathbf{F}}]\, d\Gamma = \rho\, d\Gamma \tag{2.139a}$$

$$\Rightarrow \qquad J = det[\underline{\mathbf{F}}] = \frac{d\Gamma_1}{d\Gamma} = \frac{\rho}{\rho_1} \tag{2.139b}$$

where $d\Gamma_1$ and $d\Gamma$ are the volumes in the current configuration and reference configuration, respectively; ρ_1 and ρ are the mass densities in the current and reference configuration, respectively.

Assuming that isochoric deformation takes place (volume-preserving deformation),

$$J = \det[\underline{\mathbf{F}}] = \frac{d\Gamma_1}{d\Gamma} = \frac{\rho}{\rho_1} = 1$$

. Also, we assume that the prestressed state in the reference configuration, $\underline{\mathbf{S}}^0$, is zero. Further, recall that we restrict our analysis to small deformations and small strains. Under these assumptions, it can be shown that the PK2 and Cauchy stresses coalesce. Thus, Eq. (2.137) reduces to

$$\underline{\mathbf{S}} \approx \underline{\boldsymbol{\sigma}} \tag{2.140}$$

and these stresses are usually expressed in vectorial form as follows

$$\underline{\mathbf{S}} = \left\{ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \right\} = \left\{ \begin{array}{c} S_{\text{xx}} \\ S_{\text{yy}} \\ S_{\text{zz}} \\ S_{\text{yz}} \\ S_{\text{xz}} \\ S_{\text{xy}} \end{array} \right\}$$

In short, the Cauchy stress works with the Almansi strain or Green-Cauchy strain, and the Green-Lagrange strain works with the second Piola-Kirchhoff stress tensor. For small deformation, no difference are made between the two of them.

2.8 Mechanical Behavior of Materials

Elastic behavior may be characterized by the following two conditions:

1. the stress in a material is a unique function of the strain,
2. the material has the property to complete recovery to its natural shape upon removal of the applied forces.

Nonelastic materials are known as inelastic materials. In fact, two major type of deformation that occurs in engineering materials are:

1. Elastic: associated with stretching but not breaking of chemical bonds.
2. Inelastic:
   (a) Plastic (or Non-time dependent inelastic): atoms change their relative positions.
   (b) Creep (or Time dependent inelastic): basically same as plastic but it the deformation is time dependent.

In this book, we will limit our discussion to linear elastic behavior. The elastic behavior may be linear or non-linear. Figure 2.25 shows geometrically these behavior patterns by simple stress-strain curves, with the relevant loading and unloading paths indicated.

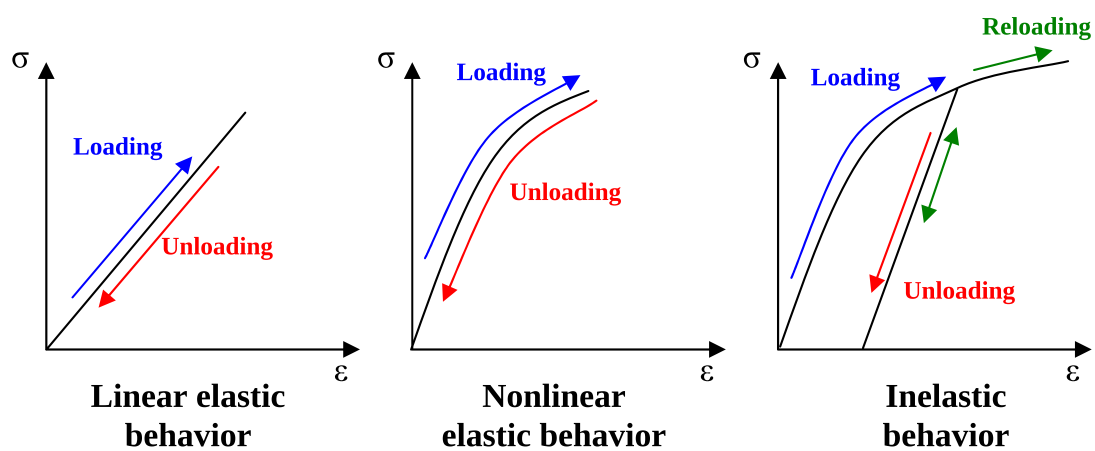


Figure 2.25: Uniaxial loading-unloading stress-strain curves.

2.9 Constitutive Equations for Elastic Materials

Equations describing stress-strain behavior are often used in engineering analysis and are usually called stress-strain relationships or ***constitutive equations***. For example, in basic mechanics of materials, elastic behavior with a linear stress-strain relationship is assumed and used in calculating stresses and deflections in simple components such as beams and shafts. More complex situations of geometry and loading can be analyzed by employing the same basic assumptions in the form of theory of elasticity, discussed in Chapter 2.

The constitutive equations take into account a three-dimensional behavior of the material. The constitutive equations may be nonlinear or linear, this is independent from elastic or inelastic behavior of the material. When analyzing thin-walled structures to determine the deflections and stresses, we often need to determine the appropriate constitutive relationships for the material involved. Here, we will limit to linear elastic stress-strain relationships.

2.9.1 Hooke's Law

Symbolically, we can write the constitutive equations for elastic behavior in its most general for as

$$\underline{\mathbf{S}} = \underline{\mathbf{G}}(\underline{\epsilon})$$

where $\underline{\mathbf{G}}$ is a symmetric tensor-valued function, $\underline{\mathbf{S}}$ and $\underline{\epsilon}$ are any of the stress and strain tensors, respectively. Throughout this book, we will assume that, in the deformed material, the displacement gradients are everywhere small compared to unity and the materials follow linear elastic behavior. Within this context the constitutive equations for linear elastic behavior are written as

$$\underline{\mathbf{S}} = \underline{\mathbf{C}}\,\underline{\epsilon}$$

where the tensor of elastic coefficients has 81 components. It can be shown that due to symmetry of both stress and strain tensors, $\underline{\mathbf{C}}$ reduces to 36 distinct coefficients (the proof is beyond the scope of this book):

$$\underline{\mathbf{S}} = \underline{\mathbf{C}}\,\underline{\epsilon}$$

$$\underline{\mathbf{S}} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.141}$$

where

$$\underline{\mathbf{S}} = \left\{ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \right\} = \left\{ \begin{array}{c} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{array} \right\} \qquad \underline{\epsilon} = \left\{ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{array} \right\} = \left\{ \begin{array}{c} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2\, e_{yz} \\ 2\, e_{xz} \\ 2\, e_{xy} \end{array} \right\}$$

The 6×6 matrix $\underline{\mathbf{C}}$ is called the elastic-constant matrix and it does not constitute a tensor. In general, $\underline{\mathbf{C}}$ may depend upon temperature. We shall ignore strain-rate effects and consider the elastic coefficients, components of $\underline{\mathbf{C}}$, at most function of position. If the elastic coefficients are constants, the material is said to be homogeneous. These constants are those describing the elastic properties of the material. Equation (2.141) is known as the Hooke's Law.

2.9.2 Internal Strain Energy

When loads are applied to a structure, the material of the structural element will deform. In the process the external work done by the loads will be converted by the action of either normal or shear stress into internal work called strain energy, provided that no energy is lost in the form of heat. Hence, the strain energy is stored in the body and we use the symbol U to designate strain energy. The unit of strain energy is [N–m] in SI and [lb–in] in English. Strain energy is always a positive scalar quantity even if the stress is compressive because stress and strain are always in the same direction. The strain energy density is expressed with u and is shown in Fig. 6.2 and has units of [Pa] in SI and [psi] in English.

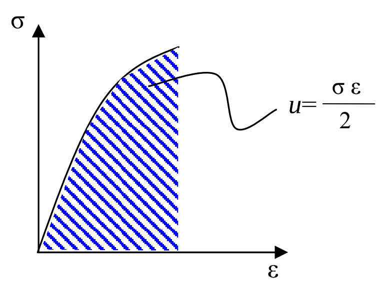


Figure 2.26: Strain energy density.

When an external force acts upon an elastic body and deforms it, the work done by the force is stored within the body in the form of strain energy. In the case of elastic deformation, the total strain energy density due to a general state of stress is

$$u = \frac{1}{2}\,\underline{\mathbf{S}}^{\mathrm{T}}\underline{\epsilon} = \frac{1}{2}\left\{S_{xx}\,\epsilon_{xx} + S_{yy}\,\epsilon_{yy} + S_{zz}\,\epsilon_{zz} + S_{xy}\,\epsilon_{xy} + S_{xz}\,\epsilon_{xz} + S_{yz}\,\epsilon_{yz}\right\} \tag{2.142}$$

Then total strain energy due to a general state of stress is

$$U = \iiint_{\mathrm{Vol}} \frac{1}{2}\left\{S_{xx}\,\epsilon_{xx} + S_{yy}\,\epsilon_{yy} + S_{zz}\,\epsilon_{zz} + S_{xy}\,\epsilon_{xy} + S_{xz}\,\epsilon_{xz} + S_{yz}\,\epsilon_{yz}\right\} d\mathrm{Vol} \tag{2.143}$$

Material that have a strain energy function are known as hyperelastic materials.

2.9.3 Anisotropic Materials

An *isotropic* material has the same material properties in all directions, opposed to an *anisotropic* material whose properties differ in various directions. A material is *homogeneous* if it has the same properties at every point. Wood is an example of a homogeneous material that can be anisotropic. A body formed of steel and aluminum portions is an example of a material that is inhomogeneous, but each portion is isotropic.

Due to the growing importance of composite materials, the linear elastic behavior of anisotropic materials will be treated here. The physical properties of anisotropic materials are directional, i.e., the physical response of the material depends on the direction in which it is acted upon. Consider, as an example, the stiffness of the unidirectional composite material: in the fiber direction the stiffness of the composite is dominated by the high stiffness of the fiber. However, in the direction transverse to the fiber, the stiffness of the composite is dominated by that of the matrix material, which is far small than that of the fiber. This contrasts with isotropic materials for which the mechanical response is identical in all directions.

Anisotropic formulation

The three-dimensional anisotropic hookean strain formulation is given by

$$\underline{\mathbf{S}} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.144}$$

Under multiaxial and isothermal (constant temperature conditions), an elastic material is one that possesses a stress-free state with all components of stress being single-valued functions of the components of strain. In other words,

$$\begin{aligned}
S_{xx} &= S_{xx}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\
S_{yy} &= S_{yy}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\
S_{zz} &= S_{zz}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\
S_{yz} &= S_{yz}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\
S_{xz} &= S_{xz}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\
S_{xy} &= S_{xy}\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right)
\end{aligned} \tag{2.145}$$

where the parenthesis implies dependence only on the current values of the quantities enclosed.

First Law of Thermodynamics of Elastic Solids

For us to fully define the response of the body, we need three additional constitutive equations. For this we assume that the following thermodynamic quantities are functions of the strain as well:

$$\begin{aligned} &\text{internal energy field} && u = u\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\ &\text{heat field} && h = h\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\ &\text{entropy field} && s = s\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \end{aligned} \tag{2.146}$$

The first Law of thermodynamics states that at every point in a body there exists an internal energy per unit volume u_l such that

$$\frac{du}{dt} = \frac{dh}{dt} + \frac{dw}{dt}$$

where h_l is the heat addition to the body per unit volume and w is the work done on the body per unit volume. Assuming that only mechanical work is done on a material element:

$$\frac{dw}{dt} = S_{xx}\,\dot{\epsilon}_{xx} + S_{yy}\,\dot{\epsilon}_{yy} + S_{zz}\,\dot{\epsilon}_{zz} + S_{yz}\,\dot{\epsilon}_{yz} + S_{x}\,\dot{\epsilon}_{xz} + S_{xy}\,\dot{\epsilon}_{xy}$$

Equations (2.146) indicate that the heat added to the body is independent of temperature. In fact, most elastic aerospace metals generate negligible heat in most cases. Thus, an elastic body under quasi-static conditions behaves as an adiabatic body (there is no heat gain or loss); i.e., $\dot{h} = 0$. Thus, the first law of thermodynamics yields to

$$\dot{u} = S_{xx}\,\dot{\epsilon}_{xx} + S_{yy}\,\dot{\epsilon}_{yy} + S_{zz}\,\dot{\epsilon}_{zz} + S_{yz}\,\dot{\epsilon}_{yz} + S_{x}\,\dot{\epsilon}_{xz} + S_{xy}\,\dot{\epsilon}_{xy} \tag{2.147}$$

Now, the time rate of change of the internal energy may be obtained from the change differentiation of

$$\begin{aligned} u &= u\left(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{yz}, \epsilon_{xz}, \epsilon_{xy}\right) \\ du &= \frac{\partial u}{\partial \epsilon_{xx}}\, d\epsilon_{xx} + \frac{\partial u}{\partial \epsilon_{yy}}\, d\epsilon_{yy} + \frac{\partial u}{\partial \epsilon_{zz}}\, d\epsilon_{zz} + \frac{\partial u}{\partial \epsilon_{yz}}\, d\epsilon_{yz} + \frac{\partial u}{\partial \epsilon_{xz}}\, d\epsilon_{xz} + \frac{\partial u}{\partial \epsilon_{xy}}\, d\epsilon_{xy} \end{aligned}$$

Thus,

$$\dot{u} = \frac{\partial u}{\partial \epsilon_{xx}}\, \dot{\epsilon}_{xx} + \frac{\partial u}{\partial \epsilon_{yy}}\, \dot{\epsilon}_{yy} + \frac{\partial u}{\partial \epsilon_{zz}}\, \dot{\epsilon}_{zz} + \frac{\partial u}{\partial \epsilon_{yz}}\, \dot{\epsilon}_{yz} + \frac{\partial u}{\partial \epsilon_{xz}}\, \dot{\epsilon}_{xz} + \frac{\partial u}{\partial \epsilon_{xy}}\, \dot{\epsilon}_{xy} \tag{2.148}$$

From Eqs. (2.147) and (2.148),

$$\left(S_{xx} - \frac{\partial u}{\partial \epsilon_{xx}}\right) \dot{\epsilon}_{xx} + \left(S_{yy} - \frac{\partial u}{\partial \epsilon_{yy}}\right) \dot{\epsilon}_{yy} + \left(S_{zz} - \frac{\partial u}{\partial \epsilon_{zz}}\right) \dot{\epsilon}_{zz} +$$

$$\left(S_{yz} - \frac{\partial u}{\partial \epsilon_{yz}}\right) \dot{\epsilon}_{yz} + \left(S_{xz} - \frac{\partial u}{\partial \epsilon_{xz}}\right) \dot{\epsilon}_{xz} + \left(S_{xy} - \frac{\partial u}{\partial \epsilon_{xy}}\right) \dot{\epsilon}_{xy} = 0$$

Since the strain rate components are independent from each other,

$$S_{\mathrm{xx}} = \frac{\partial u}{\partial \epsilon_{\mathrm{xx}}} \quad S_{\mathrm{yy}} = \frac{\partial u}{\partial \epsilon_{\mathrm{yy}}} \quad S_{\mathrm{zz}} = \frac{\partial u}{\partial \epsilon_{\mathrm{zz}}}$$
$$S_{\mathrm{yz}} = \frac{\partial u}{\partial \epsilon_{\mathrm{yz}}} \quad S_{\mathrm{xz}} = \frac{\partial u}{\partial \epsilon_{\mathrm{xz}}} \quad S_{\mathrm{xy}} = \frac{\partial u}{\partial \epsilon_{\mathrm{xy}}} \tag{2.149}$$

The above equation guarantees energy balance of the first law of thermodynamics for an elastic body under adiabatic conditions.

Second Law of Thermodynamics of Elastic Solids

The second law of thermodynamics states that there exists an entropy per unit volume s such that

$$\frac{ds}{dt} \geq \frac{1}{T}\,\frac{dh}{dt}$$

where T is the absolute temperature, s the total entropy per unit volume, and h the total heat per unit volume. We can also write this relationship as follows:

$$\dot{s}_i \equiv \dot{s} - \frac{1}{T}\,\dot{h} \geq 0 \tag{2.150}$$

where s_i is the internal entropy generation since it represents the total entropy minus a quantity that arises from the heat added to the body. For adiabatic conditions, $\dot{h} = 0$, thus,

$$\dot{s}_i = \dot{s} \geq 0 \tag{2.151}$$

Now using the change rule of differentiation of

$$s = s\left(\epsilon_{\mathrm{xx}}, \epsilon_{\mathrm{yy}}, \epsilon_{\mathrm{zz}}, \epsilon_{\mathrm{yz}}, \epsilon_{\mathrm{xz}}, \epsilon_{\mathrm{xy}}\right)$$
$$\dot{s} = \frac{\partial s}{\partial \epsilon_{\mathrm{xx}}}\,\dot{\epsilon}_{\mathrm{xx}} + \frac{\partial s}{\partial \epsilon_{\mathrm{yy}}}\,\dot{\epsilon}_{\mathrm{yy}} + \frac{\partial s}{\partial \epsilon_{\mathrm{zz}}}\,\dot{\epsilon}_{\mathrm{zz}} + \frac{\partial s}{\partial \epsilon_{\mathrm{yz}}}\,\dot{\epsilon}_{\mathrm{yz}} + \frac{\partial s}{\partial \epsilon_{\mathrm{xz}}}\,\dot{\epsilon}_{\mathrm{xz}} + \frac{\partial s}{\partial \epsilon_{\mathrm{xy}}}\,\dot{\epsilon}_{\mathrm{xy}} \geq 0 \tag{2.152}$$

Now, since all strain rates are independent of each other, we let $\dot{\epsilon}_{\mathrm{xx}}$ change in time while all other strain components don't. Thus Eq. (2.152) reduces to

$$\dot{s} = \frac{\partial s}{\partial \epsilon_{\mathrm{xx}}}\,\dot{\epsilon}_{\mathrm{xx}} \geq 0 \tag{2.153}$$

Note that $\partial s/\partial \epsilon_{\mathrm{xx}}$ and $\dot{\epsilon}_{\mathrm{xx}}$ are independent from each other because the entropy does not depend on the strain rate but on the strain value. Now, let us consider two different possible cases for the strain rate at the same time:

$$\dot{\epsilon}_{\mathrm{xx}} = c\,t \qquad \rightarrow \qquad \frac{\partial s}{\partial \epsilon_{\mathrm{xx}}}\,c\,t \geq 0$$

$$\dot{\epsilon}_{\mathrm{xx}} = -c\,t \qquad \rightarrow \qquad -\frac{\partial s}{\partial \epsilon_{\mathrm{xx}}}\,c\,t \geq 0$$

Which implies that Eq. (2.153) can be satisfied if and only if

$$\frac{\partial s}{\partial \epsilon_{\mathrm{xx}}} = 0$$

Similar, we can show that all other entropy derivatives are zero, as well. Thus, for an elastic body

$$\dot{s}_i = \dot{s} = 0$$

and the second law of thermodynamics is satisfied. This indicates that no entropy is generated in an elastic body, which is consistent with the assumption that all lines of the constitutive equation are single-valued; and, thus, all processes in an elastic body are recoverable.

Consequence of the First Law of Thermodynamics

Using the first law of thermodynamics, we can write Eq. (2.144) as follows

$$\begin{Bmatrix} S_{\mathrm{xx}} \\ S_{\mathrm{yy}} \\ S_{\mathrm{zz}} \\ S_{\mathrm{yz}} \\ S_{\mathrm{xz}} \\ S_{\mathrm{xy}} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial \epsilon_{\mathrm{xx}} \\ \partial u/\partial \epsilon_{\mathrm{yy}} \\ \partial u/\partial \epsilon_{\mathrm{zz}} \\ \partial u/\partial \epsilon_{\mathrm{yz}} \\ \partial u/\partial \epsilon_{\mathrm{xz}} \\ \partial u/\partial \epsilon_{\mathrm{xy}} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{\mathrm{xx}} \\ \epsilon_{\mathrm{yy}} \\ \epsilon_{\mathrm{zz}} \\ \gamma_{\mathrm{yz}} \\ \gamma_{\mathrm{xz}} \\ \gamma_{\mathrm{xy}} \end{Bmatrix} \tag{2.154}$$

Consider the first equation of Eq. (2.154)

$$S_{\mathrm{xx}} = \frac{\partial u}{\partial \epsilon_{\mathrm{xx}}} = C_{11}\,\epsilon_{\mathrm{xx}} + C_{12}\,\epsilon_{\mathrm{yy}} + C_{13}\,\epsilon_{\mathrm{zz}} + C_{14}\,\epsilon_{\mathrm{yz}} + C_{15}\,\epsilon_{\mathrm{xz}} + C_{16}\,\epsilon_{\mathrm{xy}}$$

Let's differentiate once respect to ϵ_{yy}:

$$\frac{\partial S_{\mathrm{xx}}}{\partial \epsilon_{\mathrm{yy}}} = \frac{\partial^2 u}{\partial \epsilon_{\mathrm{yy}}\,\partial \epsilon_{\mathrm{xx}}} = C_{12} \tag{2.155}$$

Now consider the second equation of Eq. (2.154)

$$S_{\mathrm{yy}} = \frac{\partial u}{\partial \epsilon_{\mathrm{yy}}} = C_{21}\,\epsilon_{\mathrm{xx}} + C_{22}\,\epsilon_{\mathrm{yy}} + C_{23}\,\epsilon_{\mathrm{zz}} + C_{24}\,\epsilon_{\mathrm{yz}} + C_{25}\,\epsilon_{\mathrm{xz}} + C_{26}\,\epsilon_{\mathrm{xy}}$$

Let's differentiate once respect to ϵ_{xx}:

$$\frac{\partial S_{\mathrm{yy}}}{\partial \epsilon_{\mathrm{xx}}} = \frac{\partial^2 u}{\partial \epsilon_{\mathrm{xx}}\,\partial \epsilon_{\mathrm{yy}}} = C_{21} \tag{2.156}$$

Since the order of differentiation is unimportant, Eqs. (2.155) and (2.156) are equal:

$$\frac{\partial^2 u}{\partial \epsilon_{\mathrm{yy}}\,\partial \epsilon_{\mathrm{xx}}} = \frac{\partial^2 u}{\partial \epsilon_{\mathrm{xx}}\,\partial \epsilon_{\mathrm{yy}}} \quad \rightarrow \quad C_{12} = C_{21}$$

Similarly it can be shown that the elastic-constant matrix $\underline{\mathbf{C}}$ is symmetric:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

Thus for a general linearly elastic material, the independent material constants are reduced to 21.

3-D Anisotropic Hookean Formulation

The three-dimensional anisotropic Hookean strain formulation is given by

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.157}$$

The above is known as the inverted form of the Hooke's Law. Then the Hooke's Law is defined as

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \tag{2.158}$$

where matrix $\underline{\mathbf{D}}$ is the elastic compliance material constant matrix and is defined as:

$$\underline{\mathbf{D}} = \underline{\mathbf{C}}^{-1}$$

2.9.4 Elastic Constitutive Relationship for Isotropic Materials

Now let us proceed to obtain the matrices $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ for isotropic materials. Recall that the deformation of a structure is a function of the applied loads, and the resulting strains at a point are related to the local state of stress at a point. To better understand the stress and strain relationships, let us consider a simple uniform rod subjected to a tension test. An axial load P is applied at the end of the rod, resulting in an axial deflection δ. If the rod is elastic, the relationship between axial load P and the axial deflection δ is essentially linear for sufficiently small deflections, and it is the same whether the load is increasing or decreasing. If the load P is applied at the centroid of the rod's cross section, then the axial stress and strain are uniform across all sections, except possibly in the vicinity of the ends. A plot of stress versus strain will be linear. The slope of the stress-strain diagram is the modulus of elasticity, or Young's modulus, E.

All structural materials exhibit the *Poisson effect*. To understand this effect, let us go back to our tensile test specimen: when the specimen is stretched in the axial direction, it contracts laterally; if axially compressed, it expands laterally. If a bar has a circular cross section of unloaded diameter D, and if the contraction transverse to the pull direction is d, then the transverse strain is

$$\epsilon_{\perp} = -\frac{d}{D} \tag{2.159}$$

Poisson's ratio ν is a material property that relates the axial strain to the transverse strain, as follows:

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = -\frac{\epsilon_{\perp}}{e} \tag{2.160}$$

The minus sign is important because the axial strain and accompanying transverse strain are always opposite in sign. The following properties of the elastic constants can be shown:

$$E > 0 \qquad G > 0 \qquad 0 < \nu < \frac{1}{2}$$

Materials for which

$$\nu \approx 0 \quad \text{are very compressible}$$

$$\nu \approx \frac{1}{2} \quad \text{are very incompressible}$$

Cork is an example of a very compressible material, whereas rubber is very incompressible.

The constitutive relationships for a three-dimensional state of stress can be derived by calculating the strains that accompany the normal and shear stresses considered to act separately in each direction and then adding the results together. This is an application of the *principle of superposition*, which holds for linear elastic behavior. Thus, if a plane differential element of material is subjected to stress S_{xx}, then the resulting normal strains will be

$$e_{xx} = \frac{S_{xx}}{E} \qquad e_{yy} = -\nu\, e_{xx} = -\nu\,\frac{S_{xx}}{E} \qquad e_{zz} = -\nu\, e_{xx} = -\nu\,\frac{S_{xx}}{E}$$

If a plane differential element of material is subjected to stress S_{yy}, then the resulting normal strains will be

$$e_{yy} = \frac{S_{yy}}{E} \qquad e_{xx} = -\nu\, e_{yy} = -\nu\,\frac{S_{yy}}{E} \qquad e_{zz} = -\nu\, e_{yy} = -\nu\,\frac{S_{yy}}{E}$$

If a plane differential element of material is subjected to stress S_{zz}, then the resulting normal strains will be

$$e_{zz} = \frac{S_{zz}}{E} \qquad e_{xx} = -\nu\, e_{zz} = -\nu\,\frac{S_{zz}}{E} \qquad e_{yy} = -\nu\, e_{zz} = -\nu\,\frac{S_{zz}}{E}$$

Now, if the three states of uniaxial stress are combined using the principle of superposition, we obtain a triaxial state of stress in which the total normal strain in each direction

$$
\begin{aligned}
e_{xx} &= \frac{S_{xx}}{E} - \nu\,\frac{S_{yy}}{E} - \nu\,\frac{S_{zz}}{E} \\
e_{yy} &= \frac{S_{yy}}{E} - \nu\,\frac{S_{xx}}{E} - \nu\,\frac{S_{zz}}{E} \\
e_{zz} &= \frac{S_{zz}}{E} - \nu\,\frac{S_{xx}}{E} - \nu\,\frac{S_{yy}}{E}
\end{aligned} \tag{2.161}
$$

These results are valid only for isotropic materials, that is, materials whose stiffness, strength, and other properties are the same in all directions. Young's modulus is the same in both the x and y directions of a sheet of isotropic material.

In isotropic materials, the shear stresses are independent of the normal strains. Thus

$$
\gamma_{xy} = \frac{S_{xy}}{G} \qquad \gamma_{xz} = \frac{S_{xz}}{G} \qquad \gamma_{yz} = \frac{S_{yz}}{G} \tag{2.162}
$$

The constant G is called the shear modulus. For isotropic materials it can be shown that

$$
G = \frac{E}{2(1+\nu)} \tag{2.163}
$$

The stress-strain relationship for normal components can be also expressed in matrix form as follows:

$$
\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \tag{2.164}
$$

or including all strains and stress, in matrix form,

$$
\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} \tag{2.165}
$$

or

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \tag{2.166}$$

Thus the number of independent elastic constants reduces to two and the elastic matrix is symmetric regardless of the existence of a strain energy function. The Hooke's Law relationship may be inverted to get

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{Bmatrix} \tag{2.167}$$

and

$$S_{xy} = G\,\gamma_{xy} \qquad S_{xz} = G\,\gamma_{xz} \qquad S_{yz} = G\,\gamma_{yz} \tag{2.168}$$

or including all strains and stress, in matrix form,

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2.169}$$

or

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.170}$$

2.9.5 Elastic Stress-Strain Relationship for Orthotropic Materials

Materials that possess elastic symmetry about three mutually orthogonal planes, that is, about planes oriented 90° to each other, are known as orthotropic materials. The three-dimensional orthotropic Hookean strain formulation is given by

$$\underline{\mathbf{S}} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{2.171}$$

For such materials 9 independent material constants exist. These constants are found as follows

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} \tag{2.172}$$

or

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \tag{2.173}$$

Due to symmetry the following holds:

$$\frac{\nu_{xy}}{E_{xx}} = \frac{\nu_{yx}}{E_{yy}} \qquad \frac{\nu_{xz}}{E_{xx}} = \frac{\nu_{zx}}{E_{zz}} \qquad \frac{\nu_{yz}}{E_{yy}} = \frac{\nu_{zy}}{E_{zz}} \tag{2.174}$$

2.9.6 Temperature Strains in Isotropic Materials

In the material's elastic region, behavior changes in temperature can cause two effects:

1. Changes in the elastic constants of the material (for isotropic materials E and ν).

2. Changes causes the material to strain in the absence of stress.

Thus, the total material strain can be expressed as follows

$$\underline{\epsilon}_{\text{total}} = \underline{\epsilon}_{\text{mechanical}} + \underline{\epsilon}_{\text{thermal effect due to changes in elastic constant}} + \underline{\epsilon}_{\text{thermal effect due to material changes}}$$

For many structural components, a change in temperature of few hundred degrees Fahrenheit does not result in much changes in the material's elastic constants. Thus it is a reasonable assumption to neglect this thermal effect:

$$\underline{\epsilon}_{\text{total}} = \underline{\epsilon}_{\text{mechanical}} + \underline{\epsilon}_{\text{thermal effect due to material changes}}$$

The strain caused by a temperature change in the absence of stress is called thermal strain and is denoted as $\underline{\epsilon}^t$:

$$\underline{\epsilon}_{\text{total}} = \underline{\epsilon}_{\text{mechanical}} + \underline{\epsilon}^t_{\text{thermal}} \tag{2.175}$$

For isotropic materials, the thermal strains must be a pure expansion or contraction of the material with no distortion or shear. This is because normal strain at a point is the same in all directions in isotropic materials, in which the temperature change does not induce shear strain. The strain is assumed to be a linear function of the temperature change although it may not be exactly true. However, the actual thermal strain is nearly linear with temperature change in temperature. Thus response of isotropic materials to temperature change from T_0 to T (represented by ΔT) is characterized by the linear coefficient of thermal expansion α:

$$\underline{\epsilon}^t = \alpha\,\Delta T + \text{Higher Order Terms} \approx \alpha\,\Delta T$$

where

$$\Delta T = T - T_0$$

Note that since the strains are nondimensional, the units of α must be

$$\alpha = \frac{1}{\text{units of }\Delta T}$$

Further here we will assume that the temperature change throughout the body is determined by means of a separate heat transfer analysis. This results in uncoupled thermoelasticity, in which the temperature field T, like the body forces, is prescribed for a given stress problem. Thus the strain-stress relationship for isotropic materials becomes

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} + \alpha\,\Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.176}$$

or its inverse form

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} - \frac{E\,\alpha\,\Delta T}{1-2\,\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.177}$$

2.10 Plane Stress and Plane Strain

2.10.1 Consequence of Plane Stress

Recall that for structures with a relatively small thickness, we can use the plane stress assumption. If we say the structure is confound to the x-y plane,

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{xy} & S_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.178}$$

Since $S_{zz} = S_{xz} = S_{yz} = 0$, the Hooke's Law reduced to

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ 0 \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} S_{xx} - \nu\, S_{yy} \\ S_{yy} - \nu\, S_{xx} \\ -\nu\, S_{xx} - \nu\, S_{yy} \end{Bmatrix} \tag{2.179}$$

$$\gamma_{xy} = \frac{S_{xy}}{G} \tag{2.180}$$

Note that due to Poisson effect, the plane stress is accompanied by normal strain in the z-direction. As it can be seen, the fact that $S_{zz} = 0$, $e_{zz} \neq 0$:

$$e_{zz} = -\frac{\nu}{E}\left\{S_{xx} + S_{yy}\right\}$$

The above will not be zero specially when the body undergoes temperature changes.

2.10.2 Consequence of Plane strain

Recall that for structures with a relatively large thickness, we can use the plane strain assumption. If we say the structure is in plane strain in the x-y plane,

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & e_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.181}$$

Since $e_{zz} = \gamma_{xz} = \gamma_{yz} = 0$, the inverted Hooke's Law reduced to

$$\begin{aligned} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} &= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 0 \end{Bmatrix} \\ &= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{Bmatrix} (1-\nu)\,e_{xx} + \nu\,e_{yy} \\ \nu\,e_{xx} + (1-\nu)\,e_{yy} \\ \nu\,e_{xx} + \nu\,e_{yy} \end{Bmatrix} \end{aligned} \tag{2.182}$$

$$S_{xy} = G\,\gamma_{xy} \tag{2.183}$$

Note that due to Poisson effect, the plane stress is accompanied by normal stress in the z-direction. As it can be seen, the fact that $e_{zz} = 0$, $S_{zz} \neq 0$:

$$S_{zz} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\left\{e_{xx} + e_{yy}\right\}$$

2.10.3 von Mises Stress in Plane Strain and Plane Stress

Suppose we have a structure with a plane stress state of stress at a point. For such a case:

$$S_{zz} = 0 \qquad e_{zz} = -\frac{\nu}{E}\left\{S_{xx} + S_{yy}\right\} \tag{2.184}$$

and the von Mises stress for plane stress become be

$$S_{\text{eq}}\Big|_{\text{Plane Stress}} = \sqrt{S_{\text{xx}}^2 + S_{\text{yy}}^2 - S_{\text{yy}}\,S_{\text{xx}} + 3\,S_{\text{xy}}^2} \tag{2.185}$$

Suppose we have a structure with a plane strain state of stress at a point. For such a case:

$$S_{\text{zz}} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\Big\{e_{\text{xx}} + e_{\text{yy}}\Big\} \qquad e_{\text{zz}} = 0 \tag{2.186}$$

and the von Mises stress for plane strain become

$$S_{\text{eq}}\Big|_{\text{Plane Strain}} = \sqrt{S_{\text{xx}}^2 + S_{\text{yy}}^2 - S_{\text{yy}}\,S_{\text{XX}} + 3\,S_{\text{xy}}^2 - \nu\,(1-\nu)\,\left(S_{\text{xx}} + S_{\text{yy}}\right)^2} \tag{2.187}$$

Thus by comparing the above we can conclude that von Mises stress in plane stress is higher than for the case of plane strain: S_{xx}, S_{yy}, and S_{zz},

$$S_{\text{eq}}\Big|_{\text{Plane Strain}} \leq S_{\text{eq}}\Big|_{\text{Plane Stress}}$$

Example 2.12.

Consider the following displacement field described in Example 2.8:

$$\underline{\mathbf{R}} = 0.01\,\left(x^2 + y^2\right)\,\underline{\hat{\mathbf{i}}} + 0.01\,(3 + x\,z)\,\underline{\hat{\mathbf{j}}} - \left(0.006\,z^2\right)\,\underline{\hat{\mathbf{k}}} \qquad \text{ft}$$

(2.12a) Ignoring temperature effects and assuming the material is isotropic, determine the stress tensor at (0,1,3). Assume $\nu = 0.3$ and $E = 30 \times 10^6$ psi.

The strain tensor was found as

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\,\gamma_{xy} & \frac{1}{2}\,\gamma_{xz} \\ \frac{1}{2}\,\gamma_{xy} & e_{yy} & \frac{1}{2}\,\gamma_{yz} \\ \frac{1}{2}\,\gamma_{xz} & \frac{1}{2}\,\gamma_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} 2\,x & y + \frac{z}{2} & 0 \\ y + \frac{z}{2} & 0 & \frac{x}{2} \\ 0 & \frac{x}{2} & -1.2\,z \end{bmatrix} \times 10^{-2}$$

$$= \begin{bmatrix} 0 & 0.025 & 0 \\ 0.025 & 0 & 0 \\ 0 & 0 & -0.036 \end{bmatrix}$$

Thus,

$$e_{xx} = 0.0 \qquad \gamma_{xy} = 0.05 \quad \gamma_{xz} = 0.0$$

$$\gamma_{xy} = 0.05 \quad e_{yy} = 0.0 \qquad \gamma_{yz} = 0.0$$

$$\gamma_{xz} = 0.0 \qquad \gamma_{yz} = 0.0 \qquad e_{zz} = -0.036$$

The state of stress at a point is given by the stress tensor

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix}$$

To find the stress components we can use the inverted Hooke's Law relationship

$$\left\{\begin{array}{c} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{array}\right\} = \frac{E}{(1+\nu)(1-2\,\nu)} \left[\begin{array}{cccccc} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{array}\right] \left\{\begin{array}{c} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{array}\right\}$$

Thus

$$\left\{\begin{array}{c} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{array}\right\} = \frac{3\times 10^7}{(1+0.3)(1-2\,(0.3))} \left[\begin{array}{cccccc} 0.7 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{array}\right] \left\{\begin{array}{c} 0.0 \\ 0.0 \\ -0.036 \\ 0.0 \\ 0.0 \\ 0.05 \end{array}\right\}$$

$$\left\{\begin{array}{c} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{array}\right\} = \left\{\begin{array}{c} -623077 \\ -623077 \\ -1453850 \\ 0 \\ 0 \\ 576923 \end{array}\right\} \quad \text{psi}$$

Thus the three dimensional state of stress for an isotropic related to the given state of strain is:

$$\underline{\mathbf{S}} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{array}\right] = \left[\begin{array}{rrr} -623 & 577 & 0 \\ 577 & -623 & 0 \\ 0 & 0 & -1454 \end{array}\right] \text{ ksi}$$

(2.12b) Ignoring temperature effects and assuming the material is orthotropic, determine the stress tensor at (0,1,3). The material is graphite eopxy (AS/3501).

From material tables,

$$E_{xx} = 20.00 \times 10^6 \text{ psi}$$
$$E_{yy} = 1.3 \times 10^6 \text{ psi}$$
$$E_{zz} = 1.6 \times 10^6 \text{ psi}$$
$$G_{xy} = 1.03 \times 10^6 \text{ psi}$$
$$G_{xz} = 1.03 \times 10^6 \text{ psi}$$
$$G_{yz} = 0.90 \times 10^6 \text{ psi}$$
$$\nu_{xy} = 0.30$$
$$\nu_{xz} = 0.30$$
$$\nu_{yz} = 0.49$$

From Eq. (2.174):

$$\nu_{yx} = \frac{\nu_{xy}}{E_{xx}} E_{yy} = 0.0195$$

$$\nu_{zx} = \frac{\nu_{xz}}{E_{xx}} E_{zz} = 0.024$$

$$\nu_{zy} = \frac{\nu_{yz}}{E_{yy}} E_{zz} = 0.603$$

The state of stress at a point is given by the stress tensor

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix}$$

To find the stress components we can use the Hooke's Law relationship

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} 0.0 \\ 0.0 \\ -0.036 \\ 0.0 \\ 0.0 \\ 0.05 \end{Bmatrix} = 10^{-6} \begin{bmatrix} 0.05 & -0.015 & -0.015 & 0 & 0 & 0 \\ -0.015 & 0.769231 & -0.376923 & 0 & 0 & 0 \\ -0.015 & -0.376923 & 0.625 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.11111 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.970874 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.970874 \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = 10^{6} \begin{bmatrix} 0.05 & -0.015 & -0.015 & 0 & 0 & 0 \\ -0.015 & 0.769231 & -0.376923 & 0 & 0 & 0 \\ -0.015 & -0.376923 & 0.625 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.11111 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.970874 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.970874 \end{bmatrix}^{-1} \begin{Bmatrix} 0.0 \\ 0.0 \\ -0.036 \\ 0.0 \\ 0.0 \\ 0.05 \end{Bmatrix}$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = 10^{6} \begin{bmatrix} 20.5876 & 0.913519 & 1.04502 & 0 & 0 & 0 \\ 0.913519 & 1.88584 & 1.15923 & 0 & 0 & 0 \\ 1.04502 & 1.15923 & 2.32418 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.90 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.03 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.03 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ -0.036 \\ 0.0 \\ 0.0 \\ 0.05 \end{Bmatrix}$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix} = \begin{Bmatrix} -37620.9 \\ -41732.2 \\ -83670.6 \\ 0.0 \\ 0.0 \\ 51500.0 \end{Bmatrix} \quad \text{psi}$$

Thus the three dimensional state of stress for this orthotropic related to the given state of strain is:

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} = \begin{bmatrix} -37.6211 & 51.500 & 0 \\ 51.500 & -41.732 & 0 \\ 0 & 0 & -83.671 \end{bmatrix} \text{ ksi}$$

End Example □

Example 2.13.

The state of stress at the surface of a wing is subjected to the following stresses

$$\underline{\mathbf{S}} = \begin{bmatrix} S_\circ & 0 & S_\circ \\ 0 & S_\circ & 0 \\ S_\circ & 0 & S_\circ \end{bmatrix}$$

It is known that $S_\circ = 1.0 \times 10^6$ psi.

(2.13a) Determine the principal state of stress.

The principal stresses are determined by finding the eigenvalues of the stress tensor:

$$\det \begin{bmatrix} S_{xx} - \lambda & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} - \lambda & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} - \lambda \end{bmatrix} = \begin{vmatrix} S_\circ - \lambda & 0 & S_\circ \\ 0 & S_\circ - \lambda & 0 \\ S_\circ & 0 & S_\circ - \lambda \end{vmatrix} = 0$$

which leads to the characteristic equation that can be expressed in terms of the stress invariants as follows

$$\lambda^3 - I_{\sigma_1} \lambda^2 + I_{\sigma_2} \lambda - I_{\sigma_3} = 0 \qquad (2.188)$$

where I_{σ_i}'s are the stress invariants.

$$I_{\sigma_1} = S_{xx} + S_{yy} + S_{zz} = 3\, S_\circ$$

$$I_{\sigma_2} = \det \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix} + \det \begin{bmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{bmatrix} + \det \begin{bmatrix} S_{yy} & S_{yz} \\ S_{zy} & S_{zz} \end{bmatrix}$$

$$= \begin{vmatrix} S_\circ & 0 \\ 0 & S_\circ \end{vmatrix} + \begin{vmatrix} S_\circ & S_\circ \\ S_\circ & S_\circ \end{vmatrix} + \begin{vmatrix} S_\circ & 0 \\ 0 & S_\circ \end{vmatrix} = 2\, S_\circ^2$$

$$I_{\sigma_3} = \det \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} = \begin{vmatrix} S_\circ & 0 & S_\circ \\ 0 & S_\circ & 0 \\ S_\circ & 0 & S_\circ \end{vmatrix} = 0$$

Thus, the characteristic equation becomes

$$\lambda^3 - 3\, S_\circ \lambda^2 + S_\circ^2 \lambda = (\lambda) \left(\lambda^2 - 3\, S_\circ \lambda + S_\circ^2\right) = 0$$

The three roots of the characteristic equation are

$$\lambda_1 = 2\, S_\circ \quad \lambda_2 = S_\circ \quad \lambda_3 = 0$$

and the principal stresses are

$$S_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 2\,S_\circ$$
$$S_3 = \min[\lambda_1, \lambda_2, \lambda_3] = 0$$
$$S_2 = S_\circ$$

As we can see from this problem that when $I_{\sigma_3} = 0$ it is not always a plane stress problem but if it is a plane stress problem $I_{\sigma_3} = 0$. Thus

$$\underline{\mathbf{S}}_p = \begin{bmatrix} 2\,S_\circ & 0 & 0 \\ 0 & S_\circ & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^6 \quad \text{psi}$$

(2.13b) Draw the Mohr's circle for both cases.

The Mohr's circles for both the given state of stress and the related principal state of stress will have the same Mohr's circles. The Mohr's circle is characteristic of the point of stress and not any stress transformation. Thus Fig. 2.27 shows the Mohr's circles for both cases.

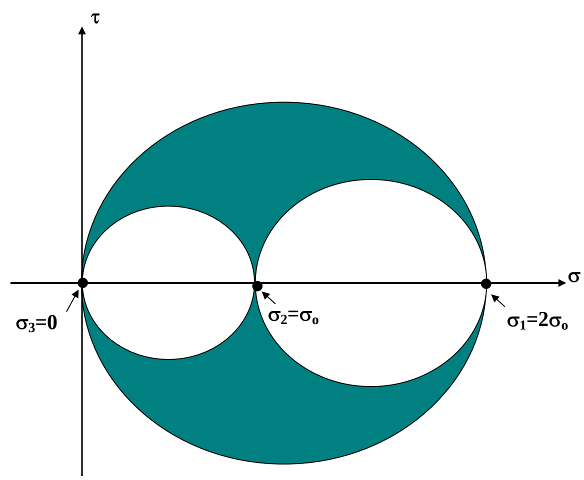


Figure 2.27: Mohr's circle case for the principal state of stress.

Furthermore, the Mohr's circle for the given state of stress will be that related to S_1 and S_3. This can be seen from the fact that the state of stress is confound to the x-z plane and only normal stresses act on the y plane.

(2.13c) If the material is isotropic, determine the principal state of strain. Assume $\nu = 0.3$ and $E = 30 \times 10^6$ psi.

Let us first express the Hooke's Law:

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}$$

Now since the strains are related to stresses, two procedures exist for isotropic materials. First obtain the state of strain for

$$\underline{\mathbf{S}} = \begin{bmatrix} S_o & 0 & S_o \\ 0 & S_o & 0 \\ S_o & 0 & S_o \end{bmatrix}$$

and then finding the eigenvalues. The second one is to note that since the principal strains are invariants and characteristics of a point, and are proportionally related to stresses for linear isotropic materials, we can obtain the state of strain for

$$\underline{\mathbf{S}}_p = \begin{bmatrix} 2\,S_o & 0 & 0 \\ 0 & S_o & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Indeed we can show that a state of principal stress will always produce a state of principal strain and viceversa for orthotropic materials (and note that isotropic materials are a special case of orthotropic materials). If x-y-z are orthogonal axes, the principal stresses can be represented as

$$\underline{\mathbf{S}}_p = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} \tag{2.189}$$

Where we can see that the only nonzero stresses present are S_1, S_2, and S_3. These are normal stresses at the new orthogonal plane 1-2-3. Note that all shear stresses are zero.

So we may apply the Hooke's Law for principal stresses:

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiplying the above we get

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{E} S_1 - \frac{\nu}{E} S_2 - \frac{\nu}{E} S_3 \\ -\frac{\nu}{E} S_1 + \frac{1}{E} S_2 - \frac{\nu}{E} S_3 \\ -\frac{\nu}{E} S_1 - \frac{\nu}{E} S_2 + \frac{1}{E} S_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus, from Hooke's Law above clearly the only nonzero strains present are the normal strains and all shear strains are zero for these axes.

By definition when principal strains are present all shear strains are zero. Then we must conclude that these normal strain are principal strains. Thus the axes of principal strain must also be principal axes of stress.

Thus

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \frac{S_\circ}{E} \begin{Bmatrix} 2-\nu \\ -2\nu+1 \\ -2\nu-\nu \\ 0 \\ 0 \\ 0 \end{Bmatrix} = S_\circ \begin{Bmatrix} 5.66667\times 10^{-8} \\ 1.33333\times 10^{-8} \\ -3.00\times 10^{-8} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.056667 \\ 0.01333 \\ -0.03000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Hence, the state of principal strain is

$$\underline{\mathbf{e}}_p = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} = \begin{bmatrix} 0.056667 & 0 & 0 \\ 0 & 0.01333 & 0 \\ 0 & 0 & -0.03000 \end{bmatrix} \qquad (2.190)$$

(2.13d) If the material is orthotropic (Boron-Epoxy), determine the principal state of strain.

From material tables[8]:

$$\begin{aligned} E_{xx} &= 30.00 \times 10^6 \text{ psi} \\ E_{yy} &= 3.0 \times 10^6 \text{ psi} \\ E_{zz} &= 3.0 \times 10^6 \text{ psi} \\ G_{xy} &= 1.00 \times 10^6 \text{ psi} \\ G_{xz} &= 1.00 \times 10^6 \text{ psi} \\ G_{yz} &= 0.60 \times 10^6 \text{ psi} \\ \nu_{xy} &= 0.30 \\ \nu_{xz} &= 0.25 \\ \nu_{yz} &= 0.25 \end{aligned}$$

From Eq. (2.174):

$$\nu_{yx} = \frac{\nu_{xy}}{E_{xx}} E_{yy} = 0.030$$

$$\nu_{zx} = \frac{\nu_{xz}}{E_{xx}} E_{zz} = 0.025$$

$$\nu_{zy} = \frac{\nu_{yz}}{E_{yy}} E_{zz} = 0.250$$

Let us first express the Hooke's Law:

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}$$

[8] http://www.matweb.com

Now since the strains are related to stresses. First let us obtain the state of strain for

$$\underline{\mathbf{S}} = \begin{bmatrix} S_o & 0 & S_o \\ 0 & S_o & 0 \\ S_o & 0 & S_o \end{bmatrix}$$

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = 10^8 \begin{bmatrix} 3.333 & -1.000 & -0.833 & 0 & 0 & 0 \\ -1.000 & 33.333 & -8.333 & 0 & 0 & 0 \\ -0.833 & -8.333 & 33.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 166.667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100.000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100.000 \end{bmatrix} \begin{Bmatrix} S_o \\ S_o \\ S_o \\ 0 \\ S_o \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = S_o \times 10^{-8} \begin{Bmatrix} 1.5 \\ 24.0 \\ 24.1667 \\ 0.0 \\ 100.00 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.015 \\ 0.24 \\ 0.241667 \\ 0.0 \\ 1.000 \\ 0.0 \end{Bmatrix}$$

The strain tensor is

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & e_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} 0.015 & 0.0 & 0.5 \\ 0.0 & 0.24 & 0.0 \\ 0.5 & 0.0 & 0.241667 \end{bmatrix}$$

And the eigenvalues are:

$$\phi_1 = -38.435 \times 10^{-8}\, S_o \qquad \phi_2 = 24 \times 10^{-8}\, S_o \qquad \phi_3 = 64.1017 \times 10^{-8}\, S_o$$

Thus the principal state of strain is

$$\underline{\mathbf{e}}_p = \begin{bmatrix} 0.64101 & 0 & 0 \\ 0 & 0.24 & 0 \\ 0 & 0 & -0.38435 \end{bmatrix}$$

Now, let us show that for orthotropic materials, the principal strains are not proportionally related to stresses. It should be highlighted that this approach is incorrect. Let us begin with the principal state of stress:

$$\underline{\mathbf{S}}_p = \begin{bmatrix} 2\,S_\text{o} & 0 & 0 \\ 0 & S_\text{o} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} e_\text{xx} \\ e_\text{yy} \\ e_\text{zz} \\ \gamma_\text{yz} \\ \gamma_\text{xz} \\ \gamma_\text{xy} \end{Bmatrix} = 10^8 \begin{bmatrix} 3.333 & -1.000 & -0.833 & 0 & 0 & 0 \\ -1.000 & 33.333 & -8.333 & 0 & 0 & 0 \\ -0.833 & -8.333 & 33.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 166.667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100.000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100.000 \end{bmatrix} \begin{Bmatrix} 2\,S_\text{o} \\ S_\text{o} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = S_\text{o} \times 10^{-8} \begin{Bmatrix} 5.667 \\ 31.33 \\ -10. \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.05667 \\ 0.3133 \\ -0.1000 \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

Thus the principal state of strain is

$$\underline{\mathbf{e}}_p = \begin{bmatrix} 5.667 & 0 & 0 \\ 0 & 31.33 & 0 \\ 0 & 0 & -10.00 \end{bmatrix} \times 10^{-8}\,S_\text{o} = \begin{bmatrix} 0.05667 & 0 & 0 \\ 0 & 0.3133 & 0 \\ 0 & 0 & -0.1000 \end{bmatrix}$$

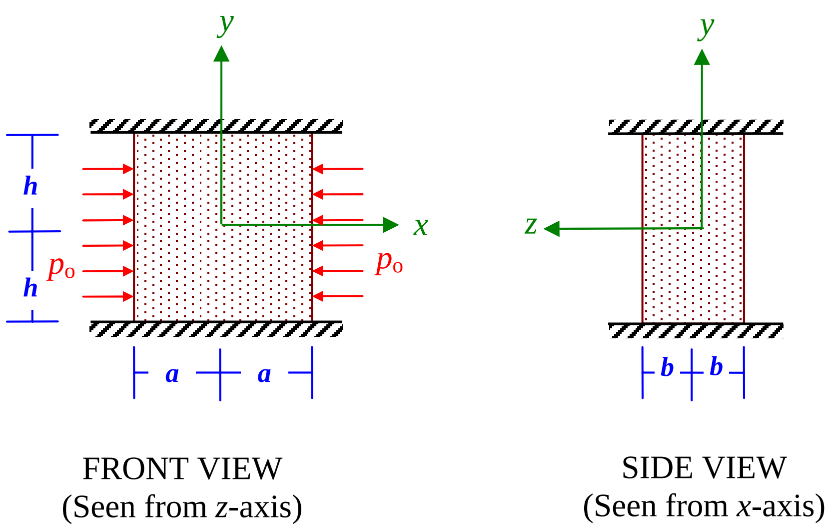
Which clearly shows that it is we cannot use the same assumption as for the case of isotropic materials. This is mainly because the material is dependent in three-mutually orthogonal planes.

End Example □

Example 2.14.

Application 1: Isothermal Isotropic Material

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed pressure p_o in the x-direction. The block is constrained to zero displacement at all points in the y-direction but is free to displace in the z-direction. Boundary conditions are such that the body is free to expand or contract in the x and z directions at both rigid interfaces.



If the bock is made of isotropic material, determine the isothermal elastic field. The mechanical properties for the isotropic material are

$$E = 210 \times 10^9 \quad \text{Pa} \qquad \nu = 0.30$$

and the geometric properties are:

$$a = h = 2\,b = 1''$$

Take the load as $p_o = 200$ lb/in^2.

For an elastic body under isothermal conditions, the problem reduces to one of characterizing the stresses, $\underline{\mathbf{S}}$, strains, $\underline{\mathbf{e}}$, and displacements, $\underline{\mathbf{R}}$. This is a set of 15 unknowns at all point in the body:

6 stress components $(S_{\rm xx}, S_{\rm yy}, S_{\rm zz}, S_{\rm yz}, S_{\rm xz}, S_{\rm xy})$

6 strain components $(e_{\rm xx}, e_{\rm yy}, e_{\rm zz}, e_{\rm yz}, e_{\rm xz}, e_{\rm xy})$

3 displacements (U, V, W)

These are solved using 15 field equations:

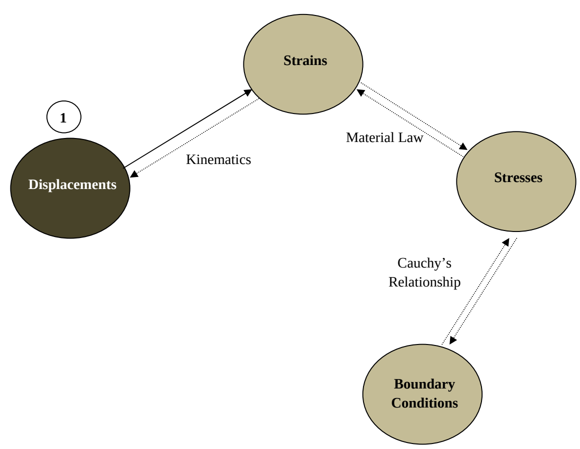
3 equilibrium equations

6 strain-displacement equations

6 stress-strain equations

Along with the boundary conditions.

2.14a) Displacement field.



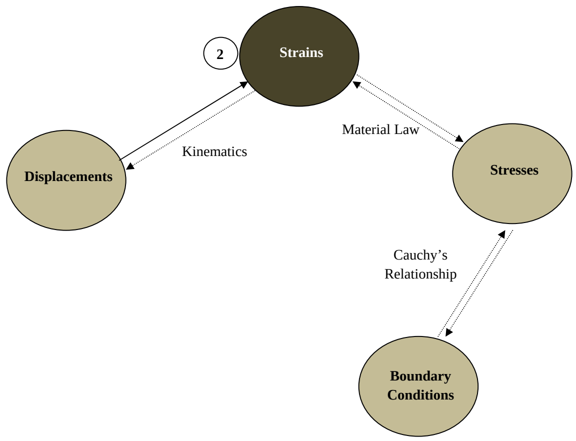
From the geometry of the problem it is assumed that at all points in the body the displacement field (displacement boundary conditions) is:

$$U(x, y, z) = u(x)$$

$$V(x, y, z) = 0$$

$$W(x, y, z) = w(z)$$

2.14b) Strain field.



The displacement gradients are then found as:

$$g_1 = \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} \qquad g_4 = \frac{\partial U}{\partial y} = 0 \qquad g_7 = \frac{\partial U}{\partial z} = 0$$

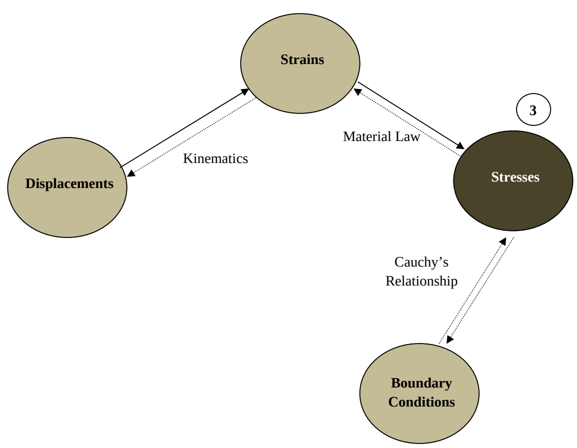
$$g_2 = \frac{\partial V}{\partial x} = 0 \qquad g_5 = \frac{\partial V}{\partial y} = 0 \qquad g_8 = \frac{\partial V}{\partial z} = 0$$

$$g_3 = \frac{\partial W}{\partial x} = 0 \qquad g_6 = \frac{\partial W}{\partial y} = 0 \qquad g_9 = \frac{\partial W}{\partial z} = \frac{\partial w}{\partial z}$$

Thus the resulting strain-displacement relationship is obtained using the Lagrange-Green equations:

$$\begin{aligned}
\epsilon_1 &= e_{xx} = g_1 = \frac{\partial u}{\partial x} \\
\epsilon_2 &= e_{\mathrm{yy}} = g_5 = 0 \\
\epsilon_3 &= e_{\mathrm{zz}} = g_9 = \frac{\partial w}{\partial z} \\
\epsilon_4 &= 2\, e_{\mathrm{yz}} = g_6 + g_8 = 0 \\
\epsilon_5 &= 2\, e_{\mathrm{xz}} = g_3 + g_7 = 0 \\
\epsilon_6 &= 2\, e_{\mathrm{xy}} = g_2 + g_4 = 0
\end{aligned}$$

2.14c) Stress field.



Now the stress-strain relationship for isotropic Hookean strain is obtained using the inverted Hooke's law:

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

$$= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ 0 \\ \epsilon_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus,

$$S_1 = S_{xx} = \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\, e_{xx} + \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, e_{zz}$$

$$S_2 = S_{yy} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, (e_{xx} + e_{zz})$$

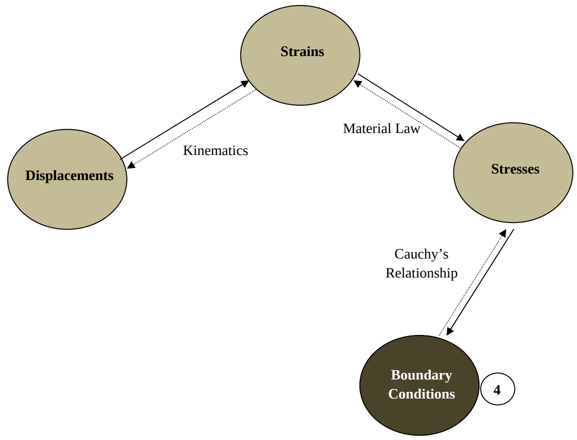
$$S_3 = S_{zz} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, e_{xx} + \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\, e_{zz}$$

$$S_4 = S_{yz} = 0$$

$$S_5 = S_{xz} = 0$$

$$S_6 = S_{xy} = 0$$

2.14d) Equilibrium Equations and Cauchy's Relationship.



Now, substituting the stress components into the three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial S_{zx}}{\partial z} + b_x = 0 \quad \rightarrow \quad \frac{\partial S_{xx}}{\partial x} = 0 \quad \rightarrow \quad S_{xx} = c_1 = c_1(y,z) = \text{constant}$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{zy}}{\partial z} + b_y = 0 \quad \rightarrow \quad \frac{\partial S_{yy}}{\partial y} = 0 \quad \rightarrow \quad S_{yy} = c_2 = c_2(x,z) = \text{constant}$$

$$\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial S_{zz}}{\partial z} + b_z = 0 \quad \rightarrow \quad \frac{\partial S_{zz}}{\partial z} = 0 \quad \rightarrow \quad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

Recall, all body forces are neglected.

Now, in other to complete find the unknowns we need to apply stress boundary conditions:

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

(a) On the surface defined by $x = a$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = -p_o\,\underline{\hat{\mathbf{i}}}$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{Bmatrix} -p_o \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix}$$

Thus,

$$S_{xx}(a,y,z) = -p_o$$

and from the stress-strain relationship,

$$S_{xy}(a,y,z) = S_{xz}(a,y,z) = 0$$

(b) On the surface defined by $x = -a$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = p_o\,\underline{\hat{\mathbf{i}}}$

$$\left\{\begin{array}{c} T_x \\ T_y \\ T_z \end{array}\right\} = \left\{\begin{array}{c} p_o \\ 0 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array}\right] \left\{\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right\} = \left\{\begin{array}{c} -S_{xx} \\ -S_{xy} \\ -S_{xz} \end{array}\right\}$$

Thus,

$$S_{xx}(-a, y, z) = -p_o$$

and from the stress-strain relationship,

$$S_{xy}(-a, y, z) = S_{xz}(-a, y, z) = 0$$

Yields the same results as in (a).

(c) On the surface defined by $z = b$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\left\{\begin{array}{c} T_x \\ T_y \\ T_z \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array}\right] \left\{\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right\} = \left\{\begin{array}{c} S_{xz} \\ S_{yz} \\ S_{zz} \end{array}\right\}$$

Thus,

$$S_{zz}(x, y, b) = 0$$

and from the stress-strain relationship,

$$S_{yz}(x, y, b) = S_{xz}(x, y, b) = 0$$

(d) On the surface defined by $z = -b$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\left\{\begin{array}{c} T_x \\ T_y \\ T_z \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array}\right] \left\{\begin{array}{c} 0 \\ 0 \\ -1 \end{array}\right\} = \left\{\begin{array}{c} -S_{xz} \\ -S_{yz} \\ -S_{zz} \end{array}\right\}$$

Thus,

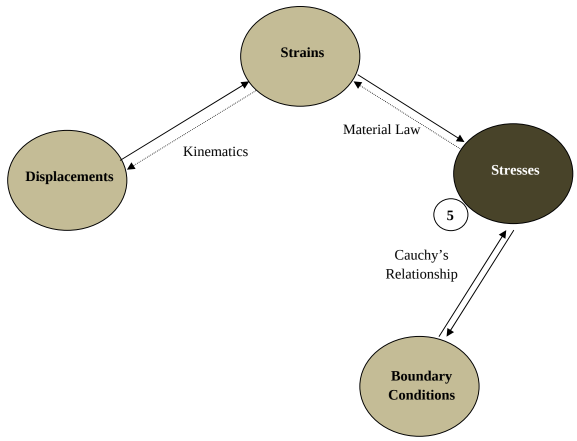
$$S_{zz}(x, y, -b) = 0$$

and from the stress-strain relationship,

$$S_{yz}(x, y, -b) = S_{xz}(x, y, -b) = 0$$

Yields the same results as in (c).

2.14e) Stress field.



We know from equilibrium conditions:

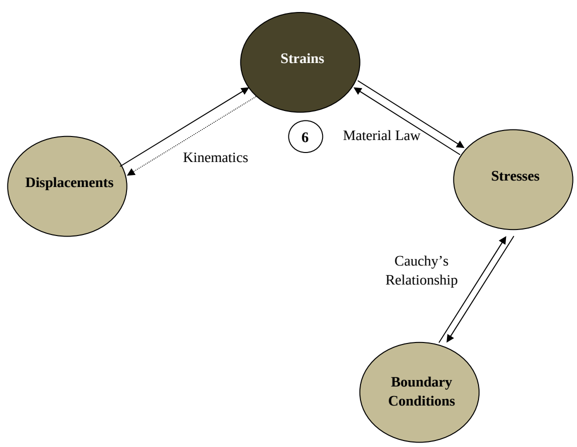
$$S_{xx} = c_1 = c_1(y,z) \qquad S_{yy} = c_2 = c_2(x,z) = \text{constant} \qquad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

From stress boundary conditions,

$$S_{xx}(a,y,z) = -p_o \qquad \text{and} \qquad c_1 = -p_o$$

$$S_{zz}(x,y,b) = 0 \qquad \text{and} \qquad c_3 = 0$$

2.14f) Strain field.



In order to obtain the constant c_2, let us use the stress-strain relationships. Let us begin with the third equation:

$$S_{zz} = 0 = \frac{\nu E}{(1+\nu)(1-2\nu)}\, e_{xx} + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\, e_{zz} \quad \rightarrow \quad e_{zz} = -\frac{\nu}{(1-\nu)}\, e_{xx}$$

Let us proceed to use the first equation:

$$S_{xx} = -p_o = \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\,e_{xx} + \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\,e_{zz}$$

$$= \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\,e_{xx} + \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\left\{-\frac{\nu}{(1-\nu)}\,e_{xx}\right\}$$

From the above equation we can obtain the value of e_{xx}:

$$e_{xx} = -\frac{(1-\nu^2)}{E}\,p_o$$

Thus e_{zz} is then

$$e_{zz} = -\frac{\nu}{(1-\nu)}\,e_{xx} = -\frac{\nu}{(1-\nu)}\left\{-\frac{(1-\nu^2)}{E}\,p_o\right\} = \frac{\nu\,(1+\nu)}{E}\,p_o$$

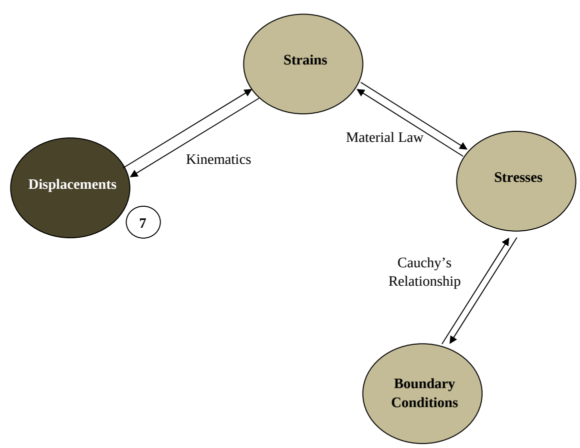
From the second stress-strain equation, we can solve for S_{yy}:

$$S_{yy} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\,(e_{xx}+e_{zz}) = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\left(-\frac{(1-\nu^2)}{E}\,p_o + \frac{\nu\,(1+\nu)}{E}\,p_o\right) = -\nu\,p_o$$

Thus,

$$c_2 = -\nu\,p_o$$

2.14g) Displacement field.



With this we have completed the stress and strain fields but the displacements are know remaining. Although we know that $V=0$, U and W are remaining. Let us use the strain-displacement equations for this purpose:

$$e_{xx} = \frac{\partial U}{\partial x} \quad\rightarrow\quad \frac{\partial U}{\partial x} = -\frac{(1-\nu^2)}{E}\,p_o \quad\rightarrow\quad U = -\frac{(1-\nu^2)}{E}\,p_o\,x + f_1(y,z)$$

Also,

$$g_4 = \frac{\partial U}{\partial y} = 0 \qquad \rightarrow \qquad U = f_2(x, z)$$

$$g_7 = \frac{\partial U}{\partial z} = 0 \qquad \rightarrow \qquad U = f_3(x, y)$$

Which implies that $f_1(y, z) = f_2(x, z) = f_3(x, y) = k_1 = \text{constant}$:

$$U(x, y, z) = -\frac{(1-\nu^2)}{E}\, p_o\, x + k_1$$

Let us assume that $U(0, 0, 0) = 0$ and thus $k_1 = 0$. Thus

$$U(x, y, z) = u(x) = -\frac{(1-\nu^2)}{E}\, p_o\, x$$

Now, we proceed to find W:

$$e_{zz} = \frac{\partial W}{\partial z} \quad \rightarrow \quad \frac{\partial W}{\partial z} = \frac{\nu\,(1+\nu)}{E}\, p_o \quad \rightarrow \quad W = \frac{\nu\,(1+\nu)}{E}\, p_o\, z + h_1(x, y)$$

Also,

$$g_3 = \frac{\partial W}{\partial x} = 0 \qquad \rightarrow \qquad W = h_2(x, z)$$

$$g_6 = \frac{\partial W}{\partial y} = 0 \qquad \rightarrow \qquad W = h_3(x, y)$$

Which implies that $h_1(y, z) = h_2(x, z) = h_3(x, y) = k_2 = \text{constant}$:

$$W(x, y, z) = \frac{\nu\,(1+\nu)}{E}\, p_o\, z + k_2$$

Let us assume that $W(0, 0, 0) = 0$ and thus $k_2 = 0$. Thus

$$W(x, y, z) = w(z) = \frac{\nu\,(1+\nu)}{E}\, p_o\, z$$

2.14h) Elastic field. In summary, the state of stress is

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} = \begin{bmatrix} -p_o & 0 & 0 \\ 0 & -\nu\, p_o & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -200 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{psi}$$

The state of strain is

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} -\dfrac{(1-\nu^2)}{E}\,p_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{\nu\,(1+\nu)}{E}\,p_o \end{bmatrix} = \begin{bmatrix} -5.97567 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2.561 \end{bmatrix} \mu$$

The displacement field is

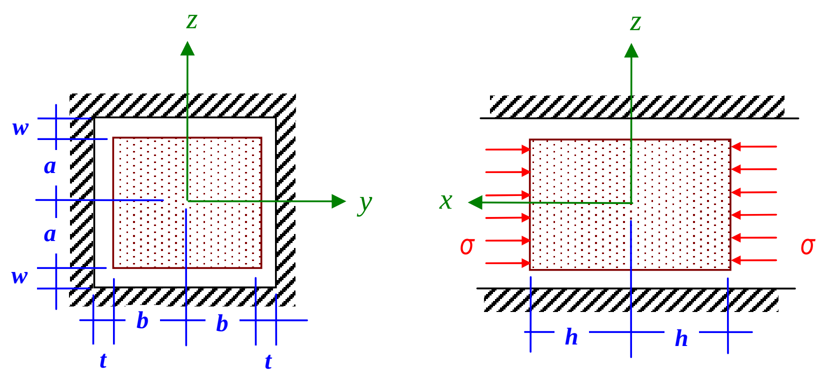
$$\underline{\mathbf{R}} = \begin{Bmatrix} U(x,y,z) \\ V(x,y,z) \\ W(x,y,z) \end{Bmatrix} = \begin{Bmatrix} -\dfrac{(1-\nu^2)}{E}\,p_o\,x \\ 0 \\ \dfrac{\nu\,(1+\nu)}{E}\,p_o\,z \end{Bmatrix} = \begin{Bmatrix} -5.975667 \times 10^{-6}\,x \\ 0 \\ 2.561 \times 10^{-6}\,z \end{Bmatrix}$$

End Example □

Example 2.15.

Application 2: Isothermal Isotropic Material

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed pressure σ in the x-direction. The block is bound in the y- and z-direction by rigid walls at $y = b + t$, $y = -b - t$, $z = a + w$, $z = -a - w$, but free to expand/contract in the x-plane.



FRONT VIEW
(Seen from x-axis)

SIDE VIEW
(Seen from y-axis)

Assume the material is isotropic with the following mechanical properties:

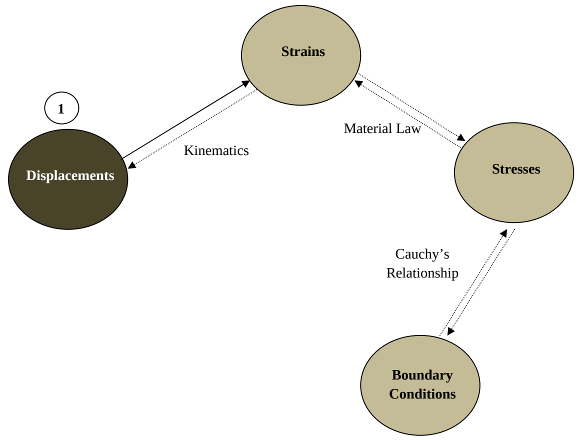
$$E = 210 \times 10^9 \quad \text{Pa} \qquad \nu = 0.30$$

The geometric properties are:

$$2\,a = 2\,b = 2\,h = 0.127 \text{ m} \qquad t = w = 0.0381 \text{ mm}$$

(2.15a) What is the needed pressure σ_1 to make contact between the solid cube structure and the rigid walls? Take $\sigma = \sigma_1$.

(1) Assumed displacement.



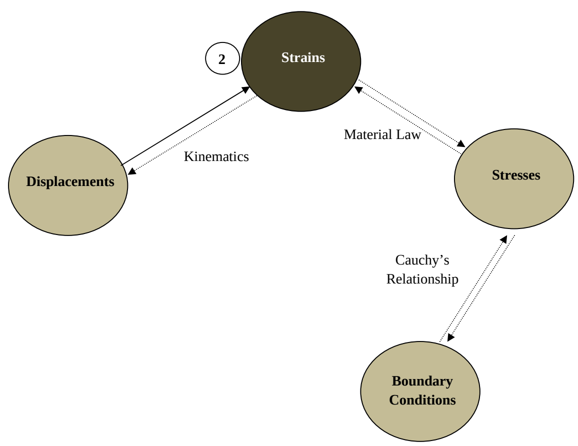
From the geometry of the problem it is assumed that at all points in the body the displacement field (displacement boundary conditions) is:

$$U_1(x, y, z) = u_1(x)$$

$$V_1(x, y, z) = v_1(y)$$

$$W_1(x, y, z) = w_1(z)$$

(2) Strains.



The displacement gradients are then found as:

$$g_1 = \frac{\partial U_1}{\partial x} = \frac{\partial u_1}{\partial x} \qquad g_4 = \frac{\partial U_1}{\partial y} = 0 \qquad g_7 = \frac{\partial U_1}{\partial z} = 0$$

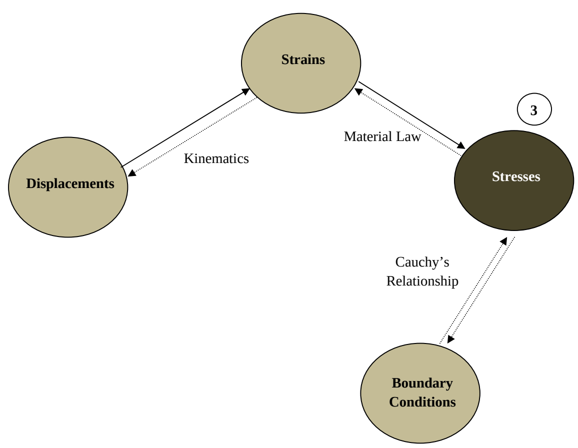
$$g_2 = \frac{\partial V_1}{\partial x} = 0 \qquad g_5 = \frac{\partial V_1}{\partial y} = \frac{\partial v_1}{\partial y} \qquad g_8 = \frac{\partial V_1}{\partial z} = 0$$

$$g_3 = \frac{\partial W_1}{\partial x} = 0 \qquad g_6 = \frac{\partial W_1}{\partial y} = 0 \qquad g_9 = \frac{\partial W_1}{\partial z} = \frac{\partial w_1}{\partial z}$$

Thus the resulting strain-displacement relationship is obtained using the Lagrange-Green equations:

$$\epsilon_1 = e_{xx} = g_1 = \frac{\partial u_1}{\partial x}$$
$$\epsilon_2 = e_{\text{yy}} = g_5 = \frac{\partial v_1}{\partial y}$$
$$\epsilon_3 = e_{\text{zz}} = g_9 = \frac{\partial w_1}{\partial z}$$
$$\epsilon_4 = 2\, e_{\text{yz}} = g_6 + g_8 = 0$$
$$\epsilon_5 = 2\, e_{\text{xz}} = g_3 + g_7 = 0$$
$$\epsilon_6 = 2\, e_{\text{xy}} = g_2 + g_4 = 0$$

(3) Stresses.



Now the stress-strain relationship for isotropic Hookean strain is obtained using the inverted Hooke's law:

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

$$= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus,

$$S_1 = S_{xx} = \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\, e_{xx} + \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, (e_{yy} + e_{zz})$$

$$S_2 = S_{yy} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, (e_{xx} + e_{zz}) + \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\, e_{yy}$$

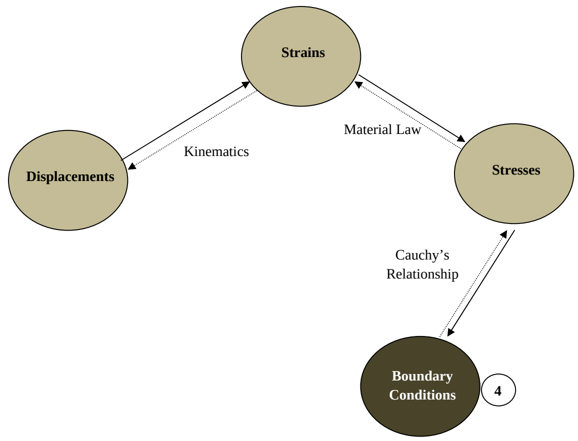
$$S_3 = S_{zz} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\, (e_{xx} + e_{yy}) + \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\, e_{zz}$$

$$S_4 = S_{yz} = 0$$

$$S_5 = S_{xz} = 0$$

$$S_6 = S_{xy} = 0$$

(4) Equilibrium Equations and Boundary Conditions.



Now, substituting the stress components into the three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial S_{zx}}{\partial z} + b_x = 0 \quad \rightarrow \quad \frac{\partial S_{xx}}{\partial x} = 0 \quad \rightarrow \quad S_{xx} = c_1 = c_1(y,z) = \text{constant}$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{zy}}{\partial z} + b_y = 0 \quad \rightarrow \quad \frac{\partial S_{yy}}{\partial y} = 0 \quad \rightarrow \quad S_{yy} = c_2 = c_2(x,z) = \text{constant}$$

$$\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial S_{zz}}{\partial z} + b_z = 0 \quad \rightarrow \quad \frac{\partial S_{zz}}{\partial z} = 0 \quad \rightarrow \quad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

Recall, all body forces are neglected.
Now in other to complete find the unknowns we need to apply stress boundary conditions:

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} n_x \\ n_y \\ n_z \end{array} \right\}$$

(a) On the surface defined by $x = h$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = -\sigma_1 \, \underline{\hat{\mathbf{i}}}$

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left\{ \begin{array}{c} -\sigma_1 \\ 0 \\ 0 \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} S_{xx} \\ S_{xy} \\ S_{xz} \end{array} \right\}$$

Thus,

$$S_{xx}(h,y,z) = -\sigma_1$$

and from the stress-strain relationship,

$$S_{xy}(h,y,z) = S_{xz}(h,y,z) = 0$$

(b) On the surface defined by $x = -h$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = \sigma_1\, \underline{\hat{\mathbf{i}}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -S_{\rm xx} \\ -S_{\rm xy} \\ -S_{\rm xz} \end{Bmatrix}$$

Thus,

$$S_{\rm xx}(-h, y, z) = -\sigma_1$$

and from the stress-strain relationship,

$$S_{\rm xy}(-h, y, z) = S_{\rm xz}(-h, y, z) = 0$$

Yields the same results as in (a).

(c) On the surface defined by $z = a$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} S_{\rm xz} \\ S_{\rm yz} \\ S_{\rm zz} \end{Bmatrix}$$

Thus,

$$S_{\rm zz}(x, y, a) = 0$$

and from the stress-strain relationship,

$$S_{\rm yz}(x, y, a) = S_{\rm xz}(x, y, a) = 0$$

(d) On the surface defined by $z = -a$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -S_{\rm xz} \\ -S_{\rm yz} \\ -S_{\rm zz} \end{Bmatrix}$$

Thus,

$$S_{\rm zz}(x, y, -a) = 0$$

and from the stress-strain relationship,

$$S_{\rm yz}(x, y, -a) = S_{\rm xz}(x, y, -a) = 0$$

Yields the same results as in (c).

(e) On the surface defined by $y = b$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{j}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S_{\rm xy} \\ S_{\rm yy} \\ S_{\rm yz} \end{Bmatrix}$$

Thus,

$$S_{\rm yy}(x, b, z) = 0$$

and from the stress-strain relationship,

$$S_{xy}(x,b,z) = S_{yz}(x,b,z) = 0$$

(f) On the surface defined by $y = -b$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{j}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -S_{xy} \\ -S_{yy} \\ -S_{yz} \end{Bmatrix}$$

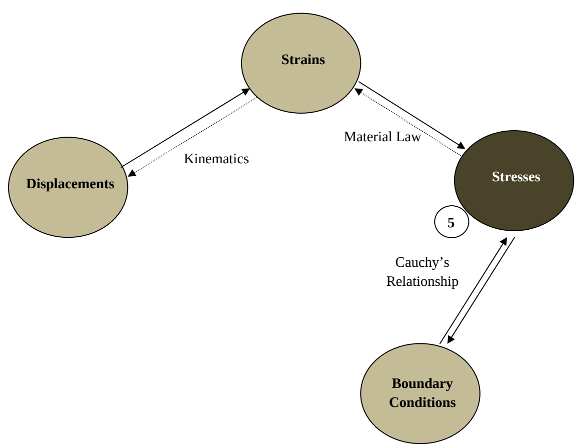
Thus,

$$S_{yy}(x,-b,z) = 0$$

and from the stress-strain relationship,

$$S_{xy}(x,-b,z) = S_{yz}(x,-b,z) = 0$$

(5) Stress field.



From equilibrium conditions:

$$S_{xx} = c_1 = c_1(y,z) = \text{constant} \qquad S_{yy} = c_2 = c_2(x,z) = \text{constant} \qquad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

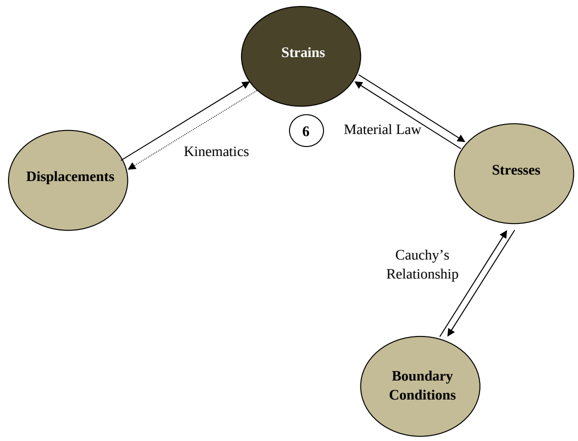
From stress boundary conditions,

$$S_{xx}(h,y,z) = -\sigma_1 \qquad \text{and} \qquad c_1 = -\sigma_1$$

$$S_{yy}(x,b,z) = 0 \qquad \text{and} \qquad c_2 = 0$$

$$S_{zz}(x,y,a) = 0 \qquad \text{and} \qquad c_3 = 0$$

(6) Strain field.



Now, let us use the stress-strain relationships (using the Hooke's Law):

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} -\sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiplying the above we get

$$\begin{Bmatrix} e_{\text{xx}} \\ e_{\text{yy}} \\ e_{\text{zz}} \\ \gamma_{\text{yz}} \\ \gamma_{\text{xz}} \\ \gamma_{\text{xy}} \end{Bmatrix} = \frac{\sigma_1}{E}\begin{Bmatrix} -1 \\ \nu \\ \nu \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus, while there is no contact between the solid structure and the rigid walls, the state of stress is

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{\text{xx}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that since the above is a principal state of stress, the result will be a principal state of strain. Thus the associated state of strain is

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & 0 & 0 \\ 0 & e_{yy} & 0 \\ 0 & 0 & e_{zz} \end{bmatrix} = \frac{\sigma_1}{E} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

We can see that $e_{yy} = e_{zz}$. The strain needed for the solid cube to make with the rigid wall is

$$e_{yy} = \frac{\text{final expansion in } y - \text{initial expansion in } y}{\text{initial expansion in } y} = \frac{(2\,b + 2\,t) - (2\,b)}{2\,b} = \frac{2\,(0.0381 \times 10^{-3})}{0.127} = 0.0006$$

Thus

$$e_{yy} = e_{zz} = 0.0006$$

The needed pressure can be calculated from the second or third equation in the Hooke's Law

$$e_{yy} = e_{zz} = \frac{\sigma_1}{E}\,\nu \qquad \rightarrow \qquad \sigma_1 = \frac{e_{yy}\,E}{\nu} = 420 \quad \text{MPa}$$

Thus for contact to occur between the cube and the rigid walls, a compression of 420 MPa need to be applied in the x-direction. The normal strain in the x-direction is

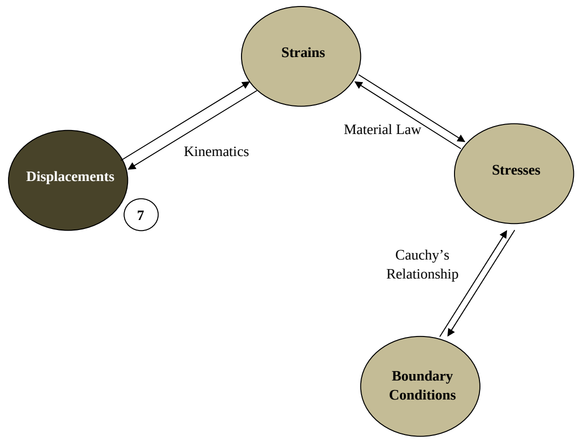
$$e_{xx} = \frac{S_{xx}}{E} = -0.002$$

Thus the stress and strain tensors are

$$\underline{\mathbf{S}}\Big|_{\text{at rigid wall}} = \begin{bmatrix} -420 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{MPa}$$

$$\underline{\mathbf{e}}\Big|_{\text{at rigid wall}} = \begin{bmatrix} -2000 & 0 & 0 \\ 0 & 600 & 0 \\ 0 & 0 & 600 \end{bmatrix} \mu$$

(7) Displacement field.



With this we have completed the stress and strain fields but the displacements are know remaining. Let us use the strain-displacement equations for this purpose:

$$e_{xx} = \frac{\partial U_1}{\partial x} \quad \longrightarrow \quad \frac{\partial U_1}{\partial x} = -0.002 \quad \longrightarrow \quad U_1(x) = -0.002\,x + f_1(y,z)$$

Also,

$$g_4 = \frac{\partial U_1}{\partial y} = 0 \qquad \longrightarrow \quad U_1 = f_2(x,z)$$

$$g_7 = \frac{\partial U_1}{\partial z} = 0 \qquad \longrightarrow \quad U_1 = f_3(x,y)$$

Which implies that $f_1(y,z) = f_2(x,z) = f_3(x,y) = k_1 = \text{constant}$:

$$U_1(x,y,z) = -0.002x + k_1$$

Let us assume that $U_1(0,0,0) = 0$ and thus $k_1 = 0$. Thus

$$U_1(x,y,z) = u_1(x) = -0.002\,x$$

Now, we proceed to find V_1:

$$e_{yy} = \frac{\partial V_1}{\partial y} \quad \longrightarrow \quad \frac{\partial V_1}{\partial y} = 0.0006 \quad \longrightarrow \quad V_1 = 0.0006\,y + h_1(x,z)$$

Also,

$$g_2 = \frac{\partial V_1}{\partial x} = 0 \qquad \longrightarrow \quad V_1 = h_2(x,z)$$

$$g_8 = \frac{\partial V_1}{\partial z} = 0 \qquad \longrightarrow \quad V_1 = h_3(x,y)$$

Which implies that $h_1(y,z) = h_2(x,z) = h_3(x,y) = k_2 = \text{constant}$:

$$V_1(x,y,z) = 0.0006\,y + k_2$$

We know that $V_1(0, b+t, 0) = V_1(0, 0.063538, 0) = 0$ and thus $k_2 = -0.00006096$.

Thus

$$V_1(x,y,z) = v_1(y)(x) = 0.0006\,y - 0.00006096$$

Now, we proceed to find W_1:

$$e_{zz} = \frac{\partial W_1}{\partial z} \quad \rightarrow \quad \frac{\partial W_1}{\partial z} = 0.0006 \quad \rightarrow \quad W_1 = 0.0006\,z + s_1(x,y)$$

Also,

$$g_3 = \frac{\partial W_1}{\partial x} = 0 \qquad \rightarrow \quad W_1 = s_2(x,z)$$

$$g_6 = \frac{\partial W_1}{\partial y} = 0 \qquad \rightarrow \quad W_1 = s_3(x,y)$$

Which implies that $s_1(y,z) = s_2(x,z) = s_3(x,y) = k_3 =$ constant:

$$W_1(x,y,z) = 0.0006\,z + k_3$$

Let us assume that $W_1(0,0,a{+}w) = W_1(0,0,0.0635) = 0$ and thus $k_3 = -0.00006096$. Thus

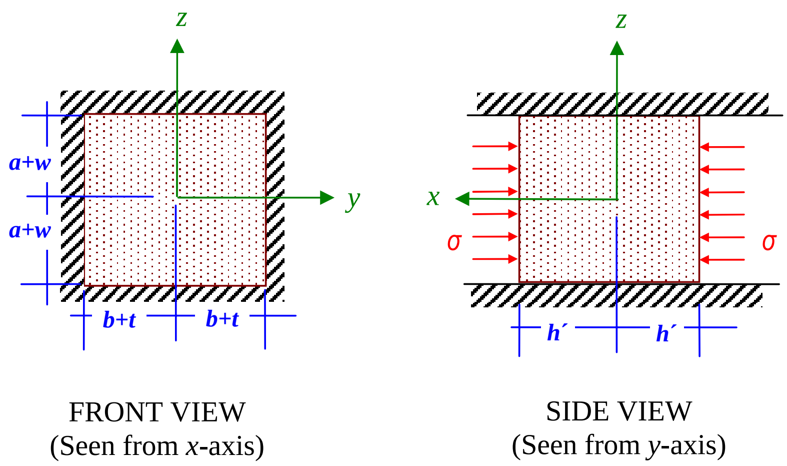
$$W_1(x,y,z) = w_1(z) = 0.0006\,z - 0.00006096$$

Thus, the displacement field is

$$\underline{\mathbf{R}}\Big|_{\text{at rigid wall}} = \left\{ \begin{array}{c} U_1(x,y,z) \\ \\ V_1(x,y,z) \\ \\ W_1(x,y,z) \end{array} \right\} = \left\{ \begin{array}{c} -0.002000\,x \\ \\ 0.000600\,y - 0.00006096 \\ \\ 0.000600\,z - 0.00006096 \end{array} \right\}$$

(2.15b) If $\sigma_2 = 600$ MPa, determine the isothermal elastic field. Take $\sigma = \sigma_2$.

A total pressure of $\sigma_1 = 420$ MPa are necessary to bring the block to have contact with the rigid walls. Once the block reaches the rigid walls, the problem changes. As the solid block reaches the rigid walls, the geometry is:



In order to find the value of h', let us use the definition of the strain in the x-direction. We know that just when the block hits the walls:

$$e_{xx}\Big|_{\text{at rigid walls}} = \frac{\text{final expansion in } x - \text{initial expansion in } x}{\text{initial expansion in } x} = \frac{2\,h' - 2\,h}{2\,h} = -0.002$$

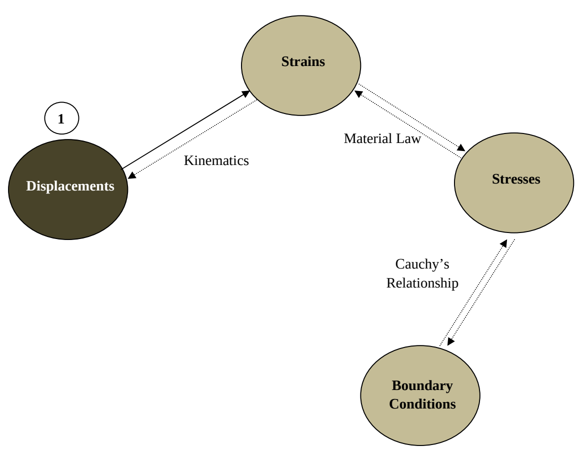
Thus

$$h' = (-0.002)(h) + h = 0.063373$$

Since $\sigma_2 > \sigma_1$, the block has reached the rigid walls, and the excessive pressure the block is experiencing will be:

$$\Delta\sigma = (600 - 420) = 180 \quad \text{MPa}$$

(1) Assumed Displacement.



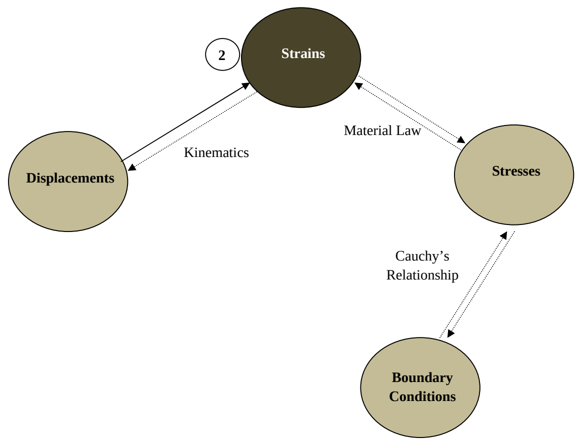
Also, there will be no further deformation in the y and z direction (because of the rigid walls) and thus the displacement field is

$$U_2(x, y, z) = u_2(x)$$

$$V_2(x, y, z) = 0$$

$$W_2(x, y, z) = 0$$

(2) Strains.



The displacement gradients are then found as:

$$g_1 = \frac{\partial U_2}{\partial x} = \frac{\partial u_2}{\partial x} \qquad g_4 = \frac{\partial U_2}{\partial y} = 0 \qquad g_7 = \frac{\partial U_2}{\partial z} = 0$$

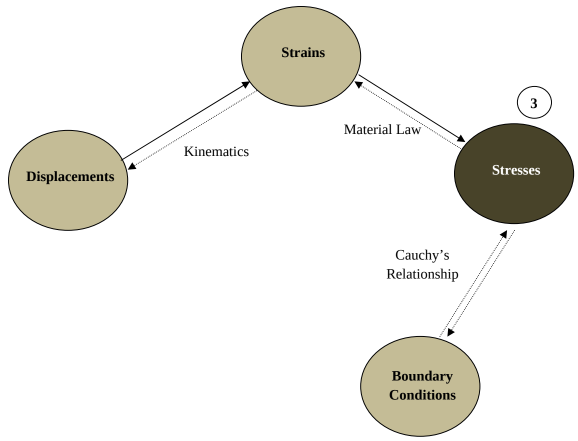
$$g_2 = \frac{\partial V_2}{\partial x} = 0 \qquad g_5 = \frac{\partial V_2}{\partial y} = 0 \qquad g_8 = \frac{\partial V_2}{\partial z} = 0$$

$$g_3 = \frac{\partial W_2}{\partial x} = 0 \qquad g_6 = \frac{\partial W_2}{\partial y} = 0 \qquad g_9 = \frac{\partial W_2}{\partial z} = 0$$

Thus the resulting strain-displacement relationship is obtained using the Lagrange-Green equations:

$$\begin{aligned}
\epsilon_1 &= e_{xx} = g_1 = \frac{\partial u_2}{\partial x} \\
\epsilon_2 &= e_{yy} = g_5 = 0 \\
\epsilon_3 &= e_{zz} = g_9 = 0 \\
\epsilon_4 &= 2\, e_{yz} = g_6 + g_8 = 0 \\
\epsilon_5 &= 2\, e_{xz} = g_3 + g_7 = 0 \\
\epsilon_6 &= 2\, e_{xy} = g_2 + g_4 = 0
\end{aligned}$$

(3) Stresses.



Now the stress-strain relationship for isotropic Hookean strain is obtained using the inverted Hooke's law:

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

$$= \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\,\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus,

$$S_1 = S_{\text{xx}} = \frac{(1-\nu)\,E}{(1+\nu)(1-2\,\nu)}\,e_{\text{xx}}$$

$$S_2 = S_{\text{yy}} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\,e_{\text{xx}}$$

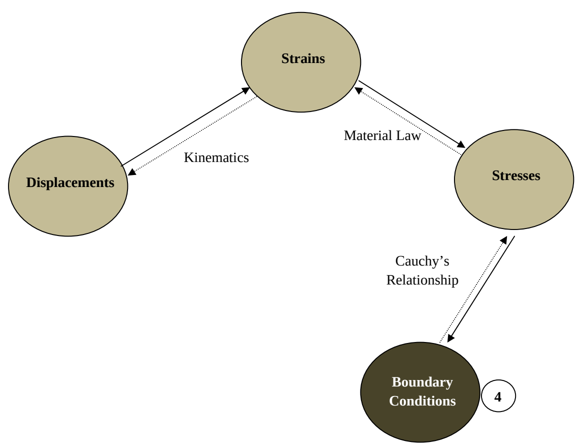
$$S_3 = S_{\text{zz}} = \frac{\nu\,E}{(1+\nu)(1-2\,\nu)}\,e_{\text{xx}}$$

$$S_4 = S_{\text{yz}} = 0$$

$$S_5 = S_{\text{xz}} = 0$$

$$S_6 = S_{\text{xy}} = 0$$

(4) Equilibrium Equations and Boundary Conditions.



Now, substituting the stress components into the three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial S_{\text{xx}}}{\partial x} + \frac{\partial S_{\text{yx}}}{\partial y} + \frac{\partial S_{\text{zx}}}{\partial z} + b_x = 0 \quad \rightarrow \quad \frac{\partial S_{\text{xx}}}{\partial x} = 0 \quad \rightarrow \quad S_{\text{xx}} = c_1 = c_1(y,z) = \text{constant}$$

$$\frac{\partial S_{\text{xy}}}{\partial x} + \frac{\partial S_{\text{yy}}}{\partial y} + \frac{\partial S_{\text{zy}}}{\partial z} + b_y = 0 \quad \rightarrow \quad \frac{\partial S_{\text{yy}}}{\partial y} = 0 \quad \rightarrow \quad S_{\text{yy}} = c_2 = c_2(x,z) = \text{constant}$$

$$\frac{\partial S_{\text{xz}}}{\partial x} + \frac{\partial S_{\text{yz}}}{\partial y} + \frac{\partial S_{\text{zz}}}{\partial z} + b_z = 0 \quad \rightarrow \quad \frac{\partial S_{\text{zz}}}{\partial z} = 0 \quad \rightarrow \quad S_{\text{zz}} = c_3 = c_3(x,y) = \text{constant}$$

Recall, all body forces are neglected.
Now in other to complete find the unknowns we need to apply stress boundary

conditions:

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

The block is only free to move in the x direction, thus:

(a) On the surface defined by $x = h'$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = -\Delta\sigma\,\underline{\hat{\mathbf{i}}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} -\Delta\sigma \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S_{\rm xx} \\ S_{\rm xy} \\ S_{\rm xz} \end{Bmatrix}$$

Thus,

$$S_{\rm xx}(h', y, z) = -\Delta\sigma$$

and from the stress-strain relationship,

$$S_{\rm xy}(h', y, z) = S_{\rm xz}(h', y, z) = 0$$

(b) On the surface defined by $x = -h'$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = \Delta\sigma\,\underline{\hat{\mathbf{i}}}$

$$\begin{Bmatrix} T_{\rm x} \\ T_{\rm y} \\ T_{\rm z} \end{Bmatrix} = \begin{Bmatrix} \Delta\sigma \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{\rm xx} & S_{\rm xy} & S_{\rm xz} \\ S_{\rm xy} & S_{\rm yy} & S_{\rm yz} \\ S_{\rm xz} & S_{\rm yz} & S_{\rm zz} \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -S_{\rm xx} \\ -S_{\rm xy} \\ -S_{\rm xz} \end{Bmatrix}$$

Thus,

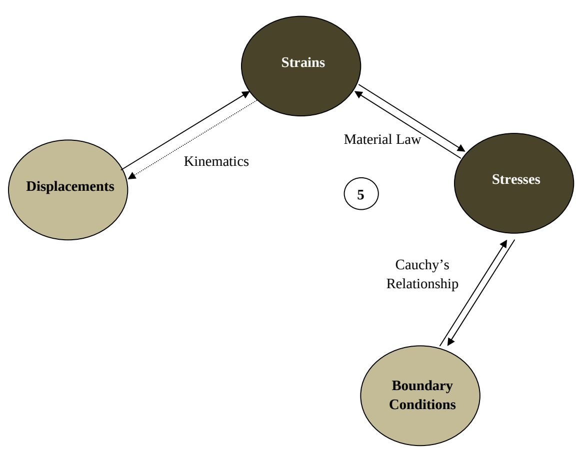
$$S_{\rm xx}(-h', y, z) = -\Delta\sigma$$

and from the stress-strain relationship,

$$S_{\rm xy}(-h', y, z) = S_{\rm xz}(-h', y, z) = 0$$

Yields the same results as in (a).

(5) Stress and strain fields.



From equilibrium conditions:

$$S_{xx} = c_1 = c_1(y,z) = \text{constant} \qquad S_{yy} = c_2 = c_2(x,z) = \text{constant} \qquad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

From stress boundary conditions,

$$S_{xx}(h,y,z) = -\Delta\sigma \qquad \text{and} \qquad c_1 = -\Delta\sigma$$

Now, using Hooke's Law:

$$S_{xx} = -\Delta\sigma = \frac{E(1-\nu)}{(1+\nu)(1-2\,\nu)}\, e_{xx} = -180 \quad \text{MPa} \quad \rightarrow \quad e_{xx} = -.000635$$

Also

$$S_{yy} = \frac{E\,\nu}{(1+\nu)(1-2\,\nu)} e_{xx} = -77 \quad \text{MPa}$$

$$S_{zz} = \frac{E\,\nu}{(1+\nu)(1-2\,\nu)} e_{xx} = -77 \quad \text{MPa}$$

Also, note that the total stress and strain are

$$\underline{\mathbf{S}} = \underline{\mathbf{S}}\Big|_{\text{before rigid wall}} + \underline{\mathbf{S}}\Big|_{\text{after rigid wall}}$$

$$\underline{\mathbf{e}} = \underline{\mathbf{e}}\Big|_{\text{before rigid wall}} + \underline{\mathbf{e}}\Big|_{\text{after rigid wall}}$$

and the total stress and strain acting in the x-direction are:

$$S_{xx}\Big|_{\text{total}} = S_{xx}\Big|_{\text{before contact}} + S_{xx}\Big|_{\text{after contact}} = -420 - 180 = -600 \text{ MPa}$$

$$e_{xx}\Big|_{\text{total}} = e_{xx}\Big|_{\text{before contact}} + e_{xx}\Big|_{\text{after contact}} = -0.00200 - 0.000635 = -0.002635$$

When $\sigma = \sigma_2 = 600$ MPa, the stress and strain fields are

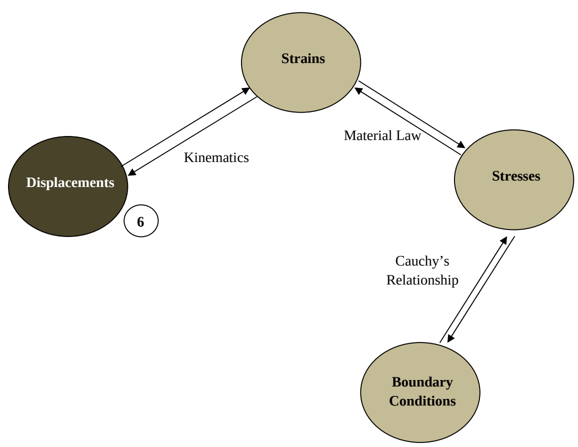
$$\underline{\mathbf{S}} = \underline{\mathbf{S}}\Big|_{\text{before rigid wall}} + \underline{\mathbf{S}}\Big|_{\text{after rigid wall}} = \begin{bmatrix} -420 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -180 & 0 & 0 \\ 0 & -77 & 0 \\ 0 & 0 & -77 \end{bmatrix} \text{ MPa}$$

$$= \begin{bmatrix} -600 & 0 & 0 \\ 0 & -77 & 0 \\ 0 & 0 & -77 \end{bmatrix} \text{ MPa}$$

$$\underline{\mathbf{e}} = \underline{\mathbf{e}}\Big|_{\text{before rigid wall}} + \underline{\mathbf{e}}\Big|_{\text{after rigid wall}} = \begin{bmatrix} -0.002 & 0 & 0 \\ 0 & 0.0006 & 0 \\ 0 & 0 & 0.0006 \end{bmatrix} + \begin{bmatrix} -0.000635 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2637 & 0 & 0 \\ 0 & 600 & 0 \\ 0 & 0 & 600 \end{bmatrix} \mu$$

(6) Displacement field.



With this we have completed the stress and strain fields but the displacements are know remaining. Let us solve this after the block has reached the rigid wall and thus

$$\underline{\mathbf{R}}\Big|_{\text{total}} = \underline{\mathbf{R}}\Big|_{\text{at rigid wall}} + \underline{\mathbf{R}}\Big|_{\text{after rigid wall}}$$

We know that $V_2 = W_2 = 0$. Only, U_2 remains. Let us use the strain-displacement equations for this purpose:

$$e_{\text{xx}} = \frac{\partial U_2}{\partial x} \quad \rightarrow \quad \frac{\partial U_2}{\partial x} = -0.000635 \quad \rightarrow \quad U_2(x) = -0.000635\, x + f_1(y,z)$$

Also,

$$g_4 = \frac{\partial U_2}{\partial y} = 0 \qquad \rightarrow \qquad U_2 = f_2(x, z)$$

$$g_7 = \frac{\partial U_2}{\partial z} = 0 \qquad \rightarrow \qquad U_2 = f_3(x, y)$$

Which implies that $f_1(y, z) = f_2(x, z) = f_3(x, y) = k_1 =$ constant:

$$U_2(x, y, z) = -0.000635\, x + k_1$$

We know that the displacement when the block just hits the y-z walls must be same to displacement when the expansion begins:

$$U_1(0.063373, 0, 0) = U_2(0.063373, 0, 0)$$

However, let us make the new displacement independent from U_1, in other words, let $U_2(0.063373, 0, 0) = 0$. As a consequence,

$$k_1 = 4.02419 \times 10^{-5}$$

Thus, the displacement field is

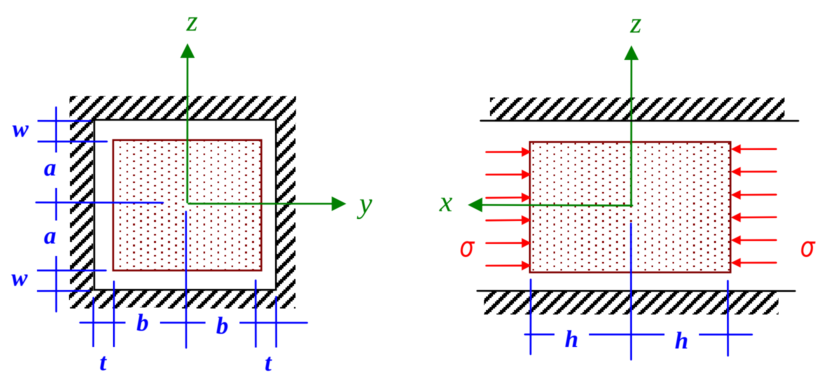
$$\underline{\mathbf{R}}\Big|_{\text{after rigid wall}} = \begin{Bmatrix} U_2(x, y, z) \\ V_2(x, y, z) \\ W_2(x, y, z) \end{Bmatrix} = \begin{Bmatrix} -0.000635\, x + 4.02419 \times 10^{-5} \\ 0.00 \\ 0.00 \end{Bmatrix}$$

End Example □

Example 2.16.

Application 3: Isothermal Orthotropic Material

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed pressure σ in the x-direction. The block is bound in the y- and z-direction by rigid walls at $y = b + t$, $y = -b - t$, $z = a + w$, $z = -a - w$, but free to expand/contract in the x-plane.



FRONT VIEW (Seen from x-axis)

SIDE VIEW (Seen from y-axis)

Assume the orthotropic material is Glass-Epoxy (Scothply 1002) . The geometric properties are:

$$2\,a = 2\,b = 2\,h = 0.127 \text{ m} \qquad t = w = 0.0381 \text{ mm}$$

What is the needed pressure, σ, to make contact between the Hookean solid cube structure and all the rigid walls?

Before we proceed let us obtain the mechanical properties for Glass-Epoxy; hence, from material tables:

$$E_{xx} = 5.6 \times 10^6 \text{ psi} \qquad E_{yy} = 1.2 \times 10^6 \text{ psi} \qquad E_{zz} = 1.3 \times 10^6 \text{ psi}$$
$$G_{xy} = 0.60 \times 10^6 \text{ psi} \qquad G_{xz} = 0.60 \times 10^6 \text{ psi} \qquad G_{yz} = 0.50 \times 10^6 \text{ psi}$$
$$\nu_{xy} = 0.26 \qquad \nu_{xz} = 0.25 \qquad \nu_{yz} = 0.34$$

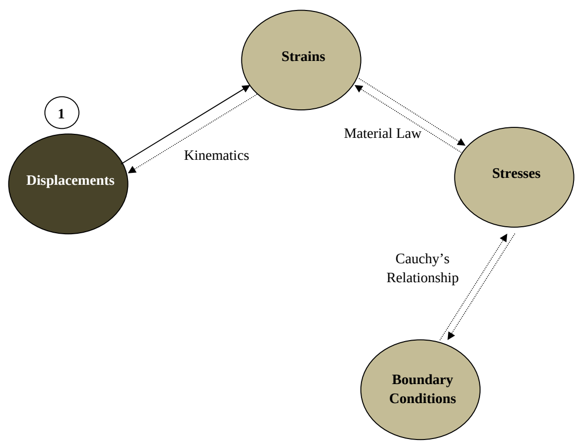
Converting to SI units[9]:

$$E_{\rm xx} = 38.612 \text{ GPa} \qquad E_{\rm yy} = 8.274 \text{ GPa} \qquad E_{\rm zz} = 8.9635 \text{ GPa}$$
$$G_{\rm xy} = 4.137 \text{ GPa} \qquad G_{\rm xz} = 4.137 \text{ GPa} \qquad G_{\rm yz} = 3.4475 \text{ GPa}$$

The remaining mechanical properties are obtained from Eq. (2.174):

$$\nu_{\rm yx} = \frac{\nu_{\rm xy}}{E_{\rm xx}} E_{\rm yy} = 0.0557143 \qquad \nu_{\rm zx} = \frac{\nu_{\rm xz}}{E_{\rm xx}} E_{\rm zz} = 0.0580357 \qquad \nu_{\rm zy} = \frac{\nu_{\rm yz}}{E_{\rm yy}} E_{\rm zz} = 0.368333$$

(2.16a) Assumed Displacement.



From the geometry of the problem it is assumed that at all points in the body the displacement field (displacement boundary conditions) is:

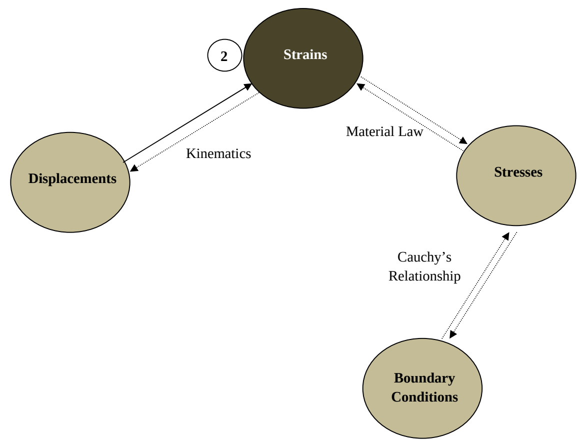
$$U_1(x, y, z) = u_1(x)$$

$$V_1(x, y, z) = v_1(y)$$

$$W_1(x, y, z) = w_1(z)$$

(2.16b) Strains.

[9] 1 psi = 6895 Pa



The displacement gradients are then found as:

$$g_1 = \frac{\partial U_1}{\partial x} = \frac{\partial u_1}{\partial x} \qquad g_4 = \frac{\partial U_1}{\partial y} = 0 \qquad g_7 = \frac{\partial U_1}{\partial z} = 0$$

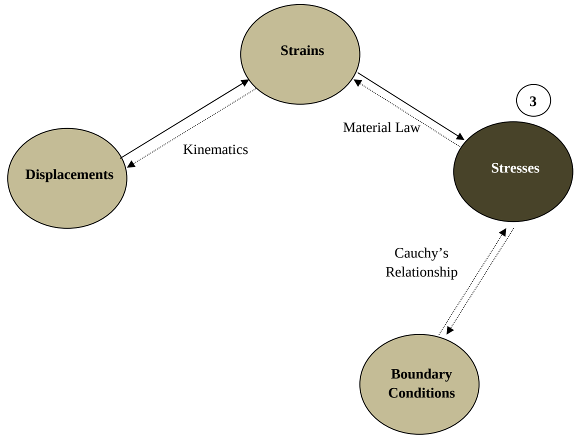
$$g_2 = \frac{\partial V_1}{\partial x} = 0 \qquad g_5 = \frac{\partial V_1}{\partial y} = \frac{\partial v_1}{\partial y} \qquad g_8 = \frac{\partial V_1}{\partial z} = 0$$

$$g_3 = \frac{\partial W_1}{\partial x} = 0 \qquad g_6 = \frac{\partial W_1}{\partial y} = 0 \qquad g_9 = \frac{\partial W_1}{\partial z} = \frac{\partial w_1}{\partial z}$$

Thus the resulting strain-displacement relationship is obtained using the Lagrange-Green equations:

$$\begin{aligned}
\epsilon_1 &= e_{xx} = g_1 = \frac{\partial u_1}{\partial x} \\
\epsilon_2 &= e_{\mathrm{yy}} = g_5 = \frac{\partial v_1}{\partial y} \\
\epsilon_3 &= e_{\mathrm{zz}} = g_9 = \frac{\partial w_1}{\partial z} \\
\epsilon_4 &= 2\, e_{\mathrm{yz}} = g_6 + g_8 = 0 \\
\epsilon_5 &= 2\, e_{\mathrm{xz}} = g_3 + g_7 = 0 \\
\epsilon_6 &= 2\, e_{\mathrm{xy}} = g_2 + g_4 = 0
\end{aligned}$$

(2.16c) Stresses.



Now the stress-strain relationship for Hookean body is obtained using the Hooke's law:

$$
\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}
$$

$$
= 10^{-9} \begin{bmatrix} 0.0258987 & -0.00673366 & -0.00647467 & 0 & 0 & 0 \\ -0.00673366 & 0.120861 & -0.0410926 & 0 & 0 & 0 \\ -0.00647467 & -0.0410926 & 0.111564 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.290065 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.241721 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.241721 \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{Bmatrix}
$$

Or in its inverted form:

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = 10^9 \begin{bmatrix} 40.4261 & 3.48665 & 3.63041 & 0 & 0 & 0 \\ 3.48665 & 9.75924 & 3.797 & 0 & 0 & 0 \\ 3.63041 & 3.797 & 10.5728 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4475 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.137 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.137 \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

Thus,

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = 10^9 \begin{bmatrix} 40.4261 & 3.48665 & 3.63041 & 0 & 0 & 0 \\ 3.48665 & 9.75924 & 3.797 & 0 & 0 & 0 \\ 3.63041 & 3.797 & 10.5728 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4475 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.137 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.137 \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

(All values in GPa)

$$S_1 = S_{\text{xx}} = 40.4261\, e_{\text{xx}} + 3.48665\, e_{\text{yy}} + 3.63041\, e_{\text{zz}}$$

$$S_2 = S_{\text{yy}} = 3.48665\, e_{\text{xx}} + 9.75924\, e_{\text{yy}} + 3.797\, e_{\text{zz}}$$

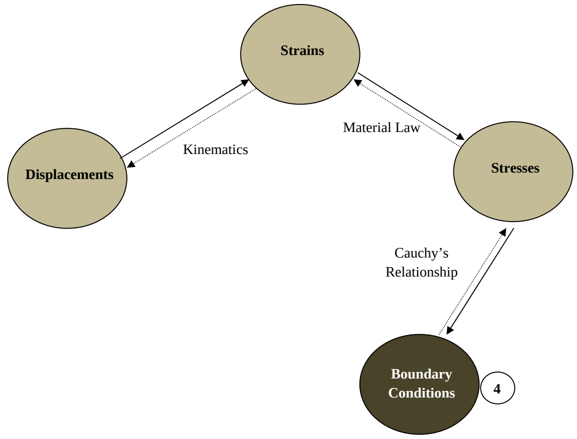
$$S_3 = S_{\text{zz}} = 3.63041\, e_{\text{xx}} + 3.797\, e_{\text{yy}} + 10.5728\, e_{\text{zz}}$$

$$S_4 = S_{\text{yz}} = 0$$

$$S_5 = S_{\text{xz}} = 0$$

$$S_6 = S_{\text{xy}} = 0$$

(2.16d) Equilibrium Equations and Boundary Conditions.



Now, substituting the stress components into the three equilibrium equations which must be satisfied at all point inside the body:

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial S_{zx}}{\partial z} + b_x = 0 \quad \rightarrow \quad \frac{\partial S_{xx}}{\partial x} = 0 \quad \rightarrow \quad S_{xx} = c_1 = c_1(y,z) = \text{constant}$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{zy}}{\partial z} + b_y = 0 \quad \rightarrow \quad \frac{\partial S_{yy}}{\partial y} = 0 \quad \rightarrow \quad S_{yy} = c_2 = c_2(x,z) = \text{constant}$$

$$\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial S_{zz}}{\partial z} + b_z = 0 \quad \rightarrow \quad \frac{\partial S_{zz}}{\partial z} = 0 \quad \rightarrow \quad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

Recall, all body forces are neglected.

Now in other to complete find the unknowns we need to apply stress boundary conditions:

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

(a) On the surface defined by $x = h$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = -\sigma_1\, \underline{\hat{\mathbf{i}}}$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{Bmatrix} -\sigma_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix}$$

Thus,

$$S_{xx}(h, y, z) = -\sigma_1$$

and from the stress-strain relationship,

$$S_{xy}(h, y, z) = S_{xz}(h, y, z) = 0$$

(b) On the surface defined by $x = -h$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{i}}}$, and $\underline{\mathbf{T}} = \sigma_1 \underline{\hat{\mathbf{i}}}$

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left\{ \begin{array}{c} \sigma_1 \\ 0 \\ 0 \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} -S_{xx} \\ -S_{xy} \\ -S_{xz} \end{array} \right\}$$

Thus,

$$S_{xx}(-h, y, z) = -\sigma_1$$

and from the stress-strain relationship,

$$S_{xy}(-h, y, z) = S_{xz}(-h, y, z) = 0$$

Yields the same results as in (a).

(c) On the surface defined by $z = a$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} S_{xz} \\ S_{yz} \\ S_{zz} \end{array} \right\}$$

Thus,

$$S_{zz}(x, y, a) = 0$$

and from the stress-strain relationship,

$$S_{yz}(x, y, a) = S_{xz}(x, y, a) = 0$$

(d) On the surface defined by $z = -a$, $\underline{\hat{\mathbf{n}}} = -\underline{\hat{\mathbf{k}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right\} = \left\{ \begin{array}{c} -S_{xz} \\ -S_{yz} \\ -S_{zz} \end{array} \right\}$$

Thus,

$$S_{zz}(x, y, -a) = 0$$

and from the stress-strain relationship,

$$S_{yz}(x, y, -a) = S_{xz}(x, y, -a) = 0$$

Yields the same results as in (c).

(e) On the surface defined by $y = b$, $\underline{\hat{\mathbf{n}}} = \underline{\hat{\mathbf{j}}}$, and $\underline{\mathbf{T}} = \underline{\mathbf{0}}$

$$\left\{ \begin{array}{c} T_x \\ T_y \\ T_z \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} = \left[\begin{array}{ccc} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{array} \right] \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} S_{xy} \\ S_{yy} \\ S_{yz} \end{array} \right\}$$

Thus,

$$S_{yy}(x, b, z) = 0$$

and from the stress-strain relationship,

$$S_{xy}(x, b, z) = S_{yz}(x, b, z) = 0$$

(f) On the surface defined by $y = -b$, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and $\mathbf{T} = \mathbf{0}$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -S_{xy} \\ -S_{yy} \\ -S_{yz} \end{Bmatrix}$$

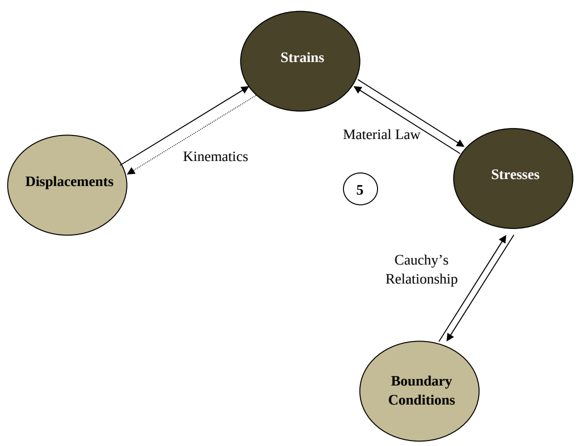
Thus,

$$S_{yy}(x, -b, z) = 0$$

and from the stress-strain relationship,

$$S_{xy}(x, -b, z) = S_{yz}(x, -b, z) = 0$$

(2.16e) Stress and strain field.



From equilibrium conditions:

$$S_{xx} = c_1 = c_1(y, z) = \text{constant} \qquad S_{yy} = c_2 = c_2(x, z) = \text{constant} \qquad S_{zz} = c_3 = c_3(x, y) = \text{constant}$$

From stress boundary conditions,

$$S_{xx}(h, y, z) = -\sigma_1 \qquad \text{and} \qquad c_1 = -\sigma_1$$

$$S_{yy}(x, b, z) = 0 \qquad \text{and} \qquad c_2 = 0$$

$$S_{zz}(x, y, a) = 0 \qquad \text{and} \qquad c_3 = 0$$

Now, let us use the stress-strain relationships (using the Hooke's Law):

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = 10^{-9} \begin{bmatrix} 0.0258987 & -0.00673366 & -0.00647467 & 0 & 0 & 0 \\ -0.00673366 & 0.120861 & -0.0410926 & 0 & 0 & 0 \\ -0.00647467 & -0.0410926 & 0.111564 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.290065 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.241721 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.241721 \end{bmatrix} \begin{Bmatrix} -\sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiplying the above we get

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \sigma_1 \begin{Bmatrix} -0.0258987 \\ 0.00673366 \\ 0.00647467 \\ 0 \\ 0 \\ 0 \end{Bmatrix} 10^{-9}$$

Thus, while there is no contact between the solid structure and the rigid walls, the state of stress is

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that since the above is a principal state of stress, the result will be a principal state of strain. Thus the associated state of strain is

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & 0 & 0 \\ 0 & e_{yy} & 0 \\ 0 & 0 & e_{zz} \end{bmatrix} = \sigma_1 \begin{bmatrix} -0.0258987 & 0 & 0 \\ 0 & 0.00673366 & 0 \\ 0 & 0 & 0.00647467 \end{bmatrix} \times 10^{-9}$$

Note that $e_{yy} \neq e_{zz}$. This implies that the block will reach two-opposite side rigid walls either in the y or in the z direction first. The strain needed for the solid cube to make with the rigid wall is

$$e_{yy} = \frac{\text{final expansion in } y - \text{initial expansion in } y}{\text{initial expansion in } y} = \frac{(2\,b + 2\,t) - (2\,b)}{2\,b}$$

$$= \frac{2\,(0.0381 \times 10^{-3})}{0.127} = 0.0006$$

$$e_{zz} = \frac{\text{final expansion in } z - \text{initial expansion in } z}{\text{initial expansion in } z} = \frac{(2\,a + 2\,w) - (2\,a)}{2\,a}$$

$$= \frac{2\,(0.0381 \times 10^{-3})}{0.127} = 0.0006$$

Thus

$$e_{yy} = e_{zz} = 0.0006$$

The needed pressure is calculated using both the second and third equation in the Hooke's Law

$$e_{yy} = 0.00673366\,\sigma_1 \times 10^{-9} \qquad \rightarrow \qquad \sigma_1 = \quad 89.1046 \text{ MPa}$$

$$e_{zz} = 0.00647467\,\sigma_1 \times 10^{-9} \qquad \rightarrow \qquad \sigma_1 = \quad 92.6688 \text{ MPa}$$

Thus, when $\sigma = 89.1046$ MPa, the block has already made contact with the rigid wall in the y-direction; and when $\sigma \geq 92.6688$ MPa the block will make contact with both walls (we need to calculate this value).

End Example □

Example 2.17.

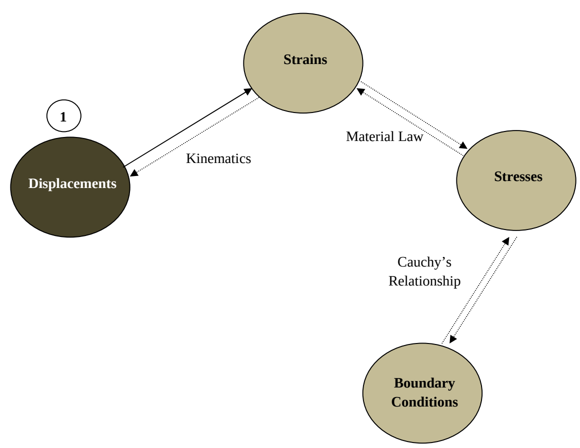
Application 4: Nonisothermal Orthotropic Material

Redo Example 2.16 but considering a decrease in temperature of 100°C from its initial temperature.

The mechanical properties for Glass-Epoxy (Scothply 1002) are:

$$E_{xx} = 38.612 \text{ GPa} \qquad E_{yy} = 8.274 \text{ GPa} \qquad E_{zz} = 8.9635 \text{ GPa}$$
$$G_{xy} = 4.137 \text{ GPa} \qquad G_{xz} = 4.137 \text{ GPa} \qquad G_{yz} = 3.4475 \text{ GPa}$$
$$\alpha_{xx} = 8.64\ \mu\text{m/m-°C} \qquad \alpha_{yy} = 22.14\ \mu\text{m/m-°C} \qquad \alpha_{zz} = 22.14\ \mu\text{m/m-°C}$$
$$\nu_{yx} = \frac{\nu_{xy}}{E_{xx}} E_{yy} = 0.0557143 \qquad \nu_{zx} = \frac{\nu_{xz}}{E_{xx}} E_{zz} = 0.0580357 \qquad \nu_{zy} = \frac{\nu_{yz}}{E_{yy}} E_{zz} = 0.368333$$

(2.17a) Assumed Displacement.



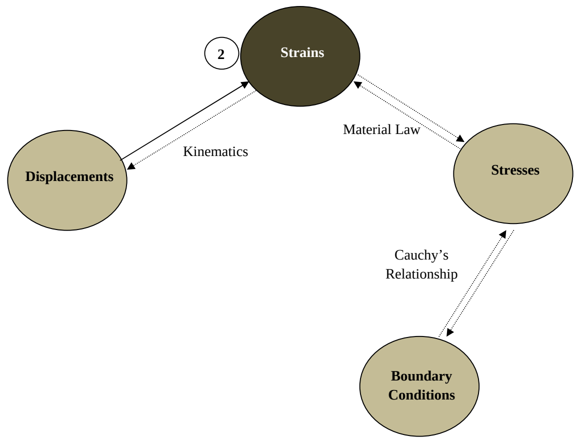
From the geometry of the problem it is assumed that at all points in the body the displacement field (displacement boundary conditions) is:

$$U_1(x, y, z) = u_1(x)$$

$$V_1(x, y, z) = v_1(y)$$

$$W_1(x, y, z) = w_1(z)$$

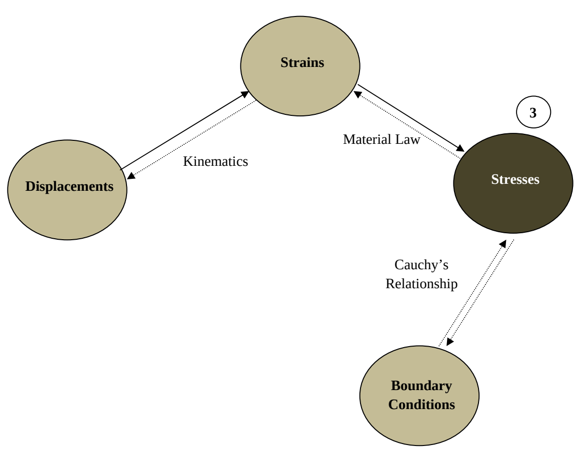
(2.17b) Strains.



The resulting strain-displacement relationship is obtained using the Lagrange-Green equations:

$$\epsilon_1 = e_{xx} = g_1 = \frac{\partial u_1}{\partial x}$$
$$\epsilon_2 = e_{\mathrm{yy}} = g_5 = \frac{\partial v_1}{\partial y}$$
$$\epsilon_3 = e_{\mathrm{zz}} = g_9 = \frac{\partial w_1}{\partial z}$$
$$\epsilon_4 = 2\, e_{\mathrm{yz}} = g_6 + g_8 = 0$$
$$\epsilon_5 = 2\, e_{\mathrm{xz}} = g_3 + g_7 = 0$$
$$\epsilon_6 = 2\, e_{\mathrm{xy}} = g_2 + g_4 = 0$$

(2.17c) Stresses.



Now the stress-strain relationship for Hookean body is obtained using the Hooke's law:

$$\left\{\begin{array}{c}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{array}\right\} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{yx}}{E_{yy}} & -\frac{\nu_{zx}}{E_{zz}} & 0 & 0 & 0\\ -\frac{\nu_{xy}}{E_{xx}} & \frac{1}{E_{yy}} & -\frac{\nu_{zy}}{E_{zz}} & 0 & 0 & 0\\ -\frac{\nu_{xz}}{E_{xx}} & -\frac{\nu_{yz}}{E_{yy}} & \frac{1}{E_{zz}} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \left\{\begin{array}{c}S_{xx}\\ S_{yy}\\ S_{zz}\\ S_{yz}\\ S_{xz}\\ S_{xy}\end{array}\right\} + \left\{\begin{array}{c}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{zz}\\ 0\\ 0\\ 0\end{array}\right\}\Delta T$$

$$\left\{\begin{array}{c}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{array}\right\} = 10^{-9}\begin{bmatrix} 0.0258987 & -0.00673366 & -0.00647467 & 0 & 0 & 0\\ -0.00673366 & 0.120861 & -0.0410926 & 0 & 0 & 0\\ -0.00647467 & -0.0410926 & 0.111564 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.290065 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.241721 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.241721 \end{bmatrix}\left\{\begin{array}{c}S_{xx}\\ S_{yy}\\ S_{zz}\\ S_{yz}\\ S_{xz}\\ S_{xy}\end{array}\right\}$$

$$+\left\{\begin{array}{c}8.64\\ 22.14\\ 22.14\\ 0\\ 0\\ 0\end{array}\right\}\times 10^{-6}\,(-100)$$

Or in its inverted form:

$$\left\{\begin{array}{c}S_1\\ S_2\\ S_3\\ S_4\\ S_5\\ S_6\end{array}\right\} = 10^{9}\begin{bmatrix} 40.4261 & 3.48665 & 3.63041 & 0 & 0 & 0\\ 3.48665 & 9.75924 & 3.797 & 0 & 0 & 0\\ 3.63041 & 3.797 & 10.5728 & 0 & 0 & 0\\ 0 & 0 & 0 & 3.4475 & 0 & 0\\ 0 & 0 & 0 & 0 & 4.137 & 0\\ 0 & 0 & 0 & 0 & 0 & 4.137 \end{bmatrix}\left\{\left\{\begin{array}{c}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ 0\\ 0\\ 0\end{array}\right\} - \left\{\begin{array}{c}8.64\\ 22.14\\ 22.14\\ 0\\ 0\\ 0\end{array}\right\}\times 10^{-6}\,(-100)\right\}$$

Thus, (All values in GPa)

$$S_1 = S_{xx} = 40.4261\, e_{xx} + 3.48665\, e_{yy} + 3.63041\, e_{zz} - 40.4261\, \alpha_{xx}\, \Delta T - 3.48665\, \alpha_{yy}\, \Delta T - 3.63041\, \alpha_{zz}\, \Delta T$$

$$= 40.4261\, e_{xx} + 3.48665\, e_{yy} + 3.63041\, e_{zz} + 0.0506854$$

$$S_2 = S_{yy} = 3.48665\, e_{xx} + 9.75924\, e_{yy} + 0.033026$$

$$= 3.48665\, e_{xx} + 9.75924\, e_{yy} + 3.797\, e_{zz} - 3.48665\, \alpha_{xx}\, \Delta T - 9.75924\, \alpha_{yy}\, \Delta T - 3.797\, \alpha_{zz}\, \Delta T$$

$$S_3 = S_{zz} = 3.63041\, e_{xx} + 3.797\, e_{yy} + 10.5728\, e_{zz} - 3.63041\, \alpha_{xx}\, \Delta T - 3.797\, \alpha_{yy}\, \Delta T - 10.5728\, \alpha_{zz}\, \Delta T$$

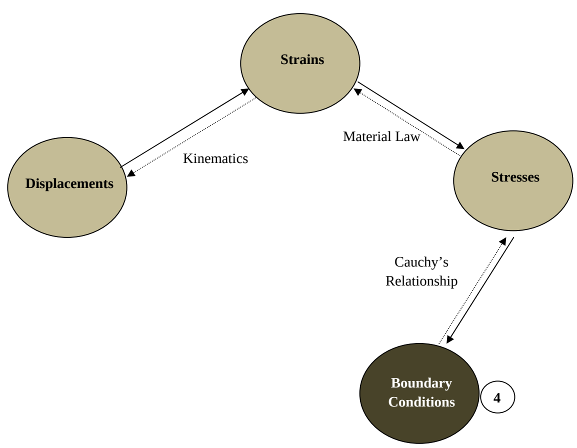
$$= 3.63041\, e_{xx} + 3.797\, e_{yy} + 10.5728\, e_{zz} + 0.034951$$

$$S_4 = S_{yz} = 0$$

$$S_5 = S_{xz} = 0$$

$$S_6 = S_{xy} = 0$$

(2.17d) Equilibrium equations and boundary conditions.



Now, substituting the stress components into the three equilibrium equations which must be satisfied at all point inside the body:

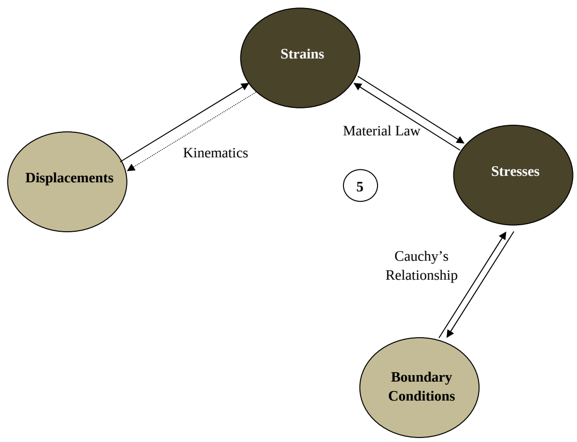
$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} + \frac{\partial S_{zx}}{\partial z} + b_x = 0 \quad \rightarrow \quad \frac{\partial S_{xx}}{\partial x} = 0 \quad \rightarrow \quad S_{xx} = c_1 = c_1(y,z) = \text{constant}$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{zy}}{\partial z} + b_y = 0 \quad \rightarrow \quad \frac{\partial S_{yy}}{\partial y} = 0 \quad \rightarrow \quad S_{yy} = c_2 = c_2(x,z) = \text{constant}$$

$$\frac{\partial S_{xz}}{\partial x} + \frac{\partial S_{yz}}{\partial y} + \frac{\partial S_{zz}}{\partial z} + b_z = 0 \quad \rightarrow \quad \frac{\partial S_{zz}}{\partial z} = 0 \quad \rightarrow \quad S_{zz} = c_3 = c_3(x,y) = \text{constant}$$

Recall, all body forces are neglected.

(2.17e) Stress and Strain fields.



From stress boundary conditions,

$$S_{\mathrm{xx}}(h, y, z) = -\sigma_1 \qquad \text{and} \qquad c_1 = -\sigma_1$$

$$S_{\mathrm{yy}}(x, b, z) = 0 \qquad \text{and} \qquad c_2 = 0$$

$$S_{\mathrm{zz}}(x, y, a) = 0 \qquad \text{and} \qquad c_3 = 0$$

Now, let us use the stress-strain relationships (using the Hooke's Law):

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = 10^{-9} \begin{bmatrix} 0.0258987 & -0.00673366 & -0.00647467 & 0 & 0 & 0 \\ -0.00673366 & 0.120861 & -0.0410926 & 0 & 0 & 0 \\ -0.00647467 & -0.0410926 & 0.111564 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.290065 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.241721 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.241721 \end{bmatrix} \begin{Bmatrix} -\sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$+ \begin{Bmatrix} 8.64 \\ 22.14 \\ 22.14 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-6} \, (-100)$$

Multiplying the above we get

$$\left\{\begin{array}{c} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{array}\right\} = \sigma_1 \left\{\begin{array}{c} -0.0258987 \\ 0.00673366 \\ 0.00647467 \\ 0 \\ 0 \\ 0 \end{array}\right\} 10^{-9} + \left\{\begin{array}{c} -0.000864 \\ -0.002214 \\ -0.002214 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

Thus, while there is no contact between the solid structure and the rigid walls, the state of stress is

$$\underline{\mathbf{S}} = \begin{bmatrix} S_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that since the above is a principal state of stress, the result will be a principal state of strain. Thus the associated state of strain is

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{xx} & 0 & 0 \\ 0 & e_{yy} & 0 \\ 0 & 0 & e_{zz} \end{bmatrix} = \sigma_1 \begin{bmatrix} -0.0258987 & 0 & 0 \\ 0 & 0.00673366 & 0 \\ 0 & 0 & 0.00647467 \end{bmatrix} \times 10^{-9}$$

$$+ \begin{bmatrix} -0.000864 & 0 & 0 \\ 0 & -0.002214 & 0 \\ 0 & 0 & -0.002214 \end{bmatrix}$$

Note that $e_{yy} \neq e_{zz}$. This implies that the block will reach two-opposite side rigid walls either in the y or in the z direction first. The strain needed for the solid cube to make with the rigid wall is

$$e_{yy} = \frac{\text{final expansion in } y - \text{initial expansion in } y}{\text{initial expansion in } y} = 0.0006$$

$$e_{zz} = \frac{\text{final expansion in } z - \text{initial expansion in } z}{\text{initial expansion in } z} = 0.0006$$

Thus

$$e_{yy} = e_{zz} = 0.0006$$

The needed pressure is calculated using both the second and third equation in the

Hooke's Law

$$e_{yy} = 0.00673366\,\sigma_1 \times 10^{-9} - 0.002214 \qquad \rightarrow \qquad \sigma_1 = \quad 417.901 \text{ MPa}$$

$$e_{zz} = 0.00647467\,\sigma_1 \times 10^{-9} - 0.002214 \qquad \rightarrow \qquad \sigma_1 = \quad 434.617 \text{ MPa}$$

Thus, when $\sigma = 417.901$ MPa, the block has already made contact with the rigid wall in the y-direction; and when $\sigma \geq 434.617$ MPa, the block will make contact with both walls (we need to calculate this value).

End Example □

2.11 References

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Keane, Andy and Nair, Prasanth, *Computational Approaches for Aerospace Design: The Pursuit of Excellence*, August 2005, John Wiley and Sons.

Shames, I. H., and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, 1985, Taylor & Francis.

Sun, C. T., *Mechanics of Aircraft Structures*, Second Edition 2006, John Wiley and Sons

2.12 Suggested Problems

Problem 2.1.

The state of stress at a point is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} -p & \tau & \tau \\ \tau & -p & \tau \\ \tau & \tau & -p \end{bmatrix} \tag{2.191}$$

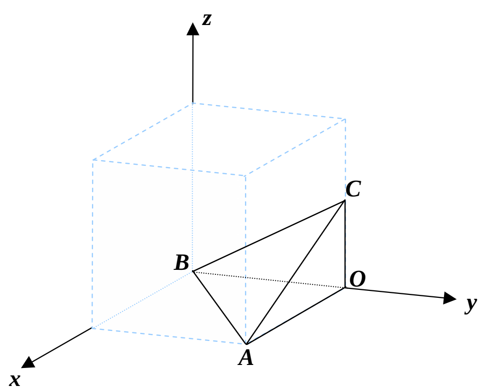
where $p > 0$ and $\tau > 0$. Determine the state of principal strain for an isotropic and for an orthotropic materials.

□

Problem 2.2.

The state of stress at a point is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{2.192}$$



Suppose the stress vector acting on the **ACB** plane is

$$\underline{\mathbf{T}}^{(\mathrm{ACB})} = \begin{Bmatrix} 50 \\ 10 \\ 20 \end{Bmatrix} \quad \text{MPa}$$

1. Determine the state of strain if the material is steel and glass-epoxy (use values used in chapter).
2. Is this a case of plane strain, plane stress, or neither of these special cases.
3. Determine the strain vector acting along the **AC** segment.

□

Problem 2.3.

The state of stress at a point is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 10 & 20 & 0 \\ 20 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^6 \quad \text{psi} \tag{2.193}$$

Find the principal strains if the material is Titanium (Ti3Al2.5V, UNS R56320; ASTM Grade 9; Half 6-4) using:

1. Eigenvalue approach.
2. Mohr's circle.

□

Problem 2.4.

Analysis of a particular body, made of Titanium (Ti3Al2.5V, UNS R56320; ASTM Grade 9; Half 6-4), indicates that stresses for orthogonal interfaces associated with reference x-y-z at a given point are

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 3000 & -1000 & 0 \\ -1000 & 2000 & 2000 \\ 0 & 2000 & 0 \end{bmatrix} \text{ kPa} \tag{2.194}$$

1. Determine the shear strain vector acting on the same interface in a direction parallel to the x-axis.
2. Determine the normal and shear strain vectors and magnitudes on the infinitesimal interface at this point whose unit normal is

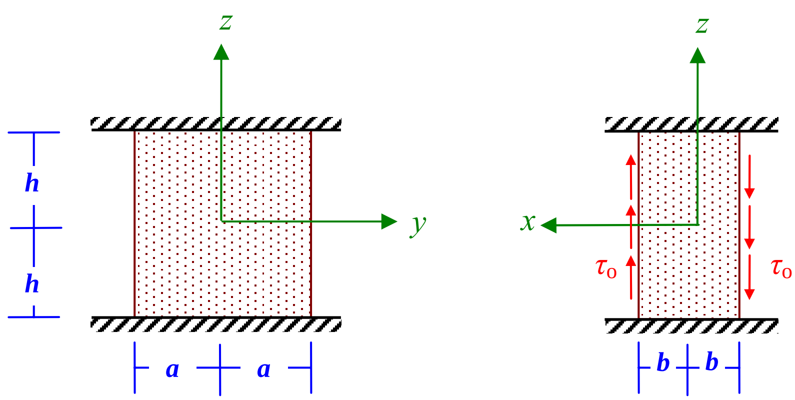
$$\underline{\hat{\mathbf{n}}}_s = 0.60\,\underline{\hat{\mathbf{j}}} + 0.8\,\underline{\hat{\mathbf{k}}} \tag{2.195}$$

3. Determine the overall maximum shear strain at the given point.

□

Problem 2.5.

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed shear τ_o acting on the x-planes only. The block is bound only in the z-direction by rigid walls at $z = h$, $z = -h$, but free to expand/contract in the x and y planes.



FRONT VIEW
(Seen from x-axis)

SIDE VIEW
(Seen from y-axis)

Assume the orthotropic material is graphite-epoxy (T300/934). The geometric properties are:

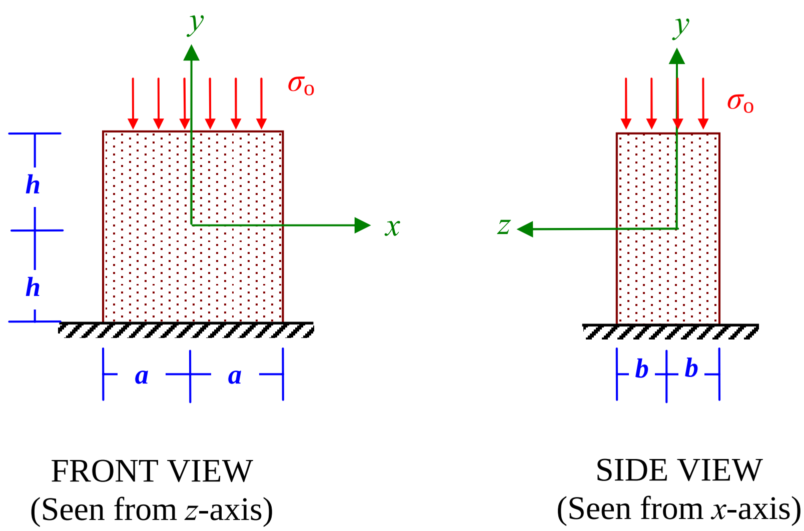
$$2\,a = 2\,b = 2\,h = 0.200 \text{ m}$$

What is the needed shear pressure τ_o if the maximum strain allowed in any given direction is 0.001?

□

Problem 2.6.

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed shear σ_o in the y-direction. The block is bound only in the y-direction by rigid wall at $y = -h$, but free to expand/contract in the all other directions.



Assume the orthotropic material is graphite-epoxy (T300/934). The geometric properties are:

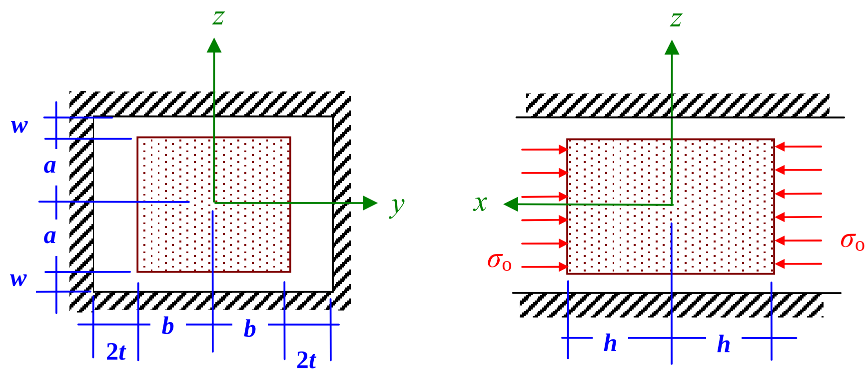
$$2\,a = 2\,b = 2\,h = 0.200 \text{ m}$$

What is the needed pressure, σ_o, if the maximum normal strain allowed in any given direction is 0.001?

□

Problem 2.7.

Consider a solid structure of a Hookean material with negligible body forces and subject to evenly distributed pressure $\sigma_\circ$ in the x-direction. The block is bound in the y- and z-direction by rigid walls at $y = b + 2t$, $y = -b - 2t$, $z = a + w$, $z = -a - w$, but free to expand/contract in the x-plane. to 20%



FRONT VIEW
(Seen from x-axis)

SIDE VIEW
(Seen from y-axis)

Considering a increase in temperature of 100°C from its initial temperature. Assume the orthotropic material is graphite-epoxy (T300/934). The geometric properties are:

$$2\,a = 2\,b = 2\,h = 0.200 \text{ m} \qquad t = w = 0.010 \text{ m}$$

What is the needed pressure, $\sigma_\circ$, to make contact between the Hookean solid cube structure and all the rigid walls?

□

Chapter 3
Material Selection

3.1 Stress-Strain Diagrams

Usually the stress-strain diagrams are based on one-dimensional tension testing. The stress-strain diagrams are characteristic of the particular material being tested and conveys important information about the mechanical properties and type of behavior. A stress-strain diagram for a typical structural steel in tension is shown in Fig. 3.1. Strains are plotted on the horizontal axis and stresses on the vertical axis. (In order to display all of the important features of this material, the strain axis in Fig. 3.1 is not drawn to scale.) From hereon all stress properties obtained from tables will be characterized with S and σ the ones we evaluate.

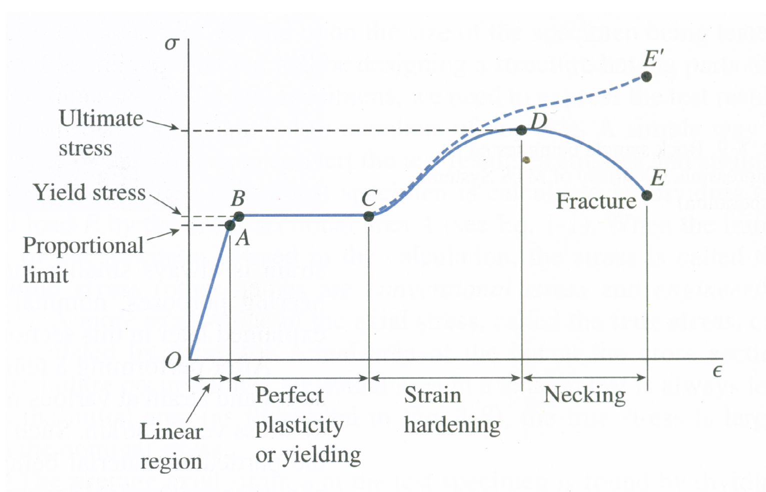


Figure 3.1: Stress-strain diagram for a typical structural steel in tension.

Proportional limit, S_p (A): *Point where the stress-strain relationship stops to be proportional.* The diagram begins with a straight line from the origin O to point A, which means that the relationship between stress and strain in this initial region is not only linear but also proportional.[1] Beyond point A, the proportionality between stress and strain no longer exists; hence the stress at A is called the proportional limit.

Yield point, S_y (B or C): *Point where as we increase the strain there is no increase in stress.* With an increase in stress beyond the proportional limit, the strain begins to increase more rapidly for each increment in stress. Consequently, the stress-strain curve has a smaller and smaller slope, until, at point B, the curve becomes horizontal. Beginning at this point, considerable elongation of the test specimen occurs with no noticeable increase in the tensile force (from B to C). This phenomenon is known as yielding of the material, and point B is called the yield point. The corresponding stress is known as the yield stress of the steel. In the region from B to C, the material becomes perfectly plastic, which means it deforms without an increase in the applied load.

Ultimate strength, S_u (D): *Maximum value in the stress-strain curve.* After undergoing the large strains that occur during yielding in the region BC, the steel begins to strain harden. During strain hardening, the material undergoes changes in its crystalline structure, resulting in increased resistance of the material to further deformation. Elongation of the test specimen in this region requires an increase in the tensile load, and therefore the stress-strain diagram has a positive slope from C to D. The load eventually reaches its maximum value, and the corresponding stress (at point D) is called the ultimate stress. Further stretching of the bar is actually accompanied by a reduction in the load, and fracture finally occurs at a point such as E.

3.2 Material Selection

Mechanical engineering design has two keystone objectives: (i) selection of the best possible material and (ii) determination of the best possible geometry for each part.

Materials engineers' has the task of developing new and better materials. However, mechanical engineers must be effective in selecting the best available material for each application, considering all important design criteria. Materials selection is typically carried out as a part of the intermediate design stage, but in some cases must be considered earlier, during the preliminary design stage. The basic steps in selection of candidate materials for any given application can be summarized as:

1. Analyzing the material-specific requirements of the application.
2. Assembling a list of requirement-responsive materials, with pertinent performance evaluation data, rank-ordered so the "best" material is at the top of the table! for each important application requirement.

[1] Two variables are said to be proportional if their ratio remains constant. Therefore, proportional relationship may be represented by a straight line through the origin. However, a proportional relationship is not the same as a linear relationship.

3. Matching the lists of materials responsive to the pertinent application requirements in order to select the "best" candidate materials for the proposed design.

Table 3.1: Potential Material-Specific Application Requirements

Requirement-Responsive Material Characteristic	Performance Evaluation Index	Special Need?
1. Strength/volume ratio	: Ultimate or yield strength	:
2. Strength/weight ratio	: Ultimate or yield strength/density	:
3. Strength at elevated temperature	: Strength loss/degree temperature	:
4. Long-term dimensional stability at elevated temperature	: Creep rate at operating temperature :	: :
5. Dimensional stability under temperature fluctuation	: Strain/degree temperature change :	: :
6. Stiffness	: Modulus of Elasticity	:
7. Ductility	: % elongation in 2 inches	:
8. Ability to store energy elastically	: Energy/unit volume at yield	:
9. Ability to dissipate energy plastically	: Energy/unit volume at rupture	:
10. Wear resistance	: Dimensionless loss at operating condition; : also hardness	:
11. Resistance to chemically reactive environment	: Dimensionless loss at operating : environment	:
12. Resistance to nuclear radiation environment	: Strength or ductility change in operating : environment	:
13. Desire to use specific manufacturing process	: Suitability for specific process	:
14. Cost constraints	: Cost/unit weight; also machinability	:
15. Procurement time constraints	: Procurement time and effort	:

3.2.1 Two-Selection Criteria

To methods are presented here: (i) Rank-Ordered-Data Table Method, (ii) Ashby Chart Method. Both methods are widely used in material selection. The steps are summarized as follows:

1. Using Table 3.1 as a guide, together with known requirements imposed by operational or functional constraints, postulated failure modes, market-driven factors, and/or management directives, establish a concise specification statement.

2. Based on the information from step 1, and the specification statement, identify all special needs for the application by writing a response of yes, no, or perhaps in Table 3.1.

3. This step is different for both methods:

   Rank-Ordered-Data Table Method: For each item receiving a yes or perhaps, use Table 3.2 to identify the corresponding performance evaluation index, and consult rank-ordered Tables 3.2–3.19 for potential material candidates.

Ashby Chart Method: For each item receiving a yes or perhaps, use Table 3.1 to identify the corresponding performance evaluation index, and consult the pertinent Ashby charts shown in Figures. Now identify a short list of highly qualified candidate materials to each selected pair of performance parameters or application constraints.

4. Comparing the results in step 3, establish the two or three better candidate materials by finding those near the tops of all the lists or charts.

5. Assign each performance index a weigh. Then from the two or three better candidate materials, use the Pugh's method to make a tentative selection for the material to be used.

Table 3.2: Strength Properties of Selected Materials

Material	Alloy	Ultimate Tensile Strength S_u (psi)	Yield Strength S_{yp} (psi)
Ultra-high-strength steel	AISI 4340	287,000	270,000
Stainless steel (age hardenable)	AM 350	206,000	173,000
High-carbon steel	AISI 1095[2]	200,000	138,000
Graphite-epoxy composite	—	200,000	—
Titanium	Ti-6Al-4V	150,000	128,000
Ceramic	Titanium carbide (bonded)	134,000	—
Nickel-based alloy	Inconel 601	102,000	35,000
Medium-carbon steel	AISI 1060 (HR)[3]	98,000	54,000
	AISI 1060 (CD)[4]	90,000	70,000
Low-carbon, low-alloy steel	AISI 4620 (HR)	87,000	63,000
	AISI 4620 (CD)	101,000	85,000
Stainless steel (austenitic)	AISI 304 (annealed)	85,000	35,000
Yellow brass	C 26800 (hard)	74,000	60,000
Commercial bronze	C 22000 (hard)	61,000	54,000
Low-carbon (mild) steel	AISI 1020 (CD)	61,000	51,000
	AISI 1020 (annealed)	57,000	43,000
	AISI 1020 (HR)	55,000	30,000
Phosphor bronze	C 52100 (annealed)	55,000	24,000
Gray cast iron	ASTM A-48 (class 50)	50,000[5]	—
Gray cast iron	ASTM A-48 (class 40)	40,000	—
Aluminum (wrought)	2024-T3 (heat treated)	70,000	50,000
Aluminum (wrought)	2024 (annealed)	27,000	11,000
Aluminum (perm. mold cast)	356.0 (sol'n. treated; aged)	38,000	27,000
Magnesium (extruded)	ASTM AZ80A-T5	50,000	35,000
Magnesium (cast)	ASTM AZ63A	29,000	14,000
Thermosetting polymer	Epoxy (glass reinforced)	—	10,000
Thermoplastic polymer	Acrylic (cast)	—	7000

[1]See, for example, ref. 1–10.
[2]Quenched and drawn to Rockwell C-42.
[3]Hot-rolled.
[4]Cold-drawn.
[5]Ultimate *compressive* strength is 170,000 psi.

Table 3.3: Strength/Weight Ratios of Selected Materials

Material	Weight Density, w (lb/in^3)	Approx. Ultimate Strength/Wt Ratio, $\frac{S_u}{w}$ (inches $\times$ 10^3)	Approx. Yield Strength/Wt Ratio, $\frac{S_{yp}}{w}$ (inches $\times$ 10^3)
Graphite-epoxy composite	0.057	3500	—
Ultra-high-strength steel	0.283	1000	950
Titanium	0.160	950	800
Stainless steel (age hardenable)	0.282	750	600
Aluminum (wrought)	0.100	700	500
Titanium carbide	0.260	500	—
Aluminum (perm. mold cast)	0.097	400	300
Medium-carbon steel	0.283	350	200
Nickel-based alloy	0.291	350	100
Stainless steel (austenitic)	0.290	290	120
Yellow brass	0.306	250	200
Low-carbon steel	0.283	200	150
Commercial bronze	0.318	200	150
Gray cast iron (class 50)	0.270	200	—
Epoxy (glass reinf.)	0.042	—	250
Acrylic (cast)	0.043	—	150

Table 3.4: Strength at Elevated Temperatures for Selected Materials

Material	Temperature Θ (°F)	Ultimate Tensile Strength $(S_u)_\Theta$ (psi)	Yield Strength $(S_{yp})_\Theta$ (psi)
Ultra-high-strength steel (4340)	−200	313,000	302,000
	RT[1]	287,000	270,000
	400	273,000	235,000
	800	221,000	186,000
	1200	103,000	62,000
Stainless steel (AM 350)	RT	206,000	173,000
	400	185,000	144,000
	800	179,000	119,000
	1000	119,000	83,000
Titanium (Ti-6Al-4V)	−200	187,000	155,000
	RT	150,000	128,000
	400	126,000	101,000
	800	90,000	75,000
	1000	81,000	59,000
Titanium carbide	RT	134,000	—
	1500	94,000	—
	1800	72,000	—
Inconel (601)	RT	102,000	35,000
	400	94,000	31,000
	800	84,000	28,000
	1200	66,000	23,000
	1600	20,000	12,000
Low-carbon steel (1020)	−200	97,000	83,000
	RT	61,000	51,000
	400	61,000	51,000
	800	45,000	38,000
	900	29,000	24,000
Aluminum (2024-T3)	−200	74,000	54,000
	RT	70,000	50,000
	400	52,000	39,000
	800	4,000	4,000
Magnesium (AZ80A-T5)	−200	63,000	53,000
	RT	50,000	35,000
	200	43,000	19,000
	400	22,000	11,000

[1]Room temperature.

Table 3.5: Stress Rupture Strength Levels (psi) Corresponding to Various Rupture Times and Temperatures for Selected Materials

Material	Alloy	Temp., Θ (°F)	Rupture Time, t, hours				
			10	100	600	1000	10,000
Stainless steel	AM 350	800	—	184,000	—	182,000	—
Iron-base superalloy	A-286	1000	120,000	100,000	—	80,000	76,000
		1200	78,000	68,000	—	50,000	34,000
		1350	50,000	35,000	—	21,000	14,000
		1500	21,000	11,000	—	—	—
Cobalt base superalloy	X-40	1500	61,000	56,000	—	51,000	—
Inconel	601	1200	—	—	—	28,000	—
		1400	—	—	—	9,100	—
		1600	—	—	—	4,200	—
		1800	—	—	—	2,000	—
Carbon steel	1050	750	—	52,500	49,000	—	—
		930	—	22,400	18,000	—	—
Aluminum	Duralumin	300	—	38,000	32,500	—	—
		480	—	11,200	8,300	—	—
		660	—	3,100	2,700	—	—
Brass	60/40	300	—	47,000	42,500	—	—
		480	—	15,700	9,000	—	—

Table 3.6: Creep Limited Maximum Stresses (psi) Corresponding to Various Strain Rates and Temperatures for Selected Materials

Material	Alloy	Temp., Θ (°F)	Strain Rate, $\dot{\delta}$, in/in/hr				
			4×10^{-7}	1×10^{-6}	4×10^{-6}	1×10^{-5}	4×10^{-5}
Strainless steel	AM 350	800	—	91,000	—	—	—
Chromium steel (Q&T)	13% Cr	840	23,500	—	33,600	—	41,500
Manganese steel	1.7% Mn	840	23,500	—	27,000	—	36,000
Carbon steel (forged)	1030	930	13,000	—	16,300	—	19,000
Stainless steel	304	1000	—	—	—	10,000	—
		1300	—	—	—	8,000	—
		1500	—	—	—	5,000	—
Phosphor bronze		440	10,000	—	15,700	—	21,300
Magnesium	HZ32A-T5	400	—	—	—	—	10,000
		500	—	—	—	—	8,000
		600	—	—	—	—	5,000
Aluminum	Duralumin	440	5,600	—	7,400	—	9,700
Brass	60/40	440	1,020	—	2,700	—	5,600

Table 3.7: Coefficients of Thermal Expansion for Selected Materials

Material	Alloy	Coefficient of Thermal Expansion, α (10^{-6} in/in/°F)	Temperature Range of Validity (°F)
Ceramic	Titanium carbide (bonded)	4.3–7.5	68–1200
Titanium	Ti-6Al-4V	5.3	68–1000
Gray cast iron	ASTM A-48 (class 50)	6.0	32–212
Steel	Most	6.3	0–200
Stainless steel	AM 350	6.3	—
Nickel-base alloy	Inconel 601	7.6	80–200
Nickel-base alloy	Inconel 600	9.3	80–1500
Cobalt base superalloy	X-40	9.2	70–1800
Stainless steel	304	9.6	32–212
Graphite-epoxy composite	—	10	—
Commercial bronze	C 22000	10.2	68–572
Iron-base superalloy	A-286	10.3	70–1000
Yellow brass	C 26800	11.3	68–572
Aluminum (cast)	356.0	11.9	68–212
Aluminum (wrought)	2024-T3	12.9	68–212
Aluminum (wrought)	2024-T3	13.7	68–572
Magnesium	Most	14.0	68
Magnesium	Most	16.0	68–750
Thermosetting polymer	Epoxy (glass reinf.)	10–20	—
Thermoplastic polymer	Acrylic	45	—

Table 3.8: Stiffness Properties of Selected Materials

Material	Young's Modulus of Elasticity, E (10^6 psi)	Shear Modulus of Elasticity, G (10^6 psi)	Poisson's Ratio, ν
Tungsten carbide	95	—	0.20
Titanium carbide	42–65 (77°F)	—	0.19
Titanium carbide	33–48 (1600–1800°F)	—	—
Molybdenum	47 (RT)[1]	—	0.29
Molybdenum	33 (1600°F)	—	—
Molybdenum	20 (2400°F)	—	—
Steel (most)	30	11.5	0.30
Stainless steel	28	10.6	0.31
Iron-base superalloy (A-286)	29.1 (RT)	—	0.31
	23.5 (1000°F)	—	—
	22.2 (1200°F)	—	—
	19.8 (1500°F)	—	—
Cobalt-base superalloy	29	—	—
Inconel	31	11.0	—
Cast iron	13–24	5.2–8.5	0.21–0.27
Commercial bronze (C 22000)	17	6.3	0.35
Titanium	16	6.2	0.31
Phosphor bronze	16	6.0	0.35
Aluminum	10.3	3.9	0.33
Magnesium	6.5	—	0.29
Graphite-epoxy composite	6.0	—	—
Acrylic thermoplastic	0.4	—	0.4

[1]Room temperature.

Table 3.9: Ductility of Selected Materials

Material	Alloy	Elongation in 2-inch Gage Length, e (2 in) (percent)
Phosphor bronze C	C 52100	70
Inconel	601	50 (RT)
Inconel	601	50 (1000°F)
Inconel	601	75 (1400°F)
Stainless steel	AISI 304	60
Copper	Oxygen-free	50
Silver		48
Gold		45
Aluminum (annealed)	1060	43
Low-carbon low-alloy steel	AISI 4620 (HR)[1]	28
	AISI 4620 (CD)[2]	22
Low-carbon steel	AISI 1020 (HR)	25
Low-carbon steel	AISI 1020 (CD)	15
Aluminum (wrought)	2024-T3	22
Stainless steel	AM 350	13
Medium-carbon steel	AISI 1060 (HR)	12
Medium-carbon steel	AISI 1060 (CD)	10
Ultra-high-strength steel	AISI 4340	11
Titanium	Ti-6Al-4V	10 (RT)
Titanium	Ti-6Al-4V	18 (800°F)
Cobalt-base superalloy	X-40	9 (RT)
Cobalt-base superalloy	X-40	12 (1200°F)
Cobalt-base superalloy	X-40	22 (1700°F)
Magnesium (forged)	AZ80A-T5	6
Aluminum (perm. mold cast)	356.0 (sol'n treated; aged)	5
Commercial bronze	C 22000 (hard)	5
Gray cast iron	All	nil

[1]Hot-rolled.
[2]Cold-drawn.

Table 3.10: Modulus of Resilience R for Selected Materials Under Tensile Loading

Material	Alloy	$R = \frac{S_{yp}^2}{2E}$ (in-lb/in^3)
Ultra-high-strength steel	AISI 4340	1220
Stainless steel	AM 350	530
Titanium	Ti-6Al-4V	510
Aluminum (wrought)	2024-T3 (heat treated)	120
Magnesium (extruded)	AZ80A-T5	90
Medium-carbon steel	AISI 1060	80
Low-carbon steel	AISI 1020	40
Stainless steel	AISI 304	21
Nickel-base alloy	Inconel 601	20
Phosphor bronze	C 52100 (annealed)	20

Table 3.11: Toughness Merit Number T for Selected Materials Under Tensile Loading

Material	Alloy	$T = S_u[e(2 \text{ in})/100]$ (in-lb/in^3)
Nickel-base alloy	Inconel 601	51,000
Stainless steel	AISI 304	51,000
Phosphor bronze	C 52100 (annealed)	38,500
Ultra-high-strength steel	AISI 4340	31,600
Stainless steel	AM 350	26,800
Aluminum (wrought)	2024-T3 (heat treated)	15,400
Low-carbon steel	AISI 1020	15,300
Titanium	Ti-6Al-4V	15,000
Medium-carbon steel	AISI 1060	9,000
Magnesium (extruded)	AZ80A-T5	3,000

Table 3.12: Hardness of Selected Materials

	Hardness Scale[1]						
Material	BHN	R_C	R_A	R_B	R_M	V	Mohs
Diamond	8500 (approx.)[2]	—	—	—	—	—	10
Sapphire	—	—	—	—	—	—	9
Tungsten carbide	1850 (approx.)[2]	—	93	—	—	—	8–9
Titanium carbide	1850 (approx.)[2]	—	93	—	—	—	8–9
Case-hardened low-carbon steel	650	62	82.5	—	—	—	—
Ultra-high-strength steel	560	56	79	—	—	—	—
Titanium	315	34	67.5	—	—	—	—
Gray cast iron	262	26	—	—	—	—	—
Low-carbon low-alloy steel	207	15	—	—	—	—	—
Medium-carbon steel (CD)[3]	183	(9)[2]	—	89.5	—	—	—
Low-carbon steel (CD)	121	—	—	68	—	127	—
Aluminum (wrought)	120	—	—	67.5	—	126	—
Nickel-base alloy	114	—	—	64	—	120	—
Magnesium (extruded)	82	—	—	49	—	—	—
Commercial bronze	70	—	—	34	—	—	—
Gold (annealed)	—	—	—	—	—	25	—
Epoxy (glass reinforced)	—	—	—	—	105	—	—
Acrylic (cast)	—	—	—	—	85	—	—

[1]BHN = Brinell hardness number
R_C = Rockwell C scale
R_A = Rockwell A scale
R_B = Rockwell B scale
R_M = Rockwell M scale
V = Vickers hardness number
Mohs = Mohs hardness number

[2]Out of normal range—information only.

[3]Cold drawn.

Table 3.13: Galvanic Corrosion Resistance in Sea Water for Selected Materials

↑ Noble or cathodic (protected end)	Platinum
	Gold
	Graphite
	Titanium
	Silver
	[Chlorimet 3 (62 Ni, 18 Cr, 18 Mo) Hastelloy C (62 Ni, 17 Cr, 15 Mo)]
	[18-8 Mo stainless steel (passive) 18-8 stainless steel (passive) Chromium stainless steel 11–30% Cr (passive)]
	[Inconel (passive) (80 Ni, 13 Cr, 7 Fe) Nickel (passive)]
	Silver solder
	[Monel (70 Ni, 30 Cu) Cupronickels (60–90 Cu, 40–10 Ni) Bronzes (Cu-Sn) Copper Brasses (Cu-Zn)]
	[Chlorimet 2 (66 Ni, 32 Mo, 1 Fe) Hastelloy B (60 Ni, 30 Mo, 6 Fe, 1 Mn)]
	[Inconel (active) Nickel (active)]
	Tin
	Lead
	Lead-tin solders
	[18-8 Mo stainless steel (active) 18-8 stainless steel (active)]
	Ni-Resist (high Ni cast iron)
	Chromium stainless steel, 13% Cr (active)
	[Cast iron Steel or iron]
	2024 aluminum (4.5 Cu, 1.5 Mg, 0.6 Mn)
	Cadmium
Active or anodic (corroded end) ↓	Commercially pure aluminum (1100)
	Zinc
	Magnesium and magnesium alloys

[1]See p. 32 of ref. 12. (Reprinted with permission of the McGraw-Hill Companies.)

Table 3.14: Corrosion-Fatigue Strength for Selected Materials

Material	Ultimate Tensile Strength (psi)	Fatigue Strength in Air (psi)	Corrosion-Fatigue Strength in Salt Spray (psi)	Cycles to Failure
Beryllium bronze[2]	94,000	36,500	38,800	5×10^7
18 Cr 8 Ni steel	148,000	53,500	35,500	5×10^7
17 Cr 1 Ni steel	122,000	73,500	27,500	5×10^7
Phosphor bronze[3]	62,000	22,000	26,000	5×10^7
Aluminum bronze	80,000	32,000	22,000	5×10^7
15 Cr steel	97,000	55,000	20,500	5×10^7
carbon steel	142,000	56,000	8,750	5×10^7
Duralumin	63,000	20,500	7,600	5×10^7
Mild steel	76,000	38,000	2,500	10×10^7

[1]See ref. 13.
[2]The apparent anomoly, that fatigue resistance with salt spray is higher than fatigue resistance in air, is acknowledged by the authors of ref. 13, but they stand by the values shown, and offer a supporting explanation.
[3]Ibid.

Table 3.15: Nuclear Radiation Exposure to Produce Significant (over 10%) Changes in Properties of Selected Materials

Material	Amount of Radiation (integrated fast neutron flux) (neutrons/cm^2)	Property Changes
Zirconium alloys	10^{21}	Little change
Stainless steels		Reduced but not greatly impaired ductility
Aluminum alloys	10^{20}	Reduced but not greatly impaired ductility
Stainless steels		Yield strength tripled
Carbon steels		Increased fracture-transition temperature; severe loss of ductility; yield strength doubled
All plastics	10^{19}	Unusable as structural materials
Ceramics		Reduced thermal conductivity, density, and crystallinity
Polystyrene		Loss of tensile strength
Carbon steel	10^{18}	Reduction of notch-impact strength
Metals		Most show significant increase in yield strength
Natural rubber		Large change; hardening
Mineral-filled phenolic	10^{17}	Loss of tensile strength
Polyethylene		Loss of tensile strength
Butyl rubber		Large change; softening
Natural and butyl rubber	10^{16}	Loss of elasticity
Polymethyl methacrylate and cellulosics	10^{15}	Loss of tensile strength
Polytetrafluroethylene		Loss of tensile strength

[1]See ref. 14.

Table 3.16: Suitability of Selected Materials for Specific Manufacturing Process

Material	Alloy	Available Forms[1]	Fabrication Properties
Ultra-high-strength steel	AISI 4340	B,b,f,p,S,s,w	Readily machinable (annealed); readily weldable (post-heat required)
Stainless steel	AM350	b,F,S,s,w,t	Readily machinable; readily weldable
Graphite-epoxy composite	—	Injection, compression, and transfer molding	—
Titanium	Ti-6A1-4V	B,b,P,S,s,w,e	Machinable (annealed), formable, weldable
Nickel-base alloy	Inconel 601	b,P,r,S,s,Sh,t	—
Medium-carbon steel	AISI 1060	b,r,f	Readily machinable; welding not recommended
Stainless steel	AISI 304	b,P,f,S,s,t,w	Readily machinable; readily weldable
Commercial bronze	C 22000 (hard)	P,r,S,s,t,w	Machinable; readily weldable
Low-carbon steel	AISI 1020	b,r,f,S,Sh	Readily machinable; readily weldable
Phosphor bronze	C 52100	r,s,w	Machinable; readily weldable
Gray cast iron	ASTM A-48 (class 50)	—	Machinable; weldable
Aluminum (wrought)	2024-T3	b,P,r,S,Sh,t,w	Easily machinable; weldable
Magnesium (extruded)	AZ80A-T5	b,r,f,Sh	Easily machinable (except fire hazard); weldable
Thermosetting polymer	Epoxy (glass reinforced)	Injection, compression, and transfer molding	—
Thermoplastic polymer	Acrylic (cast)	—	Machinable

[1]B = billets, b = bars, e = extrusions, F = foil, f = forgings, P = plates, r = rods, S = sheets, s = strip, Sh = shapes, t = tubing, w = wire

Table 3.17: Approximate Material Cost for Selected Materials

Material	Approximate cost (dollars/lb)
Gray cast iron	0.30
Low-carbon steel (HR)[2]	0.50
Low-carbon steel (CD)[3]	0.60
Ultra-high-strength steel (HR)	0.65
Zinc alloy	1.50
Acrylic	2.00
Commercial bronze	2.25
Stainless steel	2.75
Epoxy (glass reinforced)	3.00
Aluminum alloy	3.50
Magnesium alloy	5.50
Titanium alloy	9.50

[1]Material cost varies widely by year of purchase and by quantity required. Designers should always obtain specific price quotations.
[2]Hot-rolled.
[3]Cold-drawn.

Table 3.18: Relative Machinability of Selected Materials

Material	Alloy	Estimated Machinability Index[2]
Magnesium alloy	—	400
Aluminum alloy	—	300
Free-machining steel	B1112	100
Low-carbon steel	AISI 1020	65
Medium-carbon steel	AISI 1060 (annealed)	60
Ultra-high-strength steel	AISI 4340 (annealed)	50
Stainless steel alloy	(annealed)	50
Gray cast iron	—	40
Commercial bronze	—	30
Titanium alloy	(annealed)	20

[1]Machinability index is a less-than-exact evaluation of volume of material removal per hour, produced at maximum efficiency, balanced against a minimum rejection rate for reasons of surface finish or tolerance.
[2]Based on rating of 100 for B1112 resulfurized free-machining steel.

Table 3.19: Thermal Conductivity Ranges for Selected Materials

Material	Thermal Conductivity k [Btu/hr/ft/°F (W/m/°C)]
Silver	242 (419)
Copper	112 (194)–226 (391)
Pyrolytic graphite	108 (186.9)–215 (372.1)
Beryllium copper	62 (107)–150 (259)
Brass[1]	15 (26)–135 (234)
Aluminum alloys[1]	93 (161)–125 (216)
Bronze[1]	20 (35)–120 (207)
Phosphor bronze[1]	29 (50)–120 (207)
Premium graphite	65 (112)–95 (164)
Carbon graphite	18 (31)–66 (114)
Aluminum bronze[1]	39 (68)
Cast iron	25 (43)–30 (52)
Carbon steel	27 (46.7)
Silicon carbide	9 (15)–25 (43)
Lead	16 (28)–20 (35)
Stainless steel	15 (26)
Titanium	4 (7)–12 (21)
Glass	1 (1.7)–2 (3.5)
Wood composition board (tempered hardboard)	1 (1.7)–1.5 (2.6)
Silicon plastics	0.075 (0.13)–0.5 (0.87)
Phenolics	0.116 (0.201)–0.309 (0.535)
Epoxies	0.1 (0.17)–0.3 (0.52)
Teflon	0.14 (0.24)
Nylon	0.1 (0.17)–0.14 (0.24)
Plastic foam	0.009 (0.016)–0.077 (0.133)

[1]For porous metal sintered parts, thermal conductivity values are 35–65% of values shown.

Example 3.1.

It is desired to design a pressure vessel that will leak before it breaks[2]. The reason for this is that the leak can be easily detected before the onset of rapid crack propagation that might cause an explosion of the pressure vessel due to brittle behavior. To accomplish the *leak-before-break* goal, the vessel should be designed so that it can tolerate a crack having a length, a, at least equal to the wall thickness, t, of the pressure vessel without failure by rapid crack propagation. A specification statement for design of this thin-walled pressure vessel has been written as follows:

The pressure vessel should experience slow through-the thickness crack propagation to cause a leak before the onset of gross yielding of the pressure vessel wall.

From evaluation of this specification statement using Tables 3.1 and 3.2, the important evaluation indices have been deduced to be *high fracture toughness* and *high yield strength.*

By combining (2-21) and (9-5), keeping in mind the "separable" quality of the materials parameter $f_3(M)$ discussed in Example 3.2, the materials-based performance index for this case has been found to be

$$f_3(M) = \frac{K_c}{S_{yp}} \tag{3.1}$$

It is also desired to keep the vessel wall as thin as possible (corresponds to selecting materials with yield strength as high as possible).

a) Using the Ashby charts shown in Figures 3.1 through 3.6, select tentative material candidates for this application.
b) Using the rank-ordered-data tables of Table 2.1 and Tables 3.3 through 3.20, select tentative material candidates for this application.
c) Compare results of parts (a) and (b).

Use the following information to solve the problem,

$$\frac{K_c}{\sqrt{\pi}\,S} = 10\sqrt{mm} \qquad\qquad S \geq 40 \text{ MPa} \tag{3.2}$$

First note that from the problem statement:

From evaluation of this specification statement using Tables 3.1 and 3.2, the important evaluation indices have been deduced to be *high fracture toughness* and *high yield strength.*

We can deduce that we are only interested in materials with high yielding (strength to volume ratio) and ability to dissipate energy plastically (toughness). Note that the problem specification statement does not involve stiffness.

[2]By break we mean onset of nominal gross-section yielding of the pressure vessel wall.

Table 3.20: Potential Material-Specific Application Requirements

Potential Application Requirement	Special Need?
1. Strength/volume ratio	: YES
2. Strength/weight ratio	: NO
3. Strength at elevated temperature	: NO
4. Long-term dimensional stability at elevated temperature	: NO
5. Dimensional stability under temperature fluctuation	: NO
6. Stiffness	: NO
7. Ductility	: NO
8. Ability to store energy elastically	: NO
9. Ability to dissipate energy plastically	: YES
10. Wear resistance	: NO
11. Resistance to chemically reactive environment	: NO
12. Resistance to nuclear radiation environment	: NO
13. Desire to use specific manufacturing process	: NO
14. Cost constraints	: NO
15. Procurement time constraints	: NO

Solution Using Rank-Ordered-Data Tables
Thus for the performance evaluation we need Tables 3.3 and 3.12. Now let us make a short list of candidate materials from each of these tables:

Table 3.3 (Page 132) For high yield strength:

1. Ultra-high-strength steel (AISI 4340)
2. Stainless steel (AM 350)
3. High-carbon steel
4. Graphite-epoxy composite
5. Ti-6Al-4V titanium
6. Ceramic
7. Nickel-based alloy (Inconel 601)
8. Medium-carbon steel

Table 3.12 (Page 137) For high toughness:

1. Nickel-based alloy (Inconel 601)
2. Stainless steel (AISI 304)
3. Phosphor bronze
4. Ultra-high-strength Steel (AISI 4340)
5. Stainless steel (AM 350)
6. Al 2024-T3
7. Low-carbon steel
8. Ti-6Al-4V titanium

Surveying the above list, the three best candidate materials with the best potential appear to be:

1. Nickel-based alloy (Inconel 601)
2. Stainless steel (AM 350)
3. Ultra-high-strength Steel (AISI 4340)

The above list has not being giving in order of priority. Titanium was not included because was not listed in the top list of each table. However, nickel-based alloy was chosen because in Table 3.12 was on the top of the list.

Let's use the Pugh's method and take the ultra-high-strength steel as our datum. We shall use the following scoring system:

+2 meets criterion much better than datum

+1 meets criterion better than datum

0 meets criterion as well as datum

−1 meets criterion not as well as datum

−2 meets criterion much worse then the datum

Note that we take only those criteria relevant to the problem.

Table 3.21: Rank-Ordered

Criteria	Score (0-100%)	A Inconel 601	B AM 350	DATUM AISI 4340
1. Yield strength	60	-2	-1	0
9. Toughness	40	2	-1	0
Total	100	0	-2	0
Total Positive	—	2	0	0
Total Negative	—	-2	-2	0
Weighted Total	—	-40	-100	0

Thus the best candidate is the ultra-high-strength steel (AISI 4340).

Solution Using Ashby Charts

Given the performance index

$$\frac{K_{\mathrm{c}}}{\sqrt{\pi}\,S} = 10\sqrt{\mathrm{mm}} \qquad\qquad S \geq 40 \text{ MPa} \tag{3.3}$$

Figure 3.5 may be choosen to mark boundaries for the given performance index and the bound on high yield strength materials. Figure 3.5 is the two-parameter Ashby chart for plane strain fracture toughness, K/c, plotted versus failure strength, S. The content of the chart roughly corresponds to the data included in Tables 2.1 and 3.4.

We plot the red line that corresponds to:

$$\frac{K_c}{\sqrt{\pi}\, S} = 10\sqrt{mm} \qquad \rightarrow \qquad \frac{K_c^2}{\pi\, S^2} = 100\,\text{mm} \tag{3.4}$$

and we limit our search to the above of the curve since we are interested to yield before fracture.

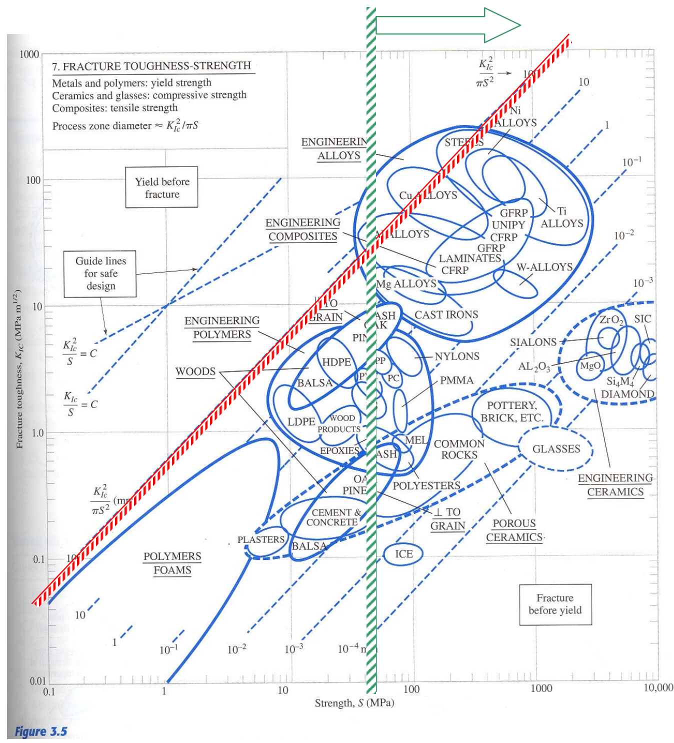


Figure 3.2: The blue line represents the line of constant toughness to strength ratio and the green line represent the strength bound.

The material candidates included within the search region are steels, Cu Alloys, and Al Alloys. These “short-name” identifiers may be interpreted from Table 2-4 (Shigley’s 9th Edition) as:

1. Steels
2. Copper alloys
3. Aluminum alloys

Since we are interested in a high yield strength, the recommended material should be a steel.

Comparison

Both procedures would agree upon Ultra-high-strength steel as a primary candidate material. The second choice could be stainless steel.

End Example □

3.3 References

Collins, J. A., *Mechanical Design of Machine Elements and Machines*, 2003, John Wiley and Sons, New York, NY.

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Juvinall, R. C., and Marsheck, K. A., *Fundamentals of Machine Component Design*, 2000, John Wiley and Sons, New York, NY.

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3.4 Suggested Problems

Problem 3.1.

It is desired to manufacture a fishing rod that is made of a light and stiff material. Using both the ranked-order-data method and the Ashby charts, determine which is better, a rod made of plastic (without fiber reinforcement) or a split-cane rod (bamboo fibers glued together).

□

Problem 3.2.

It is desired to manufacture a engine support whose fundamental natural frequency should not exceed $\omega = 2$ rpm. The engine support can be modeled as uniformly simply-supported beam. The length L of the member is 5.0 m, the width b=10.0 cm, and the depth h=20.0 cm. The cross-sectional moment of inertia is

$$I = \frac{b\,h^3}{12} = 66.67 \times 10^{-6} \quad \text{m}^4$$

The natural frequency of the support can be expressed as follows:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad k = \frac{48\,E\,I}{L^3} \qquad m = \rho\,V \tag{3.5}$$

where V is the total volume of the support, E the Young's Modulus, and ρ the material density. It is desired to suggest a reasonable material for this application.

□

Problem 3.3.

Suggested problems from chapter 3 of course textbook:

1. Problem 3.1: In addition to the original specifications, the clevis should be low-cost and capable of high production rate.
2. Problem 3.2: Choose the most appropriate material for the bridge that should be light, low volume, high static strength, high stiffness, and high corrosion resistant.
3. Problem 3.3: The support should have good strength at elevated temperature, low creep rate and good stress rupture of elevated temperature (good creep rapture resistance).

□

Chapter 4
Load Analysis

Instructional Objectives of Chapter 4

After completing this chapter, the student should be able to:

1. Fully understand the importance of units.
2. Determine load analysis of airplane structural components under both static and dynamic loading.
3. Draw load diagrams using common analytical and discrete solutions.

This chapter presents a brief review of Newton's laws and Euler's equations as applied to dynamically-loaded and steady-loaded systems in 3-D. The concepts and methods used in this chapter are usually presented in previous statics and dynamics courses. Students are encouraged to review their static and dynamic course contents.

4.1 Newton's Laws

Most of the problems in structural analysis deal with static and dynamic analyses. In fact, static loading can be considered as a special case of the dynamic one. The most popular method for the dynamic analysis is the Newtonian approach based on Newton's laws and is generally used to obtain information about internal forces. The three Newton's Laws can be briefly summarized as follows:

Newton's First Law Newton's First Law states that a body at rest tends to remain at rest and a body in motion at constant velocity will tend to maintain that velocity unless acted upon by an external force.

Newton's Second Law Newton's Second Law states that the time rate of change of momentum of a body is equal to the magnitude of the applied force and acts in the direction of the force.

Newton's Third Law Newton's Third Law states that when two particles interact, a pair of equal and opposite reaction forces will exist at their contact point. This force pair will have the same magnitude and act along the same direction line, but have opposite sense.

4.2 Units

In all engineering problems, we must deal with units carefully. Each parameter in the problem may have a specific unit system. A unit may be defined as a specified amount of a physical quantity by which through comparison another quantity of the same kind is measured. It our job to ensure that we are working in the proper unit system and make the corresponding conversions, should it be necessary.

4.2.1 Importance of Units

Equations from physics and engineering that relate physical quantities are dimensionally homogeneous. Dimensionally homogeneous equations must have the same dimensions for each term. Newton's second law relates the dimensions force, mass, length, and time:

$$\begin{gathered} \underline{\mathbf{F}} \quad \alpha \quad m\,\underline{\mathbf{a}} \\ [F] = \frac{[M]\,[L]}{[T]^2} \end{gathered} \tag{4.1}$$

If length and time are primary dimensions, Newton's second law, being dimensionally homogeneous, requires that both force and mass cannot be primary dimensions without introducing a constant of proportionality that has dimension (and units).

Because physical quantities are related by laws and definitions, a small number of physical quantities, called primary dimensions, are sufficient to conceive of and measure all others. Primary dimensions in all systems of dimensions in common use length and time. Force is selected as a primary dimension in some systems. Mass is taken as a primary dimension in others. For application in mechanics, we have four basic systems of dimensions:

1. force $[F]$, mass $[M]$, length $[L]$, time $[T]$
2. force $[F]$, length $[L]$, time $[T]$
3. mass $[M]$, length $[L]$, time $[T]$

In system 1, length $[L]$, time $[T]$, and both force $[F]$ and mass $[M]$ are selected as primary dimensions. In this system, in Newton's second law

$$F = \frac{m\,a}{g_c}$$

where the constant of proportionality, g_c, is not dimensionless. For Newton's law to be dimensionally homogeneous, the dimensions of g_c must be:

$$g_c = \frac{[M]\,[L]}{[F]\,[T]^2}$$

In system 2, mass $[M]$ is a secondary dimension, and in Newton's second law the constant of proportionality is dimensionless. In system 3, force $[F]$ is a secondary dimension, and in Newton's second law the constant of proportionality is again dimensionless. The measuring units selected for each primary physical quantities determine the numerical value of the constant of proportionality.

Secondary dimensions are those quantities measured in terms of the primary dimensions. For example, if mass, length, and time are primary dimensions, area, density, and velocity would be secondary dimensions.

4.2.2 Systems of Units

Four different systems of units can be identified:

1. **Système International d'Unitès (SI)**

   mass : kilogram (kg)
   length : meter (m)
   temperature : Celsius (°C) or Kelvin (°K)
   time : second (s)
   force : newton (N)

2. **English Engineering**

   mass : pound mass (lbm)
   length : feet (ft)
   temperature : Rankine (°R)
   time : second (s)
   force : pound force (lb or lbf)

3. **British Engineering: foot-pound-second (fps)**

   mass : slug (slug)
   length : feet (ft)
   temperature : Fahrenheit (°F)
   time : second (s)
   force : pound force (lb)

4. **British Engineering: inch-pound-second (ips)**

mass : slug (slug)
length : inch (in)
temperature : Fahrenheit (°F)
time : second (s)
force : pound force (lb)

Example 4.1.

A special payload package is to be delivered to the surface of the moon. A prototype of the package, developed, constructed, and tested near Boston, has been determined to have a mass of 24.0 kg. Assume $g_{\text{Boston}} = 9.77$ m/sec^2 and $g_{\text{moon}} = 1.7$ m/sec^2. Show all your work.

4.1a) Estimate the weight of the package, using the international system, as measured near Boston.

4.1b) Estimate the weight of the package, using the international system, on the surface of the moon.

4.1c) Reexpress the weights using *fps* and *ips* systems.

Let's use Eq. (4.1),

$$F = m\,a \qquad \rightarrow \qquad W = m\,g \tag{4.2}$$

Thus,

4.1a) Estimate the weight of the package, using the international system, as measured near Boston.

$$W_{\text{Boston}} = m\,g_{\text{Boston}} = \Big(24.0\,\text{kg}\Big)\cdot\Big(9.77\,\text{m/sec}^2\Big) = 234.48\,\frac{\text{kg}\cdot\,\text{m}}{\text{sec}^2} = 234.48\,\text{N}$$

4.1b) Estimate the weight of the package, using the international system, on the surface of the moon.

$$W_{\text{moon}} = m\,g_{\text{moon}} = \Big(24.0\,\text{kg}\Big)\cdot\Big(1.7\,\text{m/sec}^2\Big) = 40.8\,\frac{\text{kg}\cdot\,\text{m}}{\text{sec}^2} = 40.8\,\text{N}$$

4.1c) Reexpress the weights using *fps* and *ips* systems.

Using Tables,

$$W_{\text{Boston}} = \Big(234.48\,\text{N}\Big)\cdot\left(\frac{1\,\text{lb}}{4.448\,\text{N}}\right) = 52.72\,\text{lb} \qquad \text{in both units}$$

$$W_{\text{moon}} = \Big(40.8\,\text{N}\Big)\cdot\left(\frac{1\,\text{lb}}{4.448\,\text{N}}\right) = 9.17\,\text{lb} \qquad \text{in both units}$$

Alternative approach:

Note that the mass can be expressed as follows,

$$m_{fps} = \left(24.0\,\text{kg}\right)\cdot\left(\frac{2.21\,\text{lbm}}{1.0\,\text{kg}}\right)\cdot\left(\frac{1.0\,\text{slug}}{32.17\,\text{lbm}}\right) = 1.65\,\text{slug} = 1.65\,\frac{\text{lb–sec}^2}{\text{ft}}$$

$$m_{ips} = \left(1.65\,\frac{\text{lb–sec}^2}{\text{ft}}\right)\cdot\left(\frac{1.0\,\text{ft}}{12.0\,\text{in}}\right) = 0.1375\,\frac{\text{lb–sec}^2}{\text{in}}$$

and the gravitational constant at Boston and at the moon are

$$g_{\text{Boston}_{ips}} = \left(9.77\,\frac{\text{m}}{\text{sec}^2}\right)\cdot\left(\frac{1000.0\,\text{mm}}{1.0\,\text{m}}\right)\cdot\left(\frac{1.0\,\text{in}}{25.4\,\text{mm}}\right) = 384.65\,\frac{\text{in}}{\text{sec}^2}$$

$$g_{\text{Boston}_{fps}} = \left(384.65\,\frac{\text{in}}{\text{sec}^2}\right)\cdot\left(\frac{1\,\text{ft}}{12\,\text{in}}\right) = 32.05\,\frac{\text{ft}}{\text{sec}^2}$$

$$g_{\text{moon}_{ips}} = \left(1.7\,\frac{\text{m}}{\text{sec}^2}\right)\cdot\left(\frac{1000.0\,\text{mm}}{1.0\,\text{m}}\right)\cdot\left(\frac{1.0\,\text{in}}{25.4\,\text{mm}}\right) = 66.93\,\frac{\text{in}}{\text{sec}^2}$$

$$g_{\text{moon}_{fps}} = \left(66.93\,\frac{\text{in}}{\text{sec}^2}\right)\cdot\left(\frac{1\,\text{ft}}{12\,\text{in}}\right) = 5.58\,\frac{\text{ft}}{\text{sec}^2}$$

Thus we could have also obtained the results by,

$$W_{\text{moon}} = \left(1.65\,\text{slugs}\right)\cdot\left(5.58\,\frac{\text{ft}}{\text{sec}^2}\right) = 362.70\,\frac{\text{slug–ft}}{\text{sec}^2} = 9.20\,\text{lb}$$

$$W_{\text{moon}} = \left(0.1375\,\frac{\text{lb–sec}^2}{\text{in}}\right)\cdot\left(66.93\,\frac{\text{in}}{\text{sec}^2}\right) = 9.20\,\text{lb}$$

$$W_{\text{Boston}} = \left(1.65\,\text{slugs}\right)\cdot\left(32.05\,\frac{\text{ft}}{\text{sec}^2}\right) = 52.88\,\frac{\text{slug–ft}}{\text{sec}^2} = 52.88\,\text{lb}$$

$$W_{\text{Boston}} = \left(0.1375\,\frac{\text{lb–sec}^2}{\text{in}}\right)\cdot\left(384.65\,\frac{\text{in}}{\text{sec}^2}\right) = 52.88\,\text{lb}$$

End Example □

4.3 Load Analysis

Many problems deal with constant velocity, or zero velocity (static), in such cases Newton's Second Law reduces to:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$
$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0 \tag{4.3}$$

Note that the above is just a special case of the dynamic loading situation but with zero accelerations.

4.3.1 Internal Force Sign Convention

Here we will always assumes all unknown forces and moments on the system to be positive in sign as shown in Figure 4.1, regardless of what one's intuition or an inspection of the free-body diagram might indicate as to their probable directions. However, all known force components are given their proper signs to define their directions. The simultaneous solution of the set of equations that results will cause all the unknown components to have the proper signs when the solution is complete. If the loads act on the opposite direction it results in a sign reversal on that component in the solution.

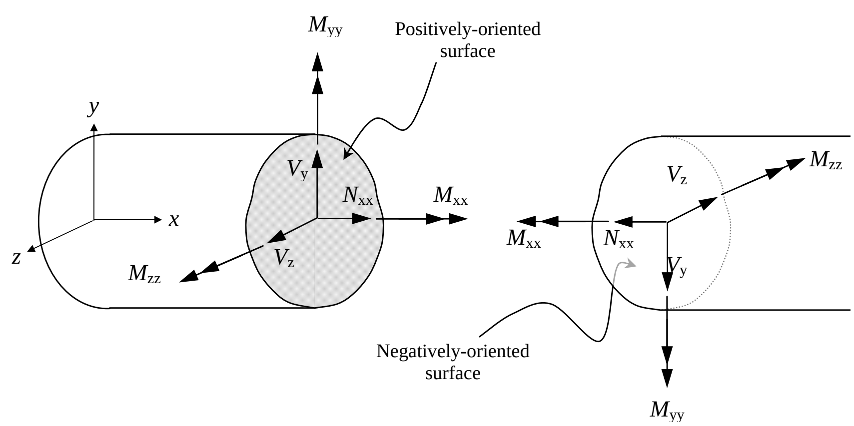


Figure 4.1: Positive sign convention.

We will need to apply the second law in order to solve for the forces on assemblies of elements that act upon one another. The six equations can be written in a 3-D system. In addition, as many (third-law) reaction force equations as are necessary will be written and the resulting set of equations solved simultaneously for the forces and moments. The number of second-law equations will be up to six times the number of individual parts in a three-dimensional system (plus the reaction equations), meaning that even simple systems result in large sets of simultaneous equations. The reaction (third-law) equations are often substituted into the second-law equations to reduce the total number of equations to be solved simultaneously.

Example 4.2.

For the given problem obtain all the reactions at point **O** (clamped-end). Use the shown sign convention (it is similar to the one used on class). The loads P and T act in the x-y plane. The length of bar CB is L, of bar BA is $2\,L$, and of bar OA is $3\,L$.

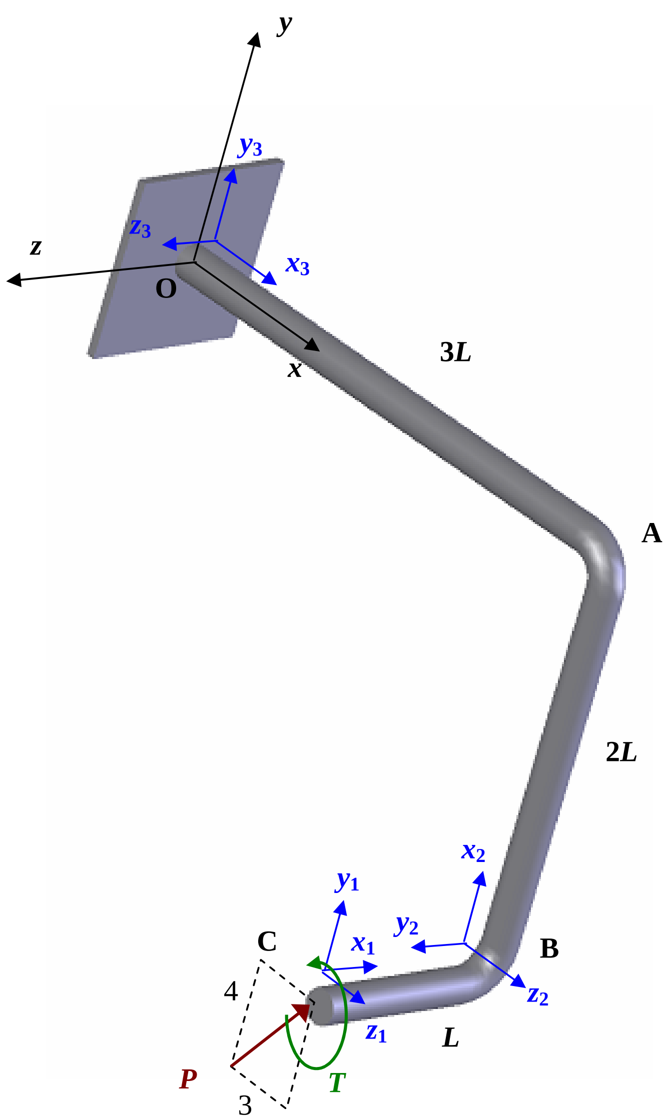


Figure 4.2: Three-dimensional bar-structure.

4.2a) Draw free-body diagrams of the each section **OA**, **BA**, **CB**.

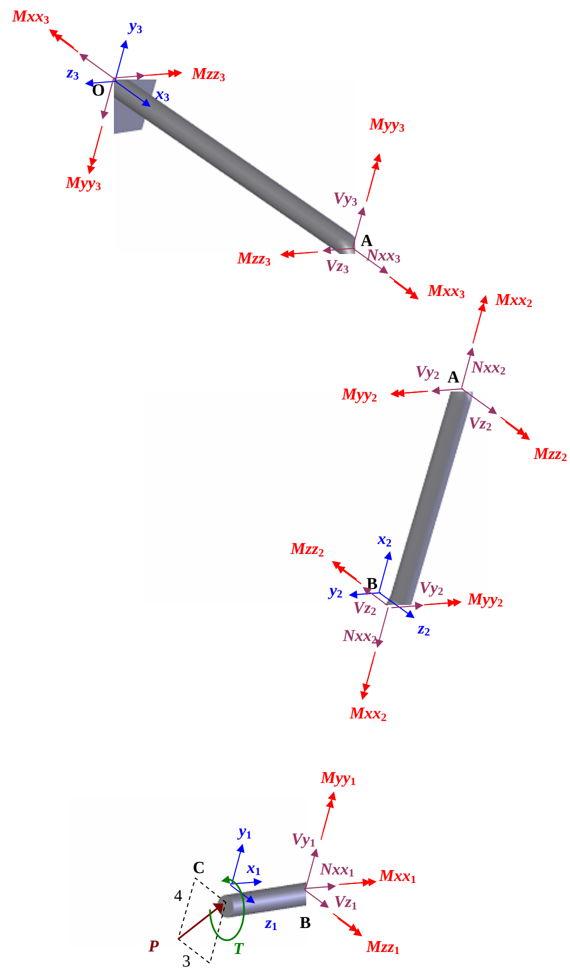


Figure 4.3: Free body diagrams for the three-dimensional bar-structure.

4.2b) Obtain the internal loads at **B**.

First of all, the coordinate system can be changed from bar to bar whenever we are consistent are clear about what we are doing. In this context, let us use a local coordinate system such that x-axis always goes along the main axis of the bar. In order to do so, let us use subscript"1" to refer to the first bar, "2" for the second bar and "3" for the third bar. This will avoid any confusion as to what coordinate system we are working with.

Next, we proceed to find the loads at **B**:

$$+\uparrow \sum F_{\mathrm{y}} = 0 \quad \Rightarrow \quad V_{\mathrm{y1}}(x_1) + \frac{4}{5}\,P = 0$$

$$V_{\mathrm{y1}}(x_1) = -\frac{4}{5}\,P$$

$$V_{\mathrm{y1}}(x_1 = L) = -\frac{4}{5}\,P$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{x}} = 0 \quad \Rightarrow \quad N_{\mathrm{xx1}}(x_1) + 0 = 0$$

$$N_{\mathrm{xx1}}(x_1) = 0$$

$$N_{\mathrm{xx1}}(x_1 = L) = 0$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{z}} = 0 \quad \Rightarrow \quad V_{\mathrm{z1}}(x_1) + \frac{3}{5}\,P = 0$$

$$V_{\mathrm{z1}}(x_1) = -\frac{3}{5}\,P$$

$$V_{\mathrm{z1}}(x_1 = L) = -\frac{3}{5}\,P$$

$$+\circlearrowleft \sum M_{y_B} = 0 \quad \Rightarrow \quad M_{yy1}(x_1) + \frac{3}{5}\, P\; x_1 = 0$$

$$M_{yy1}(x_1) = -\frac{3}{5}\, P\, x_1$$

$$M_{yy1}(x_1 = L) = -\frac{3}{5}\, P\, L$$

$$+\circlearrowleft \sum M_{x_B} = 0 \quad \Rightarrow \quad M_{xx1}(x_1) - T = 0$$

$$M_{xx1}(x_1) = T$$

$$M_{xx1}(x_1 = L) = T$$

$$+\circlearrowleft \sum M_{z_B} = 0 \quad \Rightarrow \quad M_{zz1}(x_1) - \frac{4}{5}\, P\; x_1 = 0$$

$$M_{zz1}(x_1) = \frac{4}{5}\, P\; x_1$$

$$M_{zz1}(x_1 = L) = \frac{4}{5}\, P\; L$$

4.2c) Obtain the internal loads at **A**.

From action reaction at **B**:

$$M_{xx2}(x_2 = 0) = M_{yy1}(x_1 = L) = -\frac{3}{5}\, P\, L$$

$$M_{yy2}(x_2 = 0) = -M_{xx1}(x_1 = L) = -T$$

$$M_{zz2}(x_2 = 0) = M_{zz1}(x_1 = L) = \frac{4}{5}\, P\, L$$

$$N_{xx2}(x_2 = 0) = V_{y1}(x_1 = L) = -\frac{4}{5}\, P$$

$$V_{y2}(x_2 = 0) = -N_{xx1}(x_1 = L) = 0$$

$$V_{z2}(x_2 = 0) = V_{z1}(x_1 = L) = -\frac{3}{5}\, P$$

Next, we proceed to find the loads at **A**:

$$+\uparrow \sum F_{\mathrm{y}} = 0 \quad \Rightarrow \quad -V_{\mathrm{y2}}(x_2 = 0) + V_{\mathrm{y2}}(x_2) = 0$$

$$V_{\mathrm{y2}}(x_2) = V_{\mathrm{y2}}(x_2 = 0)$$

$$V_{\mathrm{y2}}(x_2 = 2\,L) = V_{\mathrm{y2}}(x_2 = 0) = -N_{\mathrm{xx1}}(x_1 = L) = 0$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{x}} = 0 \quad \Rightarrow \quad -N_{\mathrm{xx2}}(x_2 = 0) + N_{\mathrm{xx2}}(x_2) = 0$$

$$N_{\mathrm{xx2}}(x_2) = N_{\mathrm{xx2}}(x_2 = 0)$$

$$N_{\mathrm{xx2}}(x_2 = 2\,L) = N_{\mathrm{xx2}}(x_2 = 0) = V_{\mathrm{y1}}(x_1 = L) = -\frac{4}{5}\,P$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{z}} = 0 \quad \Rightarrow \quad -V_{\mathrm{z2}}(x_2 = 0) + V_{\mathrm{z2}}(x_2) = 0$$

$$V_{\mathrm{z2}}(x_2) = V_{\mathrm{z2}}(x_2 = 0)$$

$$V_{\mathrm{z2}}(x_2 = 2\,L) = V_{\mathrm{z2}}(x_2 = 0) = V_{\mathrm{z1}}(x_1 = L) = -\frac{3}{5}\,P$$

$$+\circlearrowleft \sum M_{\mathrm{y}_A} = 0 \quad \Rightarrow \quad -M_{\mathrm{yy2}}(x_2 = 0) + M_{\mathrm{yy2}}(x_2) - V_{\mathrm{z2}}(x_2 = 0)\, x_2 = 0$$

$$M_{\mathrm{yy2}}(x_2) = M_{\mathrm{yy2}}(x_2 = 0) + V_{\mathrm{z2}}(x_2 = 0)\, x_2$$

$$M_{\mathrm{yy2}}(x_2) = -T - \frac{3}{5}\, P\, x_2$$

$$M_{\mathrm{yy2}}(x_2 = 2\,L) = -T - \frac{6}{5}\, P\, L$$

$$+\circlearrowleft \sum M_{\mathrm{x}_A} = 0 \quad \Rightarrow \quad -M_{\mathrm{xx2}}(x_2 = 0) + M_{\mathrm{xx2}}(x_2) = 0$$

$$M_{\mathrm{xx2}}(x_2) = M_{\mathrm{xx2}}(x_2 = 0) = M_{\mathrm{yy1}}(x_1 = L)$$

$$M_{\mathrm{xx2}}(x_2 = 2\,L) = -\frac{3}{5}\, P\, L$$

$$+\circlearrowleft \sum M_{\mathrm{z}_A} = 0 \quad \Rightarrow \quad -M_{\mathrm{zz2}}(x_2 = 0) + M_{\mathrm{zz2}}(x_2) + V_{\mathrm{y2}}(x_2 = 0)\, x_2 = 0$$

$$M_{\mathrm{zz2}}(x_2) = M_{\mathrm{zz2}}(x_2 = 0) + V_{\mathrm{y2}}(x_2 = 0)\, x_2$$

$$M_{\mathrm{zz2}}(x_2) = \frac{4}{5}\, P\, L + 0$$

$$M_{\mathrm{zz2}}(x_2 = 2\,L) = \frac{4}{5}\, P\, L$$

4.2d) Obtain the internal loads at **O**.

From action reaction at **A**:

$$M_{\mathrm{xx3}}(x_3 = 3\,L) = -M_{\mathrm{zz2}}(x_2 = 2\,L) = -\frac{4}{5}\, P\, L$$

$$M_{\mathrm{yy3}}(x_3 = 3\,L) = -M_{\mathrm{xx2}}(x_2 = 2\,L) = \frac{3}{5}\, P\, L$$

$$M_{\mathrm{zz3}}(x_3 = 3\,L) = -M_{\mathrm{yy2}}(x_2 = 2\,L) = T + \frac{6}{5}\, P\, L$$

$$N_{\mathrm{xx3}}(x_3 = 3\,L) = -V_{\mathrm{z2}}(x_2 = 2\,L) = \frac{3}{5}\, P$$

$$V_{\mathrm{y3}}(x_3 = 3\,L) = -N_{\mathrm{xx2}}(x_2 = 2\,L) = \frac{4}{5}\, P$$

$$V_{\mathrm{z3}}(x_3 = 3\,L) = V_{\mathrm{y2}}(x_2 = 2\,L) = 0$$

Next, we proceed to find the loads at **O**:

$$+\uparrow \sum F_{\mathrm{y}} = 0 \quad \Rightarrow \quad -V_{\mathrm{y3}}(x_3 = 3\,L) + V_{\mathrm{y3}}(x_3) = 0$$

$$V_{\mathrm{y3}}(x_3 = 3\,L) = V_{\mathrm{y3}}(x_3)$$

$$V_{\mathrm{y3}}(x_3 = 3\,L) = V_{\mathrm{y3}}(x_3 = 0) = -N_{\mathrm{xx2}}(x_2 = 2\,L) = \frac{4}{5}\,P$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{x}} = 0 \quad \Rightarrow \quad -N_{\mathrm{xx3}}(x_3 = 3\,L) + N_{\mathrm{xx3}}(x_3) = 0$$

$$N_{\mathrm{xx3}}(x_3 = 3\,L) = N_{\mathrm{xx3}}(x_3)$$

$$N_{\mathrm{xx3}}(x_3 = 3\,L) = N_{\mathrm{xx3}}(x_3 = 0) = -V_{\mathrm{z2}}(x_2 = 2\,L) = \frac{3}{5}\,P$$

$$\underset{\rightarrow}{+} \sum F_{\mathrm{z}} = 0 \quad \Rightarrow \quad -V_{\mathrm{z3}}(x_3 = 3\,L) + V_{\mathrm{z3}}(x_3) = 0$$

$$V_{\mathrm{z3}}(x_3 = 3\,L) = V_{\mathrm{z3}}(x_3)$$

$$V_{\mathrm{z3}}(x_3 = 3\,L) = V_{\mathrm{z3}}(x_3 = 0) = V_{\mathrm{y2}}(x_2 = 2\,L) = 0$$

$$+\circlearrowleft \sum M_{\mathrm{y}_O} = 0 \quad \Rightarrow \quad -M_{\mathrm{yy}3}(x_3 = 3\,L) + M_{\mathrm{yy}3}(x_3) - V_{\mathrm{z}3}(x_3 = 3\,L)\,x_3 = 3\,L$$

$$M_{\mathrm{yy}3}(x_3) = M_{\mathrm{yy}3}(x_3 = 3\,L) + V_{\mathrm{z}3}(x_3 = 3\,L)\,x_3$$

$$M_{\mathrm{yy}3}(x_3) = \frac{3}{5}\,P\,L + 0$$

$$M_{\mathrm{yy}3}(x_3 = 0) = \frac{3}{5}\,P\,L$$

$$+\circlearrowleft \sum M_{\mathrm{x}_O} = 0 \quad \Rightarrow \quad -M_{\mathrm{xx}3}(x_3 = 3\,L) + M_{\mathrm{xx}3}(x_3) = 0$$

$$M_{\mathrm{xx}3}(x_3 = 3\,L) = M_{\mathrm{xx}3}(x_3) = -M_{\mathrm{zz}2}(x_2 = 2\,L)$$

$$M_{\mathrm{xx}3}(x_3 = 3\,L) = M_{\mathrm{xx}3}(x_3 = 0) = -\frac{4}{5}\,P\,L$$

$$+\circlearrowleft \sum M_{\mathrm{z}_O} = 0 \quad \Rightarrow \quad -M_{\mathrm{zz}3}(x_3 = 3\,L) + M_{\mathrm{zz}3}(x_3) + V_{\mathrm{y}3}(x_3 = 3\,L)\,x_3 = 3\,L$$

$$M_{\mathrm{zz}3}(x_3) = M_{\mathrm{zz}3}(x_3 = 3\,L) + V_{\mathrm{y}3}(x_3 = 3\,L)\,x_3$$

$$M_{\mathrm{zz}3}(x_3) = T + \frac{6}{5}\,P\,L + \frac{4}{5}\,P\,x_3$$

$$M_{\mathrm{zz}3}(x_3 = 0) = T + \frac{18}{5}\,P\,L$$

End Example □

4.4 Load Diagrams

In the most general case, a structural component may have all type of loadings: torsional, bending and axial. Before we proceed, let us discuss three different sign conventions, which are typically used.

4.4.1 Sign Conventions

Stress Convention

In this course, problems will be solved using the following sign convention

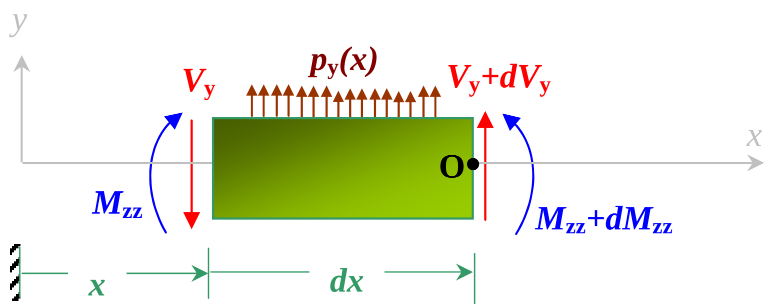


Figure 4.4: Equilibrium element supporting a general force system under the stress convention in the x-y plane.

Sum of forces in the y-direction, will give us an equation for the shear:

$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad -V_y(x) + \{V_y(x) + dV_y(x)\} + p_y(x)\,dx = 0$$

divide by dx and take $\lim_{dx\to 0}$

$$\frac{dV_y(x)}{dx} = -p_y(x) \tag{4.4}$$

Note that we can integrate the above equation over the domain where shear is interested:

$$V_y(x) = -\int p_y(x)\,dx + V_{y_0} \tag{4.5}$$

Sum of moment at O, will give us an equation for the moment:

$$+\circlearrowleft \sum M_z = 0 \qquad \Rightarrow \qquad V_y(x)dx + \{M_{zz}(x) + dM_{zz}(x)\} - M_{zz}(x) - p_y(x)\,dx\,\frac{dx}{2} = 0$$

divide by dx and take $\lim_{dx\to 0}$

$$\frac{d\,M_{zz}(x)}{dx} = -V_y(x) \tag{4.6}$$

Note that we can integrate the above equation over the domain where moment is interested:

$$M_{zz}(x) = -\int V_y(x)\,dx + M_{zz_0} \tag{4.7}$$

where M_{zz_0} is found from boundary conditions.

Structural Convention

Problems can be solved using the following sign convention

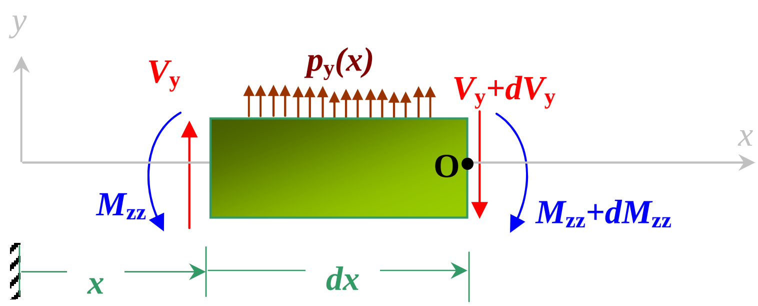


Figure 4.5: Equilibrium element supporting a general force system under the structural convention in the x-y plane

Sum of forces in the y-direction, will give us an equation for the shear:

$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad V_y(x) - \{V_y(x) + dV_y(x)\} + p_y(x)\,dx = 0$$

divide by dx and take $\lim_{dx\to 0}$

$$\frac{dV_y(x)}{dx} = p_y(x) \tag{4.8}$$

Note that we can integrate the above equation over the domain where shear is interested:

$$V_y(x) = \int p_y(x)\,dx + V_{y_0} \tag{4.9}$$

Sum of moment at O, will give us an equation for the moment:

$$+\circlearrowleft \sum M_z = 0 \qquad \Rightarrow \qquad -V_y(x)\,dx + \{M_{zz}(x) + dM_{zz}(x)\} - M_{zz}(x) - p_y(x)\,dx\,\frac{dx}{2} = 0$$

divide by dx and take $\lim_{dx\to 0}$

$$\frac{d\,M_{zz}(x)}{dx} = V_y(x) \tag{4.10}$$

Note that we can integrate the above equation over the domain where moment is interested:

$$M_{zz}(x) = \int V_y(x)\,dx + M_{zz_0} \tag{4.11}$$

where M_{zz_0} is found from boundary conditions.

Elasticity Convention

Problems can be solved using the following sign convention

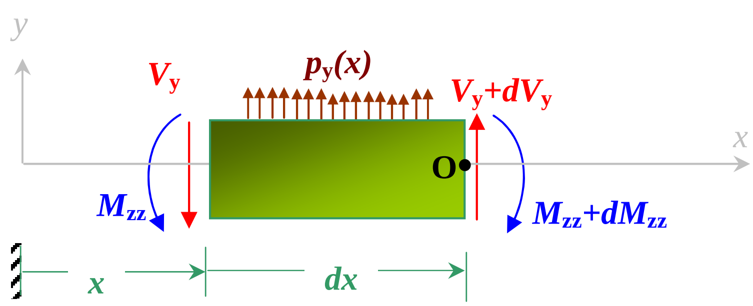


Figure 4.6: Equilibrium element supporting a general force system under the elasticity convention in the x-y plane

Sum of forces in the y-direction, will give us an equation for the shear:

$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad -V_y(x) + \{V_y(x) + dV_y(x)\} + p_y(x)\,dx = 0$$

divide by dx and take $\lim_{dx\to 0}$

$$\frac{dV_y(x)}{dx} = -p_y(x) \tag{4.12}$$

Note that we can integrate the above equation over the domain where shear is interested:

$$V_y(x) = -\int p_y(x)\,dx + V_{y_0} \tag{4.13}$$

Sum of moment at O, will give us an equation for the moment:

$$+\circlearrowleft \sum M_z = 0 \qquad \Rightarrow \qquad V_y(x)\,dx - \{M_{zz}(x) + dM_{zz}(x)\} + M_{zz}(x) - p_y(x)\,dx\,\frac{dx}{2} = 0$$

divide by dx and take $\lim_{dx \to 0}$

$$\frac{d\,M_{zz}(x)}{dx} = V_y(x) \tag{4.14}$$

Note that we can integrate the above equation over the domain where moment is interested:

$$M_{zz}(x) = \int V_y(x)\,dx + M_{zz_0} \tag{4.15}$$

where M_{zz_0} is found from boundary conditions.

4.4.2 Linear Differential Equations of Equilibrium

Consider a small differential element dx and construct a free body diagram with the actual stress distributions replaced by their statically equivalent internal resultants. Thus using the stress convention and applying Newton's Second Law the differential equations for equilibrium are found as:

$$\begin{aligned} &\frac{dN_{xx}}{dx} = -p_x(x) \qquad \frac{dV_y}{dx} = -p_y(x) \qquad \frac{dV_z}{dx} = -p_z(x) \\ &\frac{dM_{xx}}{dx} = -m_x(x) \qquad \frac{dM_{yy}}{dx} = -m_y(x) + V_z \qquad \frac{dM_{zz}}{dx} = -m_z(x) - V_y \end{aligned} \tag{4.16}$$

where $p_x(x)$ is the distributed load in the axial direction (x-axis), $p_y(x)$ the distributed load in the transverse direction (y-axis), $p_z(x)$ the distributed load in the lateral direction (z-axis), $m_x(x)$ the distributed moments about the x-axis, $m_y(x)$ the distributed moments about the y-axis, and $m_z(x)$ the distributed moments about the z-axis.

These equations are the first order ordinary differential equations that may be solved by direct integration. The solution to these equations is:

$$N_{xx}(x) = N_{xx}(x_1) - \int_{x_1}^{x} p_x(\zeta)\,d\zeta \tag{4.17}$$

$$V_y(x) = V_y(x_1) - \int_{x_1}^{x} p_y(\zeta)\,d\zeta \tag{4.18}$$

$$V_z(x) = V_z(x_1) - \int_{x_1}^{x} p_z(\zeta)\,d\zeta \tag{4.19}$$

$$M_{xx}(x) = M_{xx}(x_1) - \int_{x_1}^{x} m_x(\zeta)\,d\zeta \tag{4.20}$$

$$M_{yy}(x) = M_{yy}(x_1) - \int_{x_1}^{x} \Big\{ m_y(\zeta) - V_z(\zeta) \Big\}\,d\zeta \tag{4.21}$$

$$M_{zz}(x) = M_{zz}(x_1) - \int_{x_1}^{x} \Big\{ m_z(\zeta) + V_y(\zeta) \Big\}\,d\zeta \tag{4.22}$$

The first term on the right-hand side of the above equations are known as the boundary conditions; i.e., if the beam is statically determinate there will exist some point along the x-axis $x = x_1$ at which the resultants are known. For the case of statically indeterminate, the boundary conditions may be found using compatibility equations.

Example 4.3.

Aerospace engineers have idealized an aircraft structural component using the beam model as shown in Fig. 4.7. The cantilever beam's squared cross section is uniform. These engineers need your help to analyze this component. Take $a = 25$ mm, $b = 5$ mm. Use the stress convention and show all your steps.

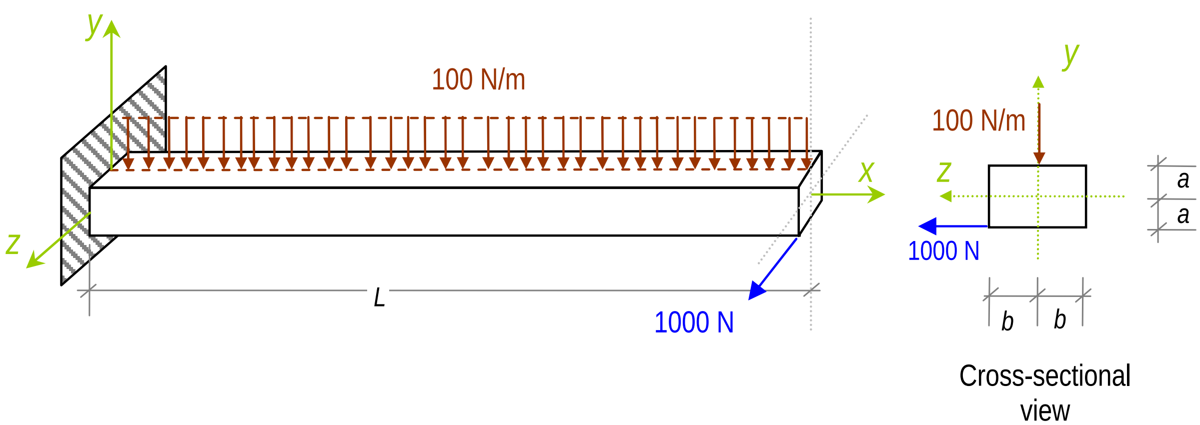
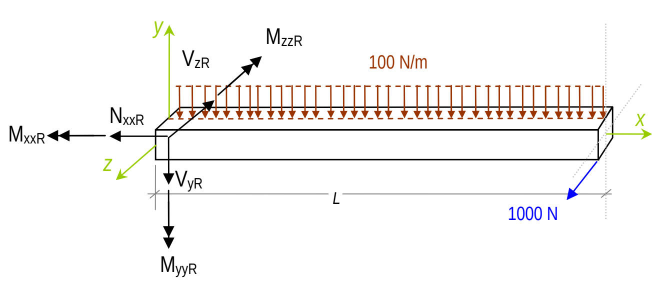


Figure 4.7: Machine component for example below.

4.3a) Obtain axial load equation for $N_{xx}(x)$ and shear equations for $V_y(x)$ and $V_z(x)$.

First obtain the reactions at the fixed end: (used positive stress convention discussed in class)



The internal shear loads at the fixed end are (all in newtons, assuming L in meters)

$$+\uparrow \sum F_y = 0 \quad \rightarrow \quad -V_{yR} - q_o\, L = -V_{yR} - 100\, L = 0 \quad \rightarrow \quad V_{yR} = -100\, L$$

$$\underset{\rightarrow}{+} \sum F_x = 0 \quad \rightarrow \quad -N_{xxR} + 0 = 0 \quad \rightarrow \quad N_{xxR} = 0$$

$$\underset{\rightarrow}{+} \sum F_z = 0 \quad \rightarrow \quad -V_{zR} + P = -V_{zR} + 1000 = 0 \quad \rightarrow \quad V_{zR} = 1000$$

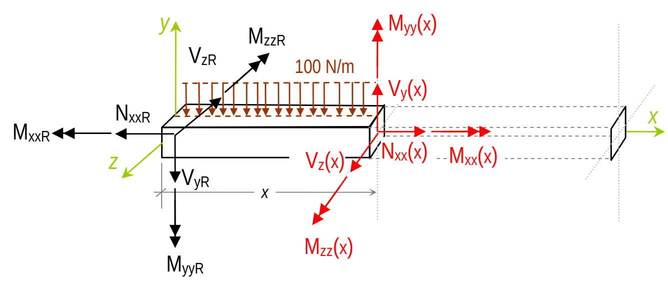
The internal moments at the fixed end are (all in N–m, assuming L in meters)

$$+ \circlearrowleft \sum M_y = 0 \quad \rightarrow \quad -M_{yyR} - P\, L = -M_{yyR} - 1000\, L = 0 \quad \rightarrow \quad M_{yyR} = -1000\, L$$

$$+ \circlearrowleft \sum M_x = 0 \quad \rightarrow \quad -M_{xxR} - P\, a = -M_{xxR} - 1000\,(0.025) = 0 \quad \rightarrow \quad M_{xxR} = -25$$

$$+ \circlearrowleft \sum M_z = 0 \quad \rightarrow \quad -M_{zzR} + q_o\, L \left(\frac{L}{2}\right) = -M_{zzR} - 100\,\frac{L^2}{2} = 0 \quad \rightarrow \quad M_{zzR} = -50\, L^2$$

Now, let us make an arbitrary cut at a distance x (used positive stress convention discussed in class)



The internal shear loads at the a distance x are (all in newtons, assuming L in meters)

$$+\uparrow \sum F_y = 0 \quad \rightarrow \quad -V_{yR} + V_y(x) + \int_0^x p_y(\zeta)\, d\zeta = 0$$

$$V_y(x) = -\int_0^x p_y(\zeta)\, d\zeta + V_{yR}$$

$$= -\int_0^x (-100)\, d\zeta - 100\, L$$

$$= 100\, x - 100\, L$$

$$V_y(x) = 100\, L \left\{-1 + \frac{x}{L}\right\}$$

$$\underset{\rightarrow}{+}\sum F_x = 0 \quad \rightarrow \quad -N_{xxR} + N_{xx}(x) = 0$$

$$N_{xx}(x) = N_{xxR}$$

$$N_{xx}(x) = 0$$

$$+\uparrow \sum F_z = 0 \quad \rightarrow \quad -V_{zR} + V_z(x) = 0$$

$$V_z(x) = V_{zR}$$

$$V_z(x) = 1000$$

4.3b) Obtain moment equations for $M_{xx}(x)$, $M_{yy}(x)$ and $M_{zz}(x)$.

The internal moments at the a distance x are (all in N–m, assuming L in meters)

$$+\circlearrowleft \sum M_y = 0 \quad \rightarrow \quad -M_{yyR} + M_{yy}(x) - \int_0^x V_z(\zeta)\, d\zeta = 0$$

$$-(-1000\,L) + M_{yy}(x) - \int_0^x (1000)\; d\zeta = 0$$

$$1000\,L + M_{yy}(x) - 1000\,x = 0$$

$$M_{yy}(x) = 1000\,L \left\{-1 + \frac{x}{L}\right\}$$

$$+\circlearrowleft \sum M_x = 0 \quad \rightarrow \quad -M_{xxR} + M_{xx}(x) = 0$$

$$25 + M_{xx}(x) = 0$$

$$M_{xx}(x) = -25$$

$$+\circlearrowleft \sum M_z = 0 \quad \rightarrow \quad -M_{zzR} + M_{zz}(x) + \int_0^x V_y(\zeta)\, d\zeta = 0$$

$$-\left(-50\,L^2\right) + M_{zz}(x) + \int_0^x (-100\,L + 100\,\zeta)\; d\zeta = 0$$

$$50\,L^2 + M_{zz}(x) - 100\,L\,x + 50\,x^2 = 0$$

$$M_{zz}(x) = 50\,L^2 \left\{-1 + 2\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right\}$$

4.3c) Plot all axial, shear, and moment equations.

In general, it is convenient to plot nondimensional quantities. Thus let the length be normalize to one:

$$\eta = \frac{x}{L} \qquad 0 < \eta < 1$$

and the nondimensional loads are:

$$\bar{M}_{xx}(\eta) = \frac{M_{xx}(x)}{1} = -25 \qquad \bar{N}_{xx}(\eta) = 0$$

$$\bar{M}_{yy}(\eta) = \frac{M_{yy}(x)}{L} = -1000 + 1000\,\eta \qquad \bar{V}_{y}(\eta) = \frac{V_{y}(x)}{L} = -100 + 100\,\eta$$

$$\bar{M}_{zz}(\eta) = \frac{M_{zz}(x)}{L^2} = -50 + 100\,\eta - 50\,\eta^2 \qquad \bar{V}_{z}(\eta) = \frac{V_{z}(x)}{1} = 1000$$

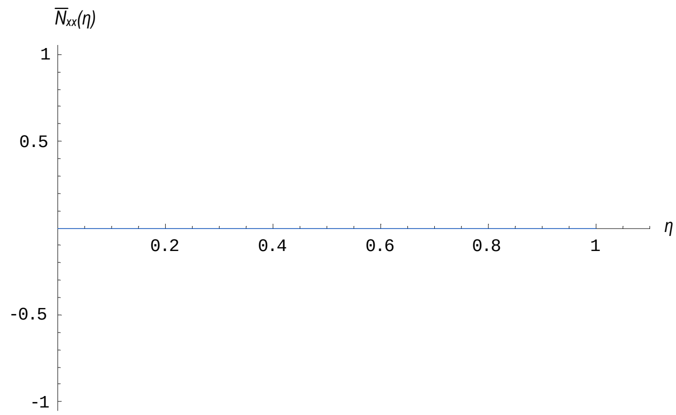


Figure 4.8: Dimensionless axial load distribution.

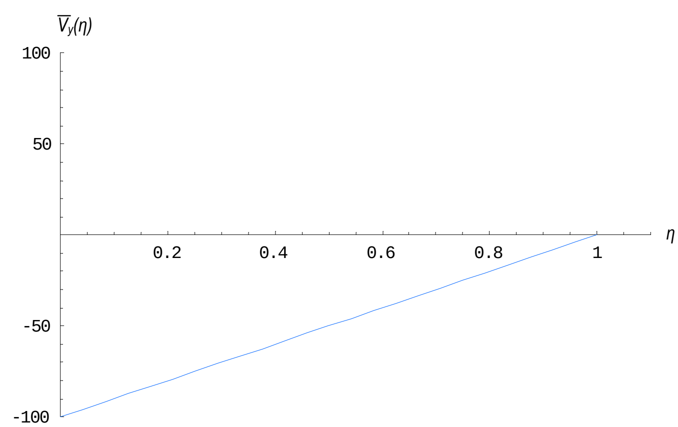


Figure 4.9: Dimensionless shear (in the y–axis) load distribution.

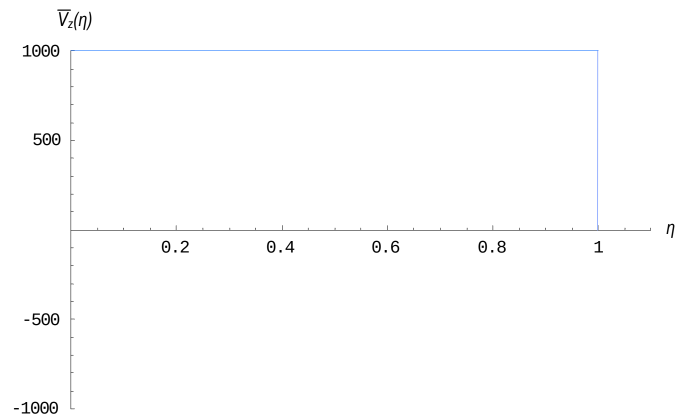


Figure 4.10: Dimensionless shear (in the z–axis) load distribution.

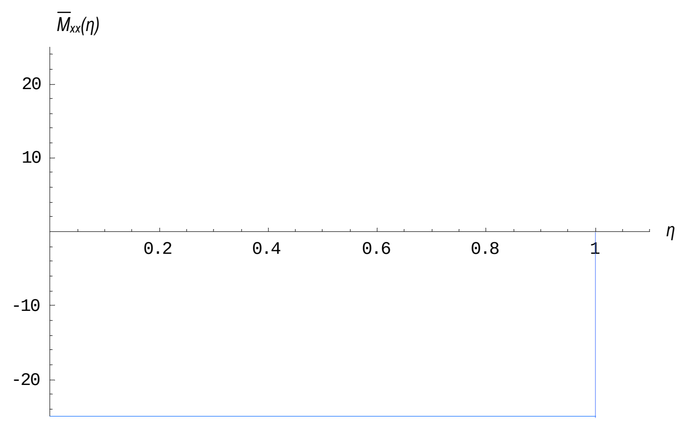


Figure 4.11: Dimensionless torsional load distribution.

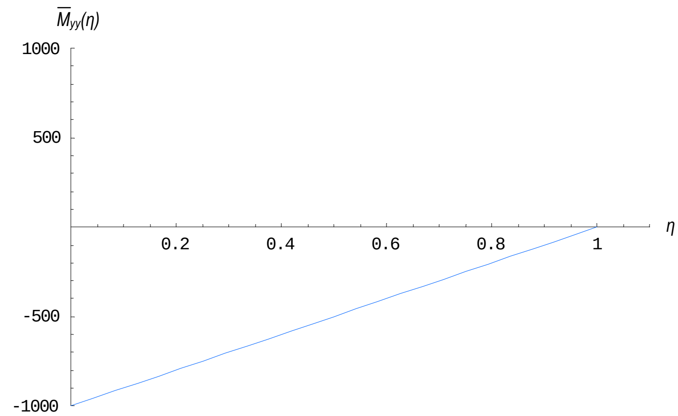


Figure 4.12: Dimensionless moment (about the y–axis) distribution.

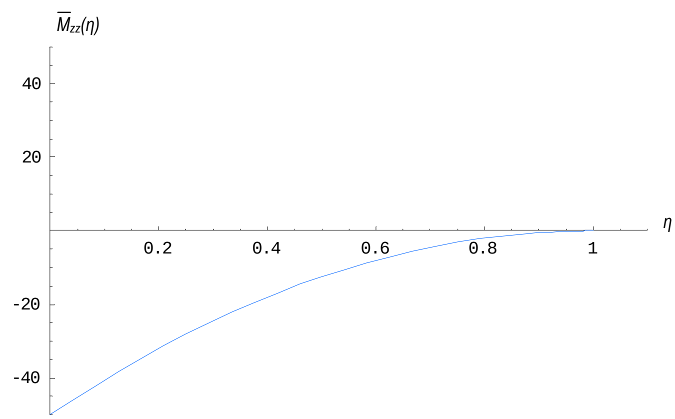


Figure 4.13: Dimensionless moment (about the z–axis) distribution.

End Example □

Example 4.4.

Consider an idealization of the helicopter blade, show in Fig. 4.14, subject to the loading shown in Fig. 4.15. The following data is given:

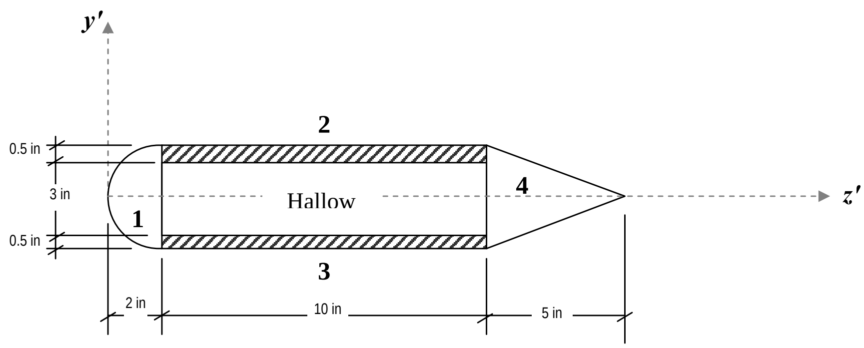


Figure 4.14: Cross-section of the helicopter blade.

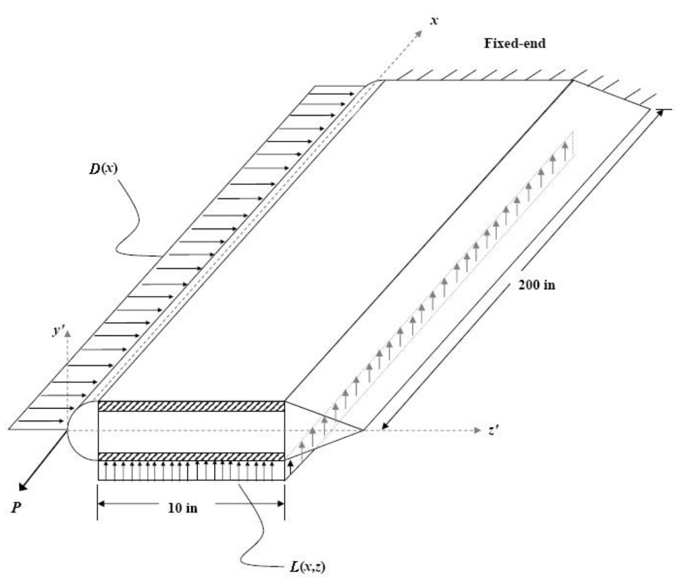


Figure 4.15: Loading on the helicopter blade.

$$L(x,z) = 4.0\left(\frac{x}{L}\right)^2 \text{ psi} \qquad D(x) = 4.0\left(\frac{x}{L}\right)^2 \text{ lb/in} \qquad P = 10000 \text{ lb}$$

where $L(x,z)$ is a pressure applied to the bottom surface. The total length of the beam is $L = 200$ in. Resolve all loads at the modulus-weighted centroid[1] (as a function of x): ($y_c^* = 0$, z_c^*=5.767 in).

First of all we concentrate the pressure load to a distributed load:

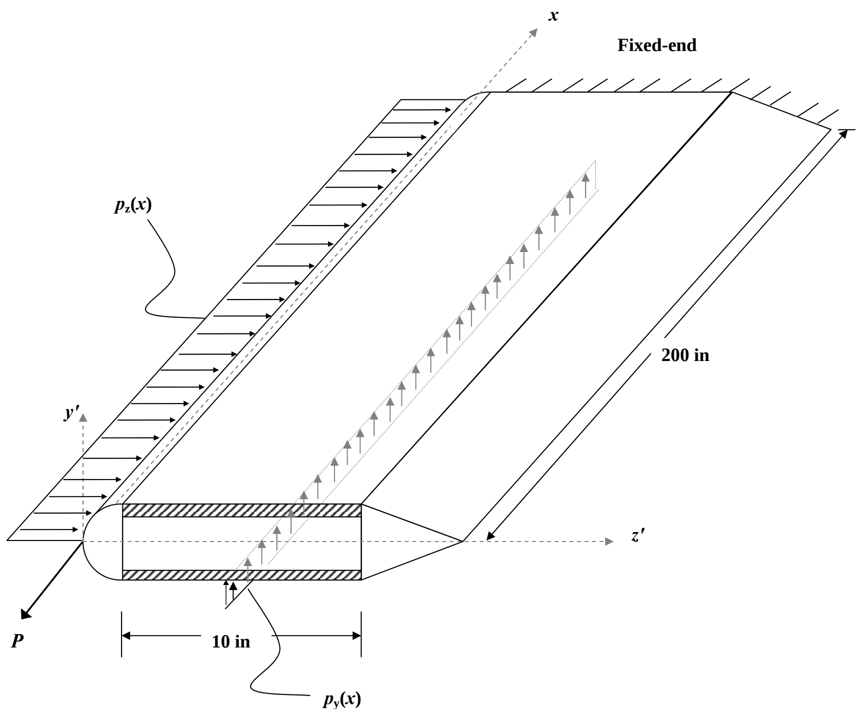


Figure 4.16: Replacing the pressure $L(x,z)$ with a distributed load $p_y(x)$.

$$p_y(x) = \int_0^{10} L(x,z)\,dz = 40.0\left(\frac{x}{L}\right)^2 \text{ lb/in}$$

[1]This will be discussed in detail in the chapter ??.

Now move all loads to the modulus-weighted centroid:

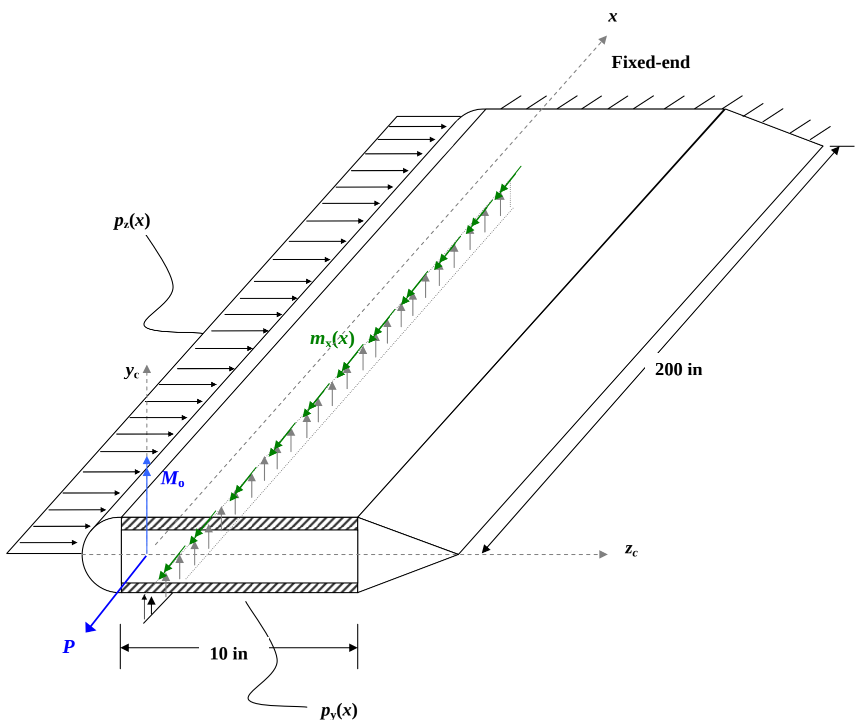


Figure 4.17: Locating all loads at the modulus-weighted centroid.

where,

$$m_x(x) = -\left(7 - z_c^*\right) p_y(x) = -0.001233\, x^2 \text{ lb-in/in} \qquad M_o = 10000(5.767) = 57670.0 \text{ lb-in}$$

Thus the loads are:

$$p_x(x) = 0 \qquad p_y(x) = 40.0 \left(\frac{x}{L}\right)^2 \text{ lb/in} \qquad p_z(x) = 4.0 \left(\frac{x}{L}\right)^2 \text{ lb/in}$$

$$m_x(x) = -49.32 \left(\frac{x}{L}\right)^2 \text{ lb-in/in} \qquad m_y(x) = 0 \qquad m_z(x) = 0$$

The loads at $x = x_1 = 0$ come from equilibrium of a differential element just after $x = 0$

$(x = 0^+)$. Using the stress convention, we get:

$$N_{xx}(x)\Big|_{x_1=0} - 10000 = 0 \qquad \rightarrow \qquad N_{xx}(x)\Big|_{x_1=0} = 10000\,\text{lb}$$

$$V_y(x)\Big|_{x_1=0} + 0 = 0 \qquad \rightarrow \qquad V_y(x)\Big|_{x_1=0} = 0$$

$$V_z(x)\Big|_{x_1=0} + 0 = 0 \qquad \rightarrow \qquad V_z(x)\Big|_{x_1=0} = 0$$

$$M_{xx}(x)\Big|_{x_1=0} + 0 = 0 \qquad \rightarrow \qquad M_{xx}(x)\Big|_{x_1=0} = 0$$

$$M_{yy}(x)\Big|_{x_1=0} + 57670 = 0 \qquad \rightarrow \qquad M_{yy}(x)\Big|_{x_1=0} = -57670\,\text{lb-in}$$

$$M_{zz}(x)\Big|_{x_1=0} + 0 = 0 \qquad \rightarrow \qquad M_{zz}(x)\Big|_{x_1=0} = 0$$

Now we proceed to obtain the internal loads. Integrating to obtain the axial internal load;

$$N_{xx}(x) = N_{xx}(x_1) - \int_{x_1}^{x} p_x(\zeta)\,d\zeta = N_{xx}(0) - \int_0^x (0)\,d\zeta = N_{xx}(0) = 10000\,\text{lb}$$

$$V_y(x) = V_y(x_1) - \int_{x_1}^{x} p_y(\zeta)\,d\zeta = V_y(0) - \int_0^x \left\{40.0\left(\frac{\zeta}{L}\right)^2\right\} d\zeta = -0.000333333\,x^3\ \text{lb}$$

$$V_z(x) = V_z(x_1) - \int_{x_1}^{x} p_z(\zeta)\,d\zeta = V_z(0) - \int_0^x \left\{4.0\left(\frac{\zeta}{L}\right)^2\right\} d\zeta = -0.0000333333\,x^3\ \text{lb}$$

$$M_{xx}(x) = M_{xx}(x_1) - \int_{x_1}^{x} m_x(\zeta)\,d\zeta = M_{xx}(0) - \int_0^x \left\{-0.001233\,\zeta^2\right\}\,d\zeta$$

$$= 0.000411\,x^3\ \text{lb-in}$$

$$M_{yy}(x) = M_{yy}(x_1) - \int_{x_1}^{x} \Big\{m_y(\zeta) - V_z(\zeta)\Big\}\,d\zeta = M_{yy}(0) - \int_0^x (0 - (-0.0000333333\,\zeta^3))\,d\zeta$$

$$= -57670 - 0.0000833333\,x^4\ \text{lb-in}$$

$$M_{zz}(x) = M_{zz}(x_1) - \int_{x_1}^{x} \Big\{m_z(\zeta) + V_y(\zeta)\Big\}\,d\zeta = M_{zz}(0) - \int_0^x (0 + (-0.000333333\,\zeta^3))\,d\zeta$$

$$= 0.0000833333\,x^4\ \text{lb-in}$$

Note that in the above equations, x is measured in inches.

End Example □

4.5 Discrete Load Diagrams

For most aircraft, an analytical load expression may not available. The only information we may have is the experimental data obtained from sensors located throughout the aircraft. For such cases we can no longer obtain close-form load diagrams, but we have to use numerical techniques to obtain the load diagrams. The points where we calculate the loads are called stations and are designated by their distance x from the centerline of the airplane. We measure these distances along the wing rather than horizontally because the air loads are perpendicular to the wing.

The method explained here can also be used when the analytical expression is available but using a numerical method is desired.

First, we need an array with all locations where the data is measured. For an example, suppose we want 10 intervals for the previous example, then we use the following locations

$$\underline{\mathbf{x}} = \{0, 20, 40, 60, \ldots, 180, 200\} \qquad \text{m}$$

This is converted into small intervals:

$$\underline{\boldsymbol{\Delta}\mathbf{x}} = \{20, 20, 20, \ldots, 20, 20\} \qquad \text{m}$$

In this example all intervals have the same interval, but they could be different. The smaller the Δx_i the better the approximation. At each location we calculate the distributed loads:

$$\begin{array}{ccc} p_{\text{x}}(x_i) & p_{\text{y}}(x_i) & p_{\text{z}}(x_i) \\ m_{\text{x}}(x_i) & m_{\text{y}}(x_i) & m_{\text{z}}(x_i) \end{array}$$

The next example will illustrate the approach.

Example 4.5.

The loads on an airplane wing cannot be represented by a simple equation. In fact, the data we get from sensors is the lift coefficient. To illustrate this consider a typical commercial airplane, shown Fig. 4.18, traveling at a cruise speed of 600 mi/hr and at an altitude of 35000 ft. The total span of the plane is 37.5 ft and the chord is 6 ft.

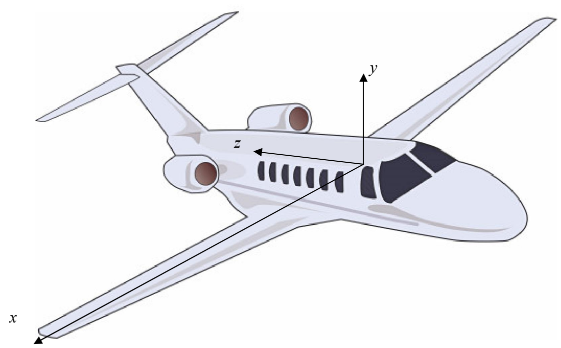


Figure 4.18: A commercial airplane travel at cruise speed.

Table 4.1: Discrete spanwise data.

Station number	Distance from the center x (in)	Airload data (lift coefficient) c_ℓ
1	0	0.876067
2	20	0.862050
3	40	0.841024
4	60	0.812990
5	80	0.777947
6	100	0.735896
7	120	0.686837
8	140	0.623760
9	160	0.560683
10	180	0.497606
11	200	0.406495
12	220	0.245299
13	225	0.000000

Determine the transverse shear and moment diagrams using the airload discrete data. The measured lift coefficient at the stations is given by Table 4.1. Since this is the only data available (no information regarding drag coefficients), loads in all other directions can be neglected.

As we can see from the example we have 13 stations (points). We compute the distance between stations (intervals) as follows:

$$\Delta x_i = x_{i+1} - x_i$$

Hence,

$$\begin{aligned}
\Delta x_1 &= x_2 - x_1 = 20 - 0 = 20 & \Delta x_2 &= x_3 - x_2 = 40 - 20 = 20 \\
\Delta x_3 &= x_4 - x_3 = 60 - 40 = 20 & \Delta x_4 &= x_5 - x_4 = 80 - 60 = 20 \\
\Delta x_5 &= x_6 - x_5 = 100 - 80 = 20 & \Delta x_6 &= x_7 - x_6 = 120 - 100 = 20 \\
\Delta x_7 &= x_8 - x_7 = 140 - 120 = 20 & \Delta x_8 &= x_9 - x_8 = 160 - 140 = 20 \\
\Delta x_9 &= x_{10} - x_9 = 180 - 160 = 20 & \Delta x_{10} &= x_{11} - x_{10} = 200 - 180 = 20 \\
\Delta x_{11} &= x_{12} - x_{11} = 220 - 200 = 20 & \Delta x_{12} &= x_{13} - x_{12} = 225 - 220 = 5
\end{aligned}$$

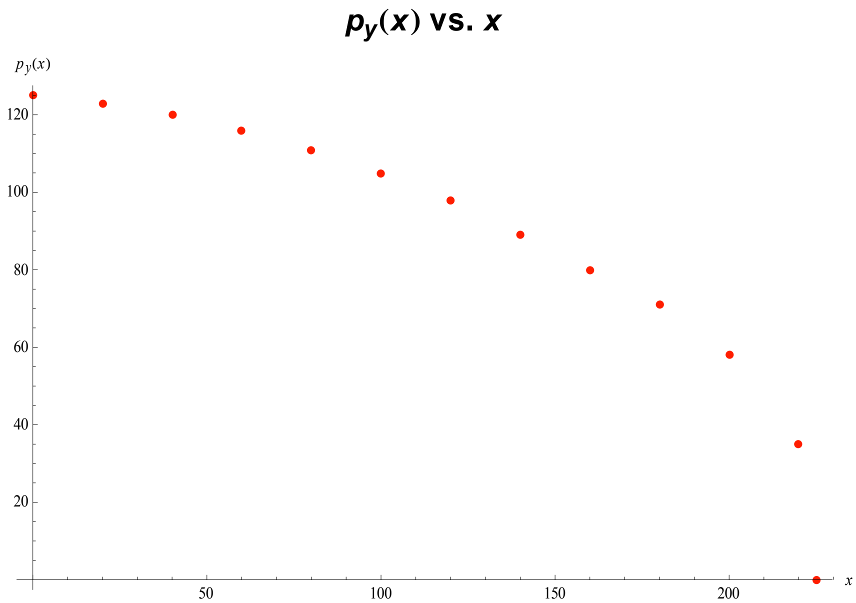
Let us proceed to obtain the discrete load per inch of span at each location x_i. This is done by calculated the lift per unit span:

$$p_y(x) = \frac{1}{2}\,\rho_\infty\, V_\infty^2\, c\, c_\ell$$

The density st 35000 ft is 0.000737 slugs/ft^3. The air speed is $V_\infty = 600$ mi/hr $= 880$ ft/sec. Hence, the load per inch of span at each station is given in Table 4.2.

Table 4.2: Discrete spanwise data.

Station number	Distance from the center x (in)	Airload data p_{y_i} (lb/in)
1	0	125.0000
2	20	123.0000
3	40	120.0000
4	60	116.0000
5	80	111.0000
6	100	105.0000
7	120	98.0001
8	140	89.0001
9	160	80.0000
10	180	71.0000
11	200	58.0000
12	220	35.0000
13	225	0.0000



Now we proceed with the solution of the shear differential equation in the y direction:

$$V_y(x) = V_y(0) - \int_0^x p_y(\zeta)\, d\zeta$$

The value of the shear at any point is the area under the load curve from that point out to the wing tip. We assume that the load curve is a series of straight lines between the known points, and we calculate the area under the curve as the sum of the areas of trapezoids:

$$V_y(x) = V_y(0) - \int_0^x p_y(\zeta)\, d\zeta \quad \rightarrow \quad V_y(x_j) \quad \approx V_{y0} - \sum_{i=1}^{j} p_{y,\text{ave}_i}\, \Delta x_i$$

We obtain the area of the trapezoids as the product of the average height, $p_{y,\text{ave}}$, and the base, Δx. Hence, we need to obtain the average values for each distributed load (all given in [lb/in]):

$$
\begin{aligned}
p_{\text{y,ave}_1} &= \frac{p_{\text{y}_1} + p_{\text{y}_2}}{2} = 124.0000 \\
p_{\text{y,ave}_2} &= \frac{p_{\text{y}_2} + p_{\text{y}_3}}{2} = 121.5000 \\
p_{\text{y,ave}_3} &= \frac{p_{\text{y}_3} + p_{\text{y}_4}}{2} = 118.0000 \\
p_{\text{y,ave}_4} &= \frac{p_{\text{y}_4} + p_{\text{y}_5}}{2} = 113.5000 \\
p_{\text{y,ave}_5} &= \frac{p_{\text{y}_5} + p_{\text{y}_6}}{2} = 108.0000 \\
p_{\text{y,ave}_6} &= \frac{p_{\text{y}_6} + p_{\text{y}_7}}{2} = 101.5000 \\
p_{\text{y,ave}_7} &= \frac{p_{\text{y}_7} + p_{\text{y}_8}}{2} = 93.5001 \\
p_{\text{y,ave}_8} &= \frac{p_{\text{y}_8} + p_{\text{y}_9}}{2} = 84.5001 \\
p_{\text{y,ave}_9} &= \frac{p_{\text{y}_9} + p_{\text{y}_{10}}}{2} = 75.5000 \\
p_{\text{y,ave}_{10}} &= \frac{p_{\text{y}_{10}} + p_{\text{y}_{11}}}{2} = 64.5000 \\
p_{\text{y,ave}_{11}} &= \frac{p_{\text{y}_{11}} + p_{\text{y}_{12}}}{2} = 46.5000 \\
p_{\text{y,ave}_{12}} &= \frac{p_{\text{y}_{12}} + p_{\text{y}_{13}}}{2} = 17.5000
\end{aligned}
$$

The change in the shear ΔV_{y} between two stations is equal to the area of the load curve between the stations. Hence, (all given in [lb]):

$$
\begin{aligned}
\Delta V_{\text{y}_1} &= -p_{\text{y,ave}_1}\,\Delta x_1 = -2480.0000 \\
\Delta V_{\text{y}_2} &= -p_{\text{y,ave}_2}\,\Delta x_2 = -2430.0000 \\
\Delta V_{\text{y}_3} &= -p_{\text{y,ave}_3}\,\Delta x_3 = -2360.0000 \\
\Delta V_{\text{y}_4} &= -p_{\text{y,ave}_4}\,\Delta x_4 = -2270.0000 \\
\Delta V_{\text{y}_5} &= -p_{\text{y,ave}_5}\,\Delta x_5 = -2160.0000 \\
\Delta V_{\text{y}_6} &= -p_{\text{y,ave}_6}\,\Delta x_6 = -2030.0000 \\
\Delta V_{\text{y}_7} &= -p_{\text{y,ave}_7}\,\Delta x_7 = -1870.0000 \\
\Delta V_{\text{y}_8} &= -p_{\text{y,ave}_8}\,\Delta x_8 = -1690.0000 \\
\Delta V_{\text{y}_9} &= -p_{\text{y,ave}_9}\,\Delta x_9 = -1510.0000 \\
\Delta V_{\text{y}_{10}} &= -p_{\text{y,ave}_{10}}\,\Delta x_{10} = -1290.0000 \\
\Delta V_{\text{y}_{11}} &= -p_{\text{y,ave}_{11}}\,\Delta x_{11} = -930.0010 \\
\Delta V_{\text{y}_{12}} &= -p_{\text{y,ave}_{12}}\,\Delta x_{12} = -87.5001
\end{aligned}
$$

We obtain the shear V_{y} by adding all ΔV_{y_i}'s values. Before we proceed we need to calculate the shear at the root. We use the equation:

$$
V_{\text{y}}(x) = V_{\text{y}}(0) - \int_0^x p_{\text{y}}(\zeta)\,d\zeta \quad \rightarrow \quad V_{\text{y}}(L)\Big|_{\text{TIP}} = V_{\text{y}}(0)\Big|_{\text{ROOT}} - \int_0^L p_{\text{y}}(\zeta)\,d\zeta
$$

There is no load applied at the tip; hence,

$$V_{\mathrm{y}}(L)\Big|_{\mathrm{TIP}} = 0$$

The shear force at the root is then calculated as follows:

$$0 = V_{\mathrm{y}}(0) - \int_0^x p_{\mathrm{y}}(\zeta)\, d\zeta \quad \rightarrow \quad V_{\mathrm{y}}(0)\Big|_{\mathrm{ROOT}} = \int_0^L p_{\mathrm{y}}(\zeta)\, d\zeta \approx \sum_{i=1}^{j} p_{\mathrm{y,ave}_i}\, \Delta x_i$$

or

$$V_{\mathrm{y}_0} = V_{\mathrm{y}}(0) = -\sum_{i=1}^{j} \Delta V_{\mathrm{y}_i}$$

Hence,

$$V_{\mathrm{y}_0} = -\sum_{i=1}^{j} \Delta V_{\mathrm{y}_i} = 21107.5 \text{ lb}$$

The shear force at each station i is (all given in [lb]):

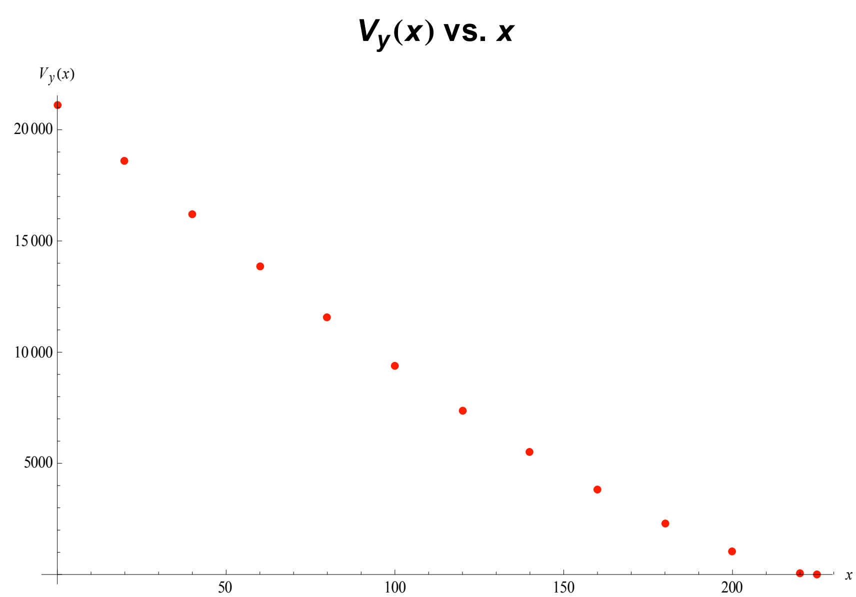
$$\begin{aligned}
x_1 &= 0 & V_{\mathrm{y}_1} &= V_{\mathrm{y}_0} = 21107.5000\\
x_2 &= 20 & V_{\mathrm{y}_2} &= \Delta V_{\mathrm{y}_1} + V_{\mathrm{y}_1} = 18627.5000\\
x_3 &= 40 & V_{\mathrm{y}_3} &= \Delta V_{\mathrm{y}_2} + V_{\mathrm{y}_2} = 16197.5000\\
x_4 &= 60 & V_{\mathrm{y}_4} &= \Delta V_{\mathrm{y}_3} + V_{\mathrm{y}_3} = 13837.5000\\
x_5 &= 80 & V_{\mathrm{y}_5} &= \Delta V_{\mathrm{y}_4} + V_{\mathrm{y}_4} = 11567.5000\\
x_6 &= 100 & V_{\mathrm{y}_6} &= \Delta V_{\mathrm{y}_5} + V_{\mathrm{y}_5} = 9407.5100\\
x_7 &= 120 & V_{\mathrm{y}_7} &= \Delta V_{\mathrm{y}_6} + V_{\mathrm{y}_6} = 7377.5000\\
x_8 &= 140 & V_{\mathrm{y}_8} &= \Delta V_{\mathrm{y}_7} + V_{\mathrm{y}_7} = 5507.5000\\
x_9 &= 160 & V_{\mathrm{y}_9} &= \Delta V_{\mathrm{y}_8} + V_{\mathrm{y}_8} = 3817.5000\\
x_{10} &= 180 & V_{\mathrm{y}_{10}} &= \Delta V_{\mathrm{y}_9} + V_{\mathrm{y}_9} = 2307.5000\\
x_{11} &= 200 & V_{\mathrm{y}_{11}} &= \Delta V_{\mathrm{y}_{10}} + V_{\mathrm{y}_{10}} = 1017.5000\\
x_{12} &= 220 & V_{\mathrm{y}_{12}} &= \Delta V_{\mathrm{y}_{11}} + V_{\mathrm{y}_{11}} = 87.5001\\
x_{13} &= 225 & V_{\mathrm{y}_{13}} &= \Delta V_{\mathrm{y}_{12}} + V_{\mathrm{y}_{12}} = 0.0000
\end{aligned}$$

In nondimensional form,

$$\eta = \frac{x}{L} \qquad \bar{V}_{\mathrm{y}_i} = \frac{V_{\mathrm{y}_i}}{\max[V_{\mathrm{y}_1}, V_{\mathrm{y}_2}, ..., V_{\mathrm{y}_{13}}]}$$

Table 4.3: Shear load values at the stations.

Station number	Distance from the center x (in)	Shear Load V_{y_i} (lb)	Nondimensional location η	Nondimensional Shear Load $\bar{V}_{y_i}$
1	0	21107.5000	0.0000	1.
2	20	18627.5000	0.0889	0.882506
3	40	16197.5000	0.1778	0.767381
4	60	13837.5000	0.2667	0.655573
5	80	11567.5000	0.3556	0.548028
6	100	9407.5100	0.4444	0.445695
7	120	7377.5000	0.5333	0.34952
8	140	5507.5000	0.6222	0.260926
9	160	3817.5000	0.7111	0.18086
10	180	2307.5000	0.8000	0.109321
11	200	1017.5000	0.8889	0.0482056
12	220	87.5001	0.9778	0.00414545
13	225	0.0000	1.0000	0.



Now, we proceed with the solution of the moment differential equation about the z axis:

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \left\{ m_z(\zeta) + V_y(\zeta) \right\} d\zeta$$

Note that there is no distributed moments about the z-axis; hence,

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \left\{ V_y(\zeta) \right\} d\zeta$$

The change in the bending moment, ΔM_{zz}, between two stations is equal to the area under the shear curve. We also assume this area is trapezoidal and we compute is by multiplying the average shear values between two stations times the distance between the stations.

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \left\{ V_y(\zeta) \right\} d\zeta \quad \rightarrow \quad M_{zz}(x_j) \ \approx M_{zz0} - \sum_{i=1}^{j} \left\{ V_{y,ave_i} \right\} \Delta x_i$$

Hence, we need to obtain the average values for shear V_{y_i} (all given in [lb]):

$$\begin{aligned}
V_{y,ave_1} &= \frac{V_{y_1} + V_{y_2}}{2} = 19867.5 \\
V_{y,ave_2} &= \frac{V_{y_2} + V_{y_3}}{2} = 17412.5 \\
V_{y,ave_3} &= \frac{V_{y_3} + V_{y_4}}{2} = 15017.5 \\
V_{y,ave_4} &= \frac{V_{y_4} + V_{y_5}}{2} = 12702.5 \\
V_{y,ave_5} &= \frac{V_{y_5} + V_{y_6}}{2} = 10487.5 \\
V_{y,ave_6} &= \frac{V_{y_6} + V_{y_7}}{2} = 8392.5 \\
V_{y,ave_7} &= \frac{V_{y_7} + V_{y_8}}{2} = 6442.5 \\
V_{y,ave_8} &= \frac{V_{y_8} + V_{y_9}}{2} = 4662.5 \\
V_{y,ave_9} &= \frac{V_{y_9} + V_{y_{10}}}{2} = 3062.5 \\
V_{y,ave_{10}} &= \frac{V_{y_{10}} + V_{y_{11}}}{2} = 1662.5 \\
V_{y,ave_{11}} &= \frac{V_{y_{11}} + V_{y_{12}}}{2} = 552.5 \\
V_{y,ave_{12}} &= \frac{V_{y_{12}} + V_{y_{13}}}{2} = 43.7501
\end{aligned}$$

We compute the change in the bending moment using the trapezoidal rule of numerical integration (all given in [lb-in]:

$$\begin{aligned}
\Delta M_{zz1} &= -V_{y,ave_1}\,\Delta x_1 = -397350.0000\\
\Delta M_{zz2} &= -V_{y,ave_2}\,\Delta x_2 = -348250.0000\\
\Delta M_{zz3} &= -V_{y,ave_3}\,\Delta x_3 = -300350.0000\\
\Delta M_{zz4} &= -V_{y,ave_4}\,\Delta x_4 = -254050.0000\\
\Delta M_{zz5} &= -V_{y,ave_5}\,\Delta x_5 = -209750.0000\\
\Delta M_{zz6} &= -V_{y,ave_6}\,\Delta x_6 = -167850.0000\\
\Delta M_{zz7} &= -V_{y,ave_7}\,\Delta x_7 = -128850.0000\\
\Delta M_{zz8} &= -V_{y,ave_8}\,\Delta x_8 = -93250.0000\\
\Delta M_{zz9} &= -V_{y,ave_9}\,\Delta x_9 = -61250.0000\\
\Delta M_{zz10} &= -V_{y,ave_{10}}\,\Delta x_{10} = -33250.0000\\
\Delta M_{zz11} &= -V_{y,ave_{11}}\,\Delta x_{11} = -11050.0000\\
\Delta M_{zz12} &= -V_{y,ave_{12}}\,\Delta x_{12} = -218.7500
\end{aligned}$$

We obtain the moment M_{zz} by adding all ΔM_{zz_i}'s values. Before we proceed we need to calculate the moment at the root. We use the equation:

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \Big\{ V_y(\zeta) \Big\}\, d\zeta$$

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \Big\{ V_y(\zeta) \Big\}\, d\zeta \quad \rightarrow \quad M_{zz}(L)\Big|_{\text{TIP}} = M_{zz}(0)\Big|_{\text{ROOT}} - \int_0^L \Big\{ V_y(\zeta) \Big\}\, d\zeta$$

There is no moment applied at the tip, hence,

$$M_{zz}(L)\Big|_{\text{TIP}} = 0$$

The moment at the root is then calculated as follows:

$$0 = M_{zz}(0) - \int_0^x V_y(\zeta)\, d\zeta \quad \rightarrow \quad M_{zz}(0)\Big|_{\text{ROOT}} = \int_0^L V_y(\zeta)\, d\zeta \approx \sum_{i=1}^{j} V_{y,ave_i}\,\Delta x_i$$

or

$$M_{zz0} = M_{zz}(0) = -\sum_{i=1}^{j} \Delta M_{zz_i}$$

Hence,

$$M_{zz0} = -\sum_{i=1}^{j} \Delta M_{zz_i} = 2.00547 \times 10^6 \text{ lb-in}$$

The moment at each station i is (all given in [lb-in]):

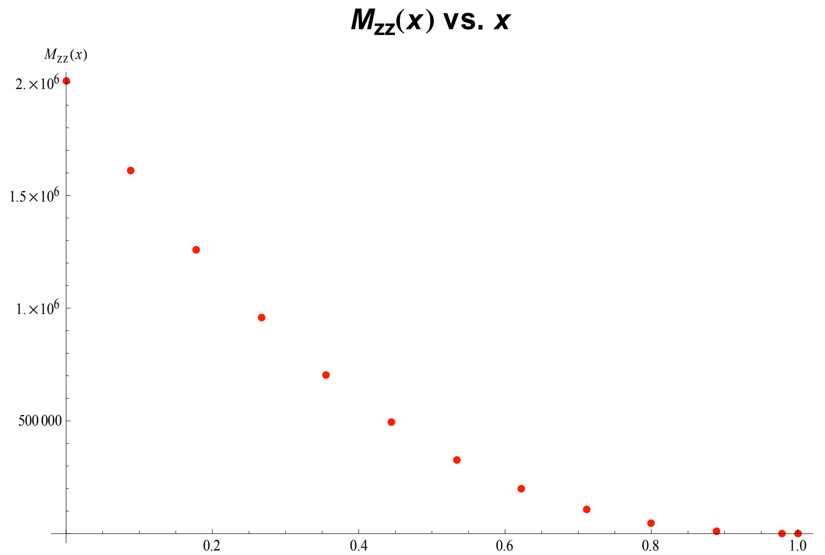
$$
\begin{aligned}
x_1 &= 0 & M_{zz1} &= M_{zz0} = 2.00547 \times 10^6 \\
x_2 &= 20 & M_{zz2} &= \Delta M_{zz1} + M_{zz1} = 1.60812 \times 10^6 \\
x_3 &= 40 & M_{zz3} &= \Delta M_{zz2} + M_{zz2} = 1.25987 \times 10^6 \\
x_4 &= 60 & M_{zz4} &= \Delta M_{zz3} + M_{zz3} = 959519 \\
x_5 &= 80 & M_{zz5} &= \Delta M_{zz4} + M_{zz4} = 705469 \\
x_6 &= 100 & M_{zz6} &= \Delta M_{zz5} + M_{zz5} = 495719 \\
x_7 &= 120 & M_{zz7} &= \Delta M_{zz6} + M_{zz6} = 327869 \\
x_8 &= 140 & M_{zz8} &= \Delta M_{zz7} + M_{zz7} = 199019 \\
x_9 &= 160 & M_{zz9} &= \Delta M_{zz8} + M_{zz8} = 105769 \\
x_{10} &= 180 & M_{zz10} &= \Delta M_{zz9} + M_{zz9} = 44518.8 \\
x_{11} &= 200 & M_{zz11} &= \Delta M_{zz10} + M_{zz10} = 11268.8 \\
x_{12} &= 220 & M_{zz12} &= \Delta M_{zz11} + M_{zz11} = 218.75 \\
x_{13} &= 225 & M_{zz13} &= \Delta M_{zz12} + M_{zz12} = 0.0000
\end{aligned}
$$

In nondimensional form,

$$\eta = \frac{x}{L} \qquad \bar{V}_{y_i} = \frac{M_{zz_i}}{\max[M_{zz1}, M_{zz2}, ..., M_{zz13}]}$$

Table 4.4: Moment load values at the stations.

Station number	Distance from the center x (in)	Moment Moment M_{zz_i} [lb-in]	Nondimensional location η	Nondimensional Moment $\bar{M}_{zz_i}$
1	0.	2.00547×10^6	0.	1.
2	20.	1.60812×10^6	0.0888889	0.801867
3	40.	1.25987×10^6	0.177778	0.628217
4	60.	959519.	0.266667	0.478451
5	80.	705469.	0.355556	0.351773
6	100.	495719.	0.444444	0.247184
7	120.	327869.	0.533333	0.163487
8	140.	199019.	0.622222	0.099238
9	160.	105769.	0.711111	0.0527402
10	180.	44518.8	0.8	0.0221987
11	200.	11268.8	0.888889	0.00561901
12	220.	218.75	0.977778	0.000109077
13	225.	0.	1.	0.


$M_{zz}(x)$ vs. x
$M_{zz}(x)$
$2.\times10^6$
1.5×10^6
$1.\times10^6$
500 000
0.2
0.4
0.6
0.8
1.0
x

End Example □

Example 4.6.

Consider the idealized helicopter blade od Example 4.4. Use five interval elements approximation, to determine the load diagrams.

From Example 4.4, we found that the loads acting on the helicopter blade are:

$$p_x(x) = 0 \qquad p_y(x) = 0.001\,x^2 \text{ lb/in} \qquad p_z(x) = 0.0001\,x^2 \text{ lb/in}$$

$$m_x(x) = -0.001233\,x^2 \text{ lb-in/in} \qquad m_y(x) = 0 \qquad m_z(x) = 0$$

Since we want five element approximation, let us divide the interval of $0 < x < 200$ into identical five elements:

$$\Delta x = \frac{x_{\text{root}} - x_{\text{tip}}}{5} = \frac{200 - 0}{5} = 40 \text{ in}$$

Hence,

$$\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \Delta x_5 = 40$$

The locations are

$$x_1 = 0$$
$$x_2 = 40$$
$$x_3 = 80$$
$$x_4 = 120$$
$$x_5 = 160$$
$$x_6 = 200$$

Let us proceed to obtain the discrete distributed loads at each location x_i:

$x_1 = 0$	$p_{x_1} = p_x(x_1) = 0$	$p_{y_1} = p_y(x_1) = 0.0$	$p_{z_1} = p_z(x_1) = 0.00$
$x_2 = 40$	$p_{x_2} = p_x(x_2) = 0$	$p_{y_2} = p_y(x_2) = 1.6$	$p_{z_2} = p_z(x_2) = 0.16$
$x_3 = 80$	$p_{x_3} = p_x(x_3) = 0$	$p_{y_3} = p_y(x_3) = 6.4$	$p_{z_3} = p_z(x_3) = 0.64$
$x_4 = 120$	$p_{x_4} = p_x(x_4) = 0$	$p_{y_4} = p_y(x_4) = 14.4$	$p_{z_4} = p_z(x_4) = 1.44$
$x_5 = 160$	$p_{x_5} = p_x(x_5) = 0$	$p_{y_5} = p_y(x_5) = 25.6$	$p_{z_5} = p_z(x_5) = 2.56$
$x_6 = 200$	$p_{x_6} = p_x(x_6) = 0$	$p_{y_6} = p_y(x_6) = 40.0$	$p_{z_6} = p_z(x_6) = 4.00$

Let us proceed to obtain the discrete distributed moments at each location x_i:

$$
\begin{array}{llll}
x_1 = 0 & m_{x_1} = m_x(x_1) = 0.0000 & m_{y_1} = m_y(x_1) = 0 & m_{z_1} = m_z(x_1) = 0 \\
x_2 = 40 & m_{x_2} = m_x(x_2) = -1.9728 & m_{y_2} = m_y(x_2) = 0 & m_{z_2} = m_z(x_2) = 0 \\
x_3 = 80 & m_{x_3} = m_x(x_3) = -7.8912 & m_{y_3} = m_y(x_3) = 0 & m_{z_3} = m_z(x_3) = 0 \\
x_4 = 120 & m_{x_4} = m_x(x_4) = -17.7552 & m_{y_4} = m_y(x_4) = 0 & m_{z_4} = m_z(x_4) = 0 \\
x_5 = 160 & m_{x_5} = m_x(x_5) = -31.5648 & m_{y_5} = m_y(x_5) = 0 & m_{z_5} = m_z(x_5) = 0 \\
x_6 = 200 & m_{x_6} = m_x(x_6) = -49.3200 & m_{y_6} = m_y(x_6) = 0 & m_{z_6} = m_z(x_6) = 0
\end{array}
$$

The tip load values are

$$P = 10000 \text{ lb} \qquad M_o = 10000(5.767) = 57670 \text{ lb-in}$$

Now in order to calculate the axial load along the wing's major axis, let us use Simpson's integration rule. We start we the solution of the axial differential equation:

$$N_{xx}(x) = N_{xx}(0) - \int_0^x p_x(\zeta)\, d\zeta \quad \rightarrow \quad N_{xx}(x_j) \ \approx N_{xx0} - \sum_{i=1}^{j} p_{x,ave_i}\, \Delta x_i$$

Hence, we need to obtain the average values for each distributed load:

$$
\begin{aligned}
p_{x,ave_1} &= \frac{p_{x_1} + p_{x_2}}{2} = 0 \\
p_{x,ave_2} &= \frac{p_{x_2} + p_{x_3}}{2} = 0 \\
p_{x,ave_3} &= \frac{p_{x_3} + p_{x_4}}{2} = 0 \\
p_{x,ave_4} &= \frac{p_{x_4} + p_{x_5}}{2} = 0 \\
p_{x,ave_5} &= \frac{p_{x_5} + p_{x_6}}{2} = 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta N_{xx_1} &= -p_{x,ave_1}\, \Delta x_1 = 0 \\
\Delta N_{xx_2} &= -p_{x,ave_2}\, \Delta x_2 = 0 \\
\Delta N_{xx_3} &= -p_{x,ave_3}\, \Delta x_3 = 0 \\
\Delta N_{xx_4} &= -p_{x,ave_4}\, \Delta x_4 = 0 \\
\Delta N_{xx_5} &= -p_{x,ave_5}\, \Delta x_5 = 0
\end{aligned}
$$

Hence,

$$N_{xx0} = N_{xx}(0) = P = 10000$$

$$
\begin{aligned}
x_1 &= 0 & N_{xx_1} &= N_{xx_0} = 10000 \\
x_2 &= 40 & N_{xx_2} &= \Delta N_{xx_1} + N_{xx_1} = 10000 \\
x_3 &= 80 & N_{xx_3} &= \Delta N_{xx_2} + N_{xx_2} = 10000 \\
x_4 &= 120 & N_{xx_4} &= \Delta N_{xx_3} + N_{xx_3} = 10000 \\
x_5 &= 160 & N_{xx_5} &= \Delta N_{xx_4} + N_{xx_4} = 10000 \\
x_6 &= 200 & N_{xx_6} &= \Delta N_{xx_5} + N_{xx_5} = 10000
\end{aligned}
$$

In nondimensional form,

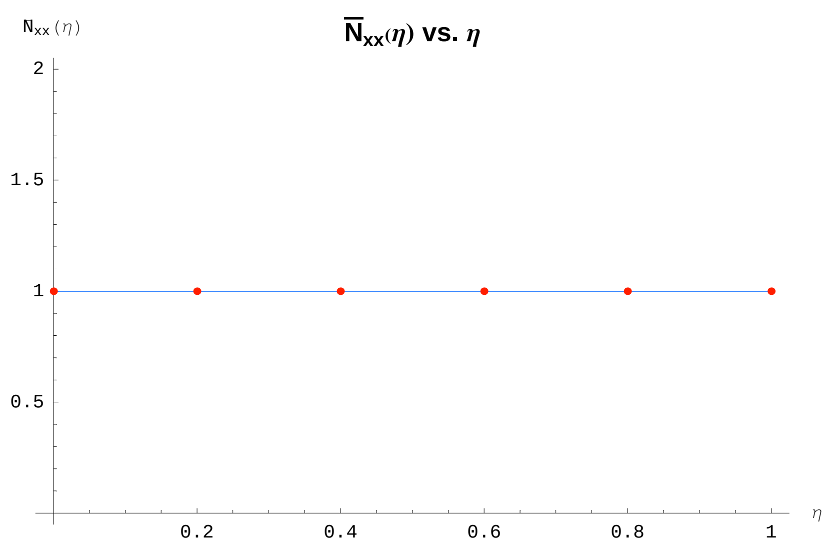
$$
\eta = \frac{x}{L} \qquad \bar{N}_{xx_i} = \frac{N_{xx_i}}{P}
$$

$$
\begin{aligned}
\eta_1 &= 0.0 & \bar{N}_{xx_1} &= 1 \\
\eta_2 &= 0.2 & \bar{N}_{xx_2} &= 1 \\
\eta_3 &= 0.4 & \bar{N}_{xx_3} &= 1 \\
\eta_4 &= 0.6 & \bar{N}_{xx_4} &= 1 \\
\eta_5 &= 0.8 & \bar{N}_{xx_5} &= 1 \\
\eta_6 &= 1.0 & \bar{N}_{xx_6} &= 1
\end{aligned}
$$

Compare to the exact nondimensional equation

$$
\bar{N}_{xx}(\eta) = \frac{N_{xx}(x)}{P} = 1
$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



Now we proceed with the solution of the shear differential equation in the y direction:

$$
V_y(x) = V_y(0) - \int_0^x p_y(\zeta)\, d\zeta \quad \rightarrow \quad V_y(x_j) \approx V_{y_0} - \sum_{i=1}^{j} p_{y,\text{ave}_i}\, \Delta x_i
$$

Hence, we need to obtain the average values for each distributed load:

$$p_{\text{y,ave}_1} = \frac{p_{\text{y}_1} + p_{\text{y}_2}}{2} = 0.8$$
$$p_{\text{y,ave}_2} = \frac{p_{\text{y}_2} + p_{\text{y}_3}}{2} = 4.0$$
$$p_{\text{y,ave}_3} = \frac{p_{\text{y}_3} + p_{\text{y}_4}}{2} = 10.4$$
$$p_{\text{y,ave}_4} = \frac{p_{\text{y}_4} + p_{\text{y}_5}}{2} = 20.0$$
$$p_{\text{y,ave}_5} = \frac{p_{\text{y}_5} + p_{\text{y}_6}}{2} = 32.80$$

Thus

$$\Delta V_{\text{y}_1} = -p_{\text{y,ave}_1}\,\Delta x_1 = -32.0$$
$$\Delta V_{\text{y}_2} = -p_{\text{y,ave}_2}\,\Delta x_2 = -160.0$$
$$\Delta V_{\text{y}_3} = -p_{\text{y,ave}_3}\,\Delta x_3 = -416.0$$
$$\Delta V_{\text{y}_4} = -p_{\text{y,ave}_4}\,\Delta x_4 = -800.0$$
$$\Delta V_{\text{y}_5} = -p_{\text{y,ave}_5}\,\Delta x_5 = -1312.0$$

Hence,

$$V_{\text{y}_0} = V_{\text{y}}(0) = 0$$

$$\begin{array}{ll} x_1 = 0 & V_{\text{y}_1} = V_{\text{y}_0} = 0 \\ x_2 = 40 & V_{\text{y}_2} = \Delta V_{\text{y}_1} + V_{\text{y}_1} = -32.0 \\ x_3 = 80 & V_{\text{y}_3} = \Delta V_{\text{y}_2} + V_{\text{y}_2} = -192.0 \\ x_4 = 120 & V_{\text{y}_4} = \Delta V_{\text{y}_3} + V_{\text{y}_3} = -608.0 \\ x_5 = 160 & V_{\text{y}_5} = \Delta V_{\text{y}_4} + V_{\text{y}_4} = -1408.0 \\ x_6 = 200 & V_{\text{y}_6} = \Delta V_{\text{y}_5} + V_{\text{y}_5} = -2720.0 \end{array}$$

In nondimensional form,

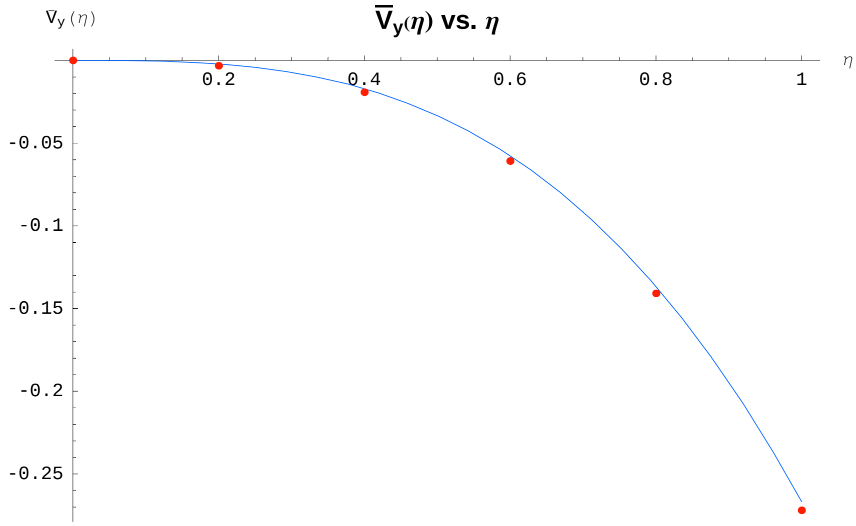
$$\eta = \frac{x}{L} \qquad \bar{V}_{\text{y}_i} = \frac{V_{\text{y}_i}}{P}$$

$$\begin{array}{ll} \eta_1 = 0 & \bar{V}_{\text{y}_1} = 0.0 \\ \eta_2 = 0.2 & \bar{V}_{\text{y}_2} = -0.0032 \\ \eta_3 = 0.4 & \bar{V}_{\text{y}_3} = -0.0192 \\ \eta_4 = 0.6 & \bar{V}_{\text{y}_4} = -0.0608 \\ \eta_5 = 0.8 & \bar{V}_{\text{y}_5} = -0.1408 \\ \eta_6 = 1.0 & \bar{V}_{\text{y}_6} = -0.2720 \end{array}$$

Compare to the exact nondimensional equation

$$\bar{V}_{\text{y}}(\eta) = \frac{V_{\text{y}}(x)}{P} = -0.266667\,\eta^3$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



Now, we proceed with the solution of the shear differential equation in the z direction:

$$V_z(x) = V_z(0) - \int_0^x p_z(\zeta)\, d\zeta \quad \rightarrow \quad V_z(x_j) \approx V_{z0} - \sum_{i=1}^{j} p_{z,\text{ave}_i}\, \Delta x_i$$

Hence, we need to obtain the average values for each distributed load:

$$\begin{aligned} p_{z,\text{ave}_1} &= \frac{p_{z_1} + p_{z_2}}{2} = 0.08 \\ p_{z,\text{ave}_2} &= \frac{p_{z_2} + p_{z_3}}{2} = 0.40 \\ p_{z,\text{ave}_3} &= \frac{p_{z_3} + p_{z_4}}{2} = 1.04 \\ p_{z,\text{ave}_4} &= \frac{p_{z_4} + p_{z_5}}{2} = 2.00 \\ p_{z,\text{ave}_5} &= \frac{p_{z_5} + p_{z_6}}{2} = 3.28 \end{aligned}$$

Thus

$$\begin{aligned} \Delta V_{z_1} &= -p_{z,\text{ave}_1}\, \Delta x_1 = -3.20 \\ \Delta V_{z_2} &= -p_{z,\text{ave}_2}\, \Delta x_2 = -16.00 \\ \Delta V_{z_3} &= -p_{z,\text{ave}_3}\, \Delta x_3 = -41.60 \\ \Delta V_{z_4} &= -p_{z,\text{ave}_4}\, \Delta x_4 = -80.00 \\ \Delta V_{z_5} &= -p_{z,\text{ave}_5}\, \Delta x_5 = -131.20 \end{aligned}$$

Hence,

$$V_{z0} = V_z(0) = 0$$

$$
\begin{aligned}
x_1 &= 0 & V_{z_1} &= V_{z_0} = 0 \\
x_2 &= 40 & V_{z_2} &= \Delta V_{z_1} + V_{z_1} = -3.20 \\
x_3 &= 80 & V_{z_3} &= \Delta V_{z_2} + V_{z_2} = -19.20 \\
x_4 &= 120 & V_{z_4} &= \Delta V_{z_3} + V_{z_3} = -60.80 \\
x_5 &= 160 & V_{z_5} &= \Delta V_{z_4} + V_{z_4} = -140.80 \\
x_6 &= 200 & V_{z_6} &= \Delta V_{z_5} + V_{z_5} = -272.00
\end{aligned}
$$

In nondimensional form,

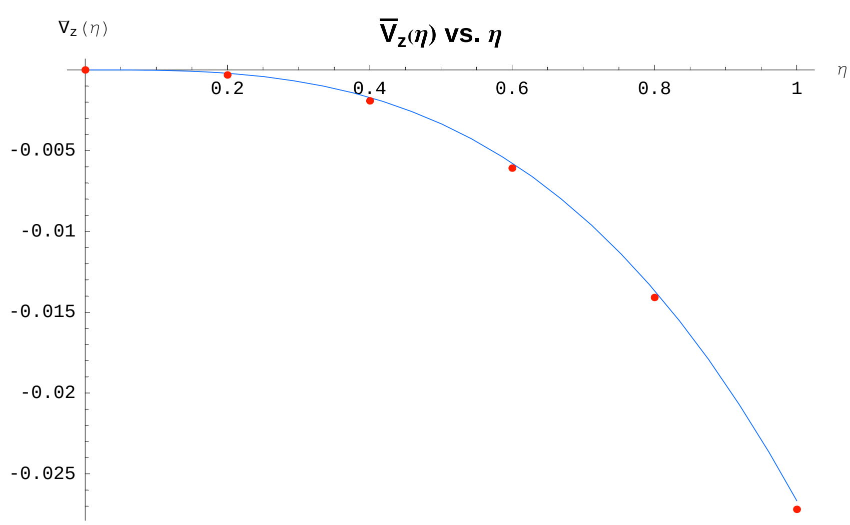
$$
\eta = \frac{x}{L} \qquad \bar{V}_{z_i} = \frac{V_{z_i}}{P}
$$

$$
\begin{aligned}
\eta_1 &= 0.0 & \bar{V}_{z_1} &= 0 \\
\eta_2 &= 0.2 & \bar{V}_{z_2} &= -0.00032 \\
\eta_3 &= 0.4 & \bar{V}_{z_3} &= -0.00192 \\
\eta_4 &= 0.6 & \bar{V}_{z_4} &= -0.00608 \\
\eta_5 &= 0.8 & \bar{V}_{z_5} &= -0.01408 \\
\eta_6 &= 1.0 & \bar{V}_{z_6} &= -0.02720
\end{aligned}
$$

Compare to the exact nondimensional equation

$$
\bar{V}_z(\eta) = \frac{V_z(x)}{P} = -0.026667\,\eta^3
$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



Now, we proceed with the solution of the torsional differential equation about the x axis:

$$
M_{xx}(x) = M_{xx}(0) - \int_0^x m_x(\zeta)\, d\zeta \quad \rightarrow \quad M_{xx}(x_j) \;\approx M_{xx_0} - \sum_{i=1}^{j} m_{x,\text{ave}_i}\, \Delta x_i
$$

Hence, we need to obtain the average values for each distributed moment:

$$m_{x,ave_1} = \frac{m_{x_1} + m_{x_2}}{2} = -0.9864$$
$$m_{x,ave_2} = \frac{m_{x_2} + m_{x_3}}{2} = -4.932$$
$$m_{x,ave_3} = \frac{m_{x_3} + m_{x_4}}{2} = -12.8232$$
$$m_{x,ave_4} = \frac{m_{x_4} + m_{x_5}}{2} = -24.66$$
$$m_{x,ave_5} = \frac{m_{x_5} + m_{x_6}}{2} = -40.4424$$

Thus

$$\Delta M_{xx1} = -m_{x,ave_1}\,\Delta x_1 = 39.456$$
$$\Delta M_{xx2} = -m_{x,ave_2}\,\Delta x_2 = 197.28$$
$$\Delta M_{xx3} = -m_{x,ave_3}\,\Delta x_3 = 512.928$$
$$\Delta M_{xx4} = -m_{x,ave_4}\,\Delta x_4 = 986.4$$
$$\Delta M_{xx5} = -m_{x,ave_5}\,\Delta x_5 = 1617.7$$

Hence,

$$M_{xx0} = M_{xx}(0) = 0$$

$$x_1 = 0 \qquad M_{xx1} = M_{xx0} = 0$$
$$x_2 = 40 \qquad M_{xx2} = \Delta M_{xx1} + M_{xx1} = 39.456$$
$$x_3 = 80 \qquad M_{xx3} = \Delta M_{xx2} + M_{xx2} = 236.736$$
$$x_4 = 120 \qquad M_{xx4} = \Delta M_{xx3} + M_{xx3} = 749.664$$
$$x_5 = 160 \qquad M_{xx5} = \Delta M_{xx4} + M_{xx4} = 1736.06$$
$$x_6 = 200 \qquad M_{xx6} = \Delta M_{xx5} + M_{xx5} = 3353.76$$

In nondimensional form,

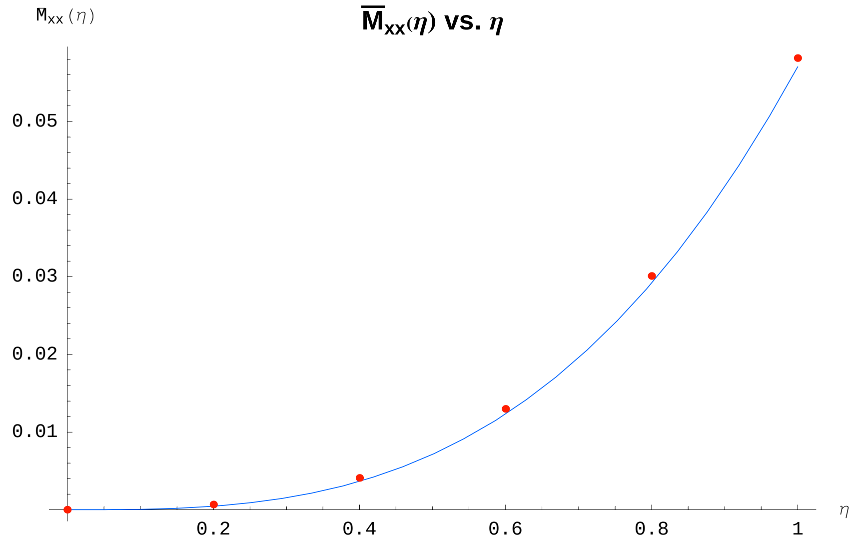
$$\eta = \frac{x}{L} \qquad \bar{M}_{xx_i} = \frac{M_{xx_i}}{M_o}$$

$$\eta_1 = 0.0 \qquad \bar{M}_{xx_1} = 0$$
$$\eta_2 = 0.2 \qquad \bar{M}_{xx_2} = 0.000684169$$
$$\eta_3 = 0.4 \qquad \bar{M}_{xx_3} = 0.00410501$$
$$\eta_4 = 0.6 \qquad \bar{M}_{xx_4} = 0.0129992$$
$$\eta_5 = 0.8 \qquad \bar{M}_{xx_5} = 0.0301034$$
$$\eta_6 = 1.0 \qquad \bar{M}_{xx_6} = 0.0581543$$

Compare to the exact nondimensional equation

$$\bar{M}_{xx}(\eta) = \frac{M_{xx}(x)}{M_o} = 0.057014\,\eta^3$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



Now, we proceed with the solution of the moment differential equation about the y axis:

$$M_{yy}(x) = M_{yy}(0) - \int_0^x \left\{ m_y(\zeta) - V_z(\zeta) \right\} d\zeta \quad \rightarrow \quad M_{yy}(x_j) \;\approx M_{yy_0} - \sum_{i=1}^{j} \left\{ m_{y,ave_i} - V_{z,ave_i} \right\} \Delta x_i$$

Hence, we need to obtain the average values for each distributed moment and Shear V_z:

$$m_{y,ave_1} = \frac{m_{y_1} + m_{y_2}}{2} = 0.0 \qquad V_{z,ave_1} = \frac{V_{z_1} + V_{z_2}}{2} = -1.6$$
$$m_{y,ave_2} = \frac{m_{y_2} + m_{y_3}}{2} = 0.0 \qquad V_{z,ave_2} = \frac{V_{z_2} + V_{z_3}}{2} = -11.20$$
$$m_{y,ave_3} = \frac{m_{y_3} + m_{y_4}}{2} = 0.0 \qquad V_{z,ave_3} = \frac{V_{z_3} + V_{z_4}}{2} = -40.00$$
$$m_{y,ave_4} = \frac{m_{y_4} + m_{y_5}}{2} = 0.0 \qquad V_{z,ave_4} = \frac{V_{z_4} + V_{z_5}}{2} = -100.80$$
$$m_{y,ave_5} = \frac{m_{y_5} + m_{y_6}}{2} = 0.0 \qquad V_{z,ave_5} = \frac{V_{z_5} + V_{z_6}}{2} = -206.4$$

Thus

$$\Delta M_{yy_1} = -\left\{ m_{y,ave_1} - V_{z,ave_1} \right\} \Delta x_1 = -64.0$$
$$\Delta M_{yy_2} = -\left\{ m_{y,ave_2} - V_{z,ave_2} \right\} \Delta x_2 = -448.0$$
$$\Delta M_{yy_3} = -\left\{ m_{y,ave_3} - V_{z,ave_3} \right\} \Delta x_3 = -1600.0$$
$$\Delta M_{yy_4} = -\left\{ m_{y,ave_4} - V_{z,ave_4} \right\} \Delta x_4 = -4032.0$$
$$\Delta M_{yy_5} = -\left\{ m_{y,ave_5} - V_{z,ave_5} \right\} \Delta x_5 = -8256.0$$

Hence,

$$M_{yy_0} = M_{yy}(0) = -57670$$

$$\begin{aligned}
x_1 &= 0 & M_{yy_1} &= M_{yy_0} = -57670.0 \\
x_2 &= 40 & M_{yy_2} &= \Delta M_{yy_1} + M_{yy_1} = -57734.0 \\
x_3 &= 80 & M_{yy_3} &= \Delta M_{yy_2} + M_{yy_2} = -58182.0 \\
x_4 &= 120 & M_{yy_4} &= \Delta M_{yy_3} + M_{yy_3} = -59782.0 \\
x_5 &= 160 & M_{yy_5} &= \Delta M_{yy_4} + M_{yy_4} = -63814.0 \\
x_6 &= 200 & M_{yy_6} &= \Delta M_{yy_5} + M_{yy_5} = -72070.
\end{aligned}$$

In nondimensional form,

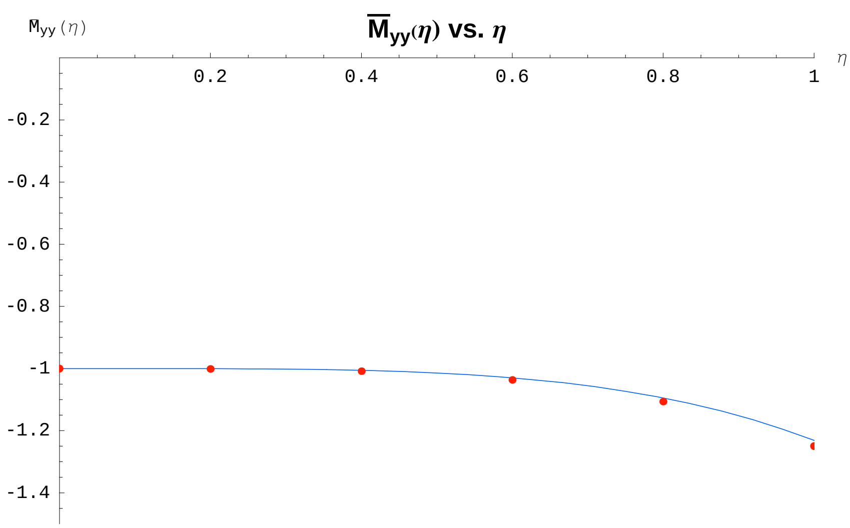
$$\eta = \frac{x}{L} \qquad \bar{M}_{yy_i} = \frac{M_{yy_i}}{M_o}$$

$$\begin{aligned}
\eta_1 &= 0.0 & \bar{M}_{yy_1} &= -1.00000 \\
\eta_2 &= 0.2 & \bar{M}_{yy_2} &= -1.00111 \\
\eta_3 &= 0.4 & \bar{M}_{yy_3} &= -1.00888 \\
\eta_4 &= 0.6 & \bar{M}_{yy_4} &= -1.03662 \\
\eta_5 &= 0.8 & \bar{M}_{yy_5} &= -1.10654 \\
\eta_6 &= 1.0 & \bar{M}_{yy_6} &= -1.2497
\end{aligned}$$

Compare to the exact nondimensional equation

$$\bar{M}_{yy}(\eta) = \frac{M_{yy}(x)}{M_o} = -1 - 0.231201\,\eta^4$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



Lastly, we proceed with the solution of the moment differential equation about the z axis:

$$M_{zz}(x) = M_{zz}(0) - \int_0^x \left\{ m_z(\zeta) + V_y(\zeta) \right\} d\zeta \quad \rightarrow \quad M_{zz}(x_j) \;\approx M_{zz0} - \sum_{i=1}^{j} \left\{ m_{z,\text{ave}_i} + V_{y,\text{ave}_i} \right\} \Delta x_i$$

Hence, we need to obtain the average values for each distributed moment and Shear V_{y}:

$$
\begin{aligned}
m_{\text{z,ave}_1} &= \frac{m_{\text{z}_1} + m_{\text{z}_2}}{2} = 0 & V_{\text{y,ave}_1} &= \frac{V_{\text{y}_1} + V_{\text{y}_2}}{2} = -16 \\
m_{\text{z,ave}_2} &= \frac{m_{\text{z}_2} + m_{\text{z}_3}}{2} = 0 & V_{\text{y,ave}_2} &= \frac{V_{\text{y}_2} + V_{\text{y}_3}}{2} = -112 \\
m_{\text{z,ave}_3} &= \frac{m_{\text{z}_3} + m_{\text{z}_4}}{2} = 0 & V_{\text{y,ave}_3} &= \frac{V_{\text{y}_3} + V_{\text{y}_4}}{2} = -400 \\
m_{\text{z,ave}_4} &= \frac{m_{\text{z}_4} + m_{\text{z}_5}}{2} = 0 & V_{\text{y,ave}_4} &= \frac{V_{\text{y}_4} + V_{\text{y}_5}}{2} = -1008 \\
m_{\text{z,ave}_5} &= \frac{m_{\text{z}_5} + m_{\text{z}_6}}{2} = 0 & V_{\text{y,ave}_5} &= \frac{V_{\text{y}_5} + V_{\text{y}_6}}{2} = -2064
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta M_{\text{zz}_1} &= -\left\{ m_{\text{z,ave}_1} + V_{\text{y,ave}_1} \right\} \Delta x_1 = 640 \\
\Delta M_{\text{zz}_2} &= -\left\{ m_{\text{z,ave}_2} + V_{\text{y,ave}_2} \right\} \Delta x_2 = 4480 \\
\Delta M_{\text{zz}_3} &= -\left\{ m_{\text{z,ave}_3} + V_{\text{y,ave}_3} \right\} \Delta x_3 = 16000 \\
\Delta M_{\text{zz}_4} &= -\left\{ m_{\text{z,ave}_4} + V_{\text{y,ave}_4} \right\} \Delta x_4 = 40320 \\
\Delta M_{\text{zz}_5} &= -\left\{ m_{\text{z,ave}_5} + V_{\text{y,ave}_5} \right\} \Delta x_5 = 82560
\end{aligned}
$$

Hence,

$$M_{\text{zz}_0} = M_{\text{zz}}(0) = 0$$

$$
\begin{aligned}
x_1 &= 0 & M_{\text{zz}_1} &= M_{\text{zz}_0} = 0 \\
x_2 &= 40 & M_{\text{zz}_2} &= \Delta M_{\text{zz}_1} + M_{\text{zz}_1} = 640 \\
x_3 &= 80 & M_{\text{zz}_3} &= \Delta M_{\text{zz}_2} + M_{\text{zz}_2} = 5120 \\
x_4 &= 120 & M_{\text{zz}_4} &= \Delta M_{\text{zz}_3} + M_{\text{zz}_3} = 21120 \\
x_5 &= 160 & M_{\text{zz}_5} &= \Delta M_{\text{zz}_4} + M_{\text{zz}_4} = 61440 \\
x_6 &= 200 & M_{\text{zz}_6} &= \Delta M_{\text{zz}_5} + M_{\text{zz}_5} = 144000
\end{aligned}
$$

In nondimensional form,

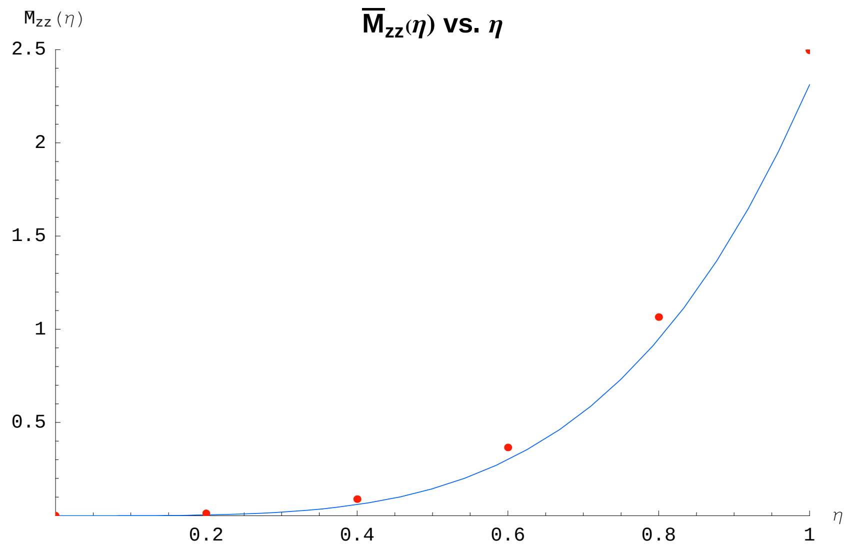
$$\eta = \frac{x}{L} \qquad \bar{M}_{\text{zz}_i} = \frac{M_{\text{zz}_i}}{M_{\text{o}}}$$

$$
\begin{aligned}
\eta_1 &= 0.0 & \bar{M}_{\text{zz}_1} &= 0.00000 \\
\eta_2 &= 0.2 & \bar{M}_{\text{zz}_2} &= 0.0110976 \\
\eta_3 &= 0.4 & \bar{M}_{\text{zz}_3} &= 0.088781 \\
\eta_4 &= 0.6 & \bar{M}_{\text{zz}_4} &= 0.366222 \\
\eta_5 &= 0.8 & \bar{M}_{\text{zz}_5} &= 1.06537 \\
\eta_6 &= 1.0 & \bar{M}_{\text{zz}_6} &= 2.49697
\end{aligned}
$$

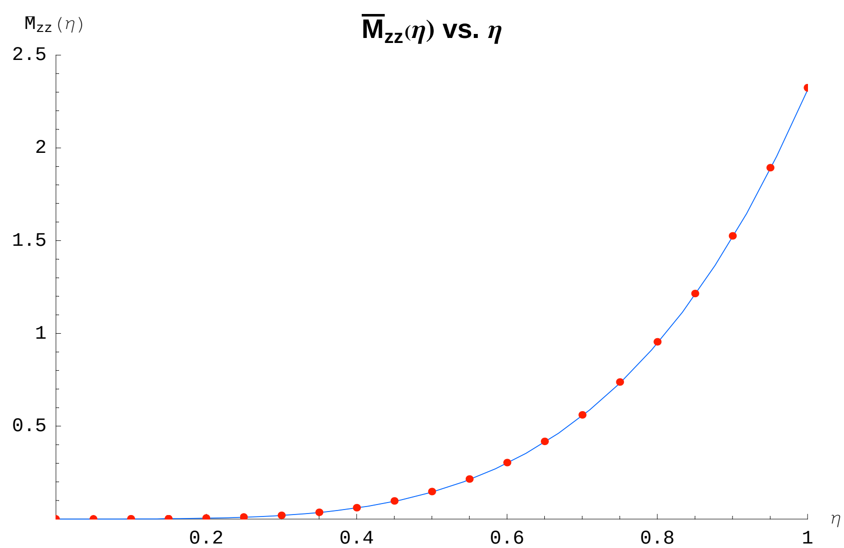
Compare to the exact nondimensional equation

$$\bar{M}_{\text{zz}}(\eta) = \frac{M_{\text{zz}}(x)}{M_{\text{o}}} = 2.31201\, \eta^4$$

The following plot shows the discrete method (points in red) and the exact solution (line in blue):



As we increase the number of intervals, the solutions approaches to the exact solution. As for an example, consider the plot for $\bar{M}_{zz}$ with 20 intervals:



End Example □

4.6 References

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4.7 Suggested Problems

Problem 4.1.

At a small planet of an unknown galaxy called TEXTBOOK, there is a group of engineering working with a different unit system. The system is called the word-unit-system (*WUS*). The force is measured in LETTER and time is measured in WORD. It is known that:

$$1\,\text{LETTER} = 1\,\frac{\text{PAGE}\cdot\text{CHAPTER}}{\text{WORD}^2} \tag{4.23}$$

where

$$1\,\text{PAGE} = 1\,\frac{\text{LETTER}\cdot\text{WORD}^2}{\text{CHAPTER}} \tag{4.24}$$

At TEXTBOOK, Newton's Second Law holds,

$$\underline{\mathbf{F}} = m\,\underline{\mathbf{a}}$$

The conversion to our unit system can be obtained by using the following data:

$$\begin{aligned} 1\,\text{LETTER} &= 1.5\ \text{Newtons} \\ 1\,\text{PAGE} &= 2.0\,\text{slugs} \\ 1\,\text{CHAPTER} &= 10\,\text{inches} \end{aligned}$$

For the known information, obtain the following,

a) Determine the units for mass in *WUS*.

b) Determine the units for length in *WUS*.

c) Find the relationship between the acceleration in the *SI* and the *WUS*.

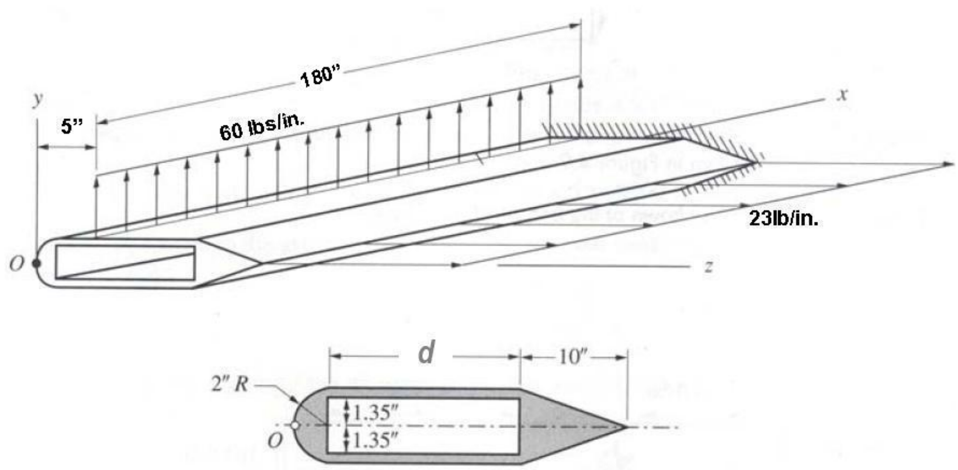
d) Find the relationship between the mass moment of inertia in the *fps* and the *WUS*.

e) The gravitational constant at our planet is known as $9.81\,\text{m/sec}^2$. Convert our gravitational constant to the *WUS*.

□

Problem 4.2.

The wing of the aircraft can be modeled as a cantilever beam and the wing's uniform cross section is given. If $d = 18$ in, determine the load profiles about the aerodynamic center of the idealized wing box.



Assume point **O** enters first in contact with the wing.

□

Problem 4.3.

Solve problem 3.2 but change the loads to:

$$p_y(x) = 100\sqrt{1 - \left(\frac{x}{L}\right)^2} \text{ lb/in}$$

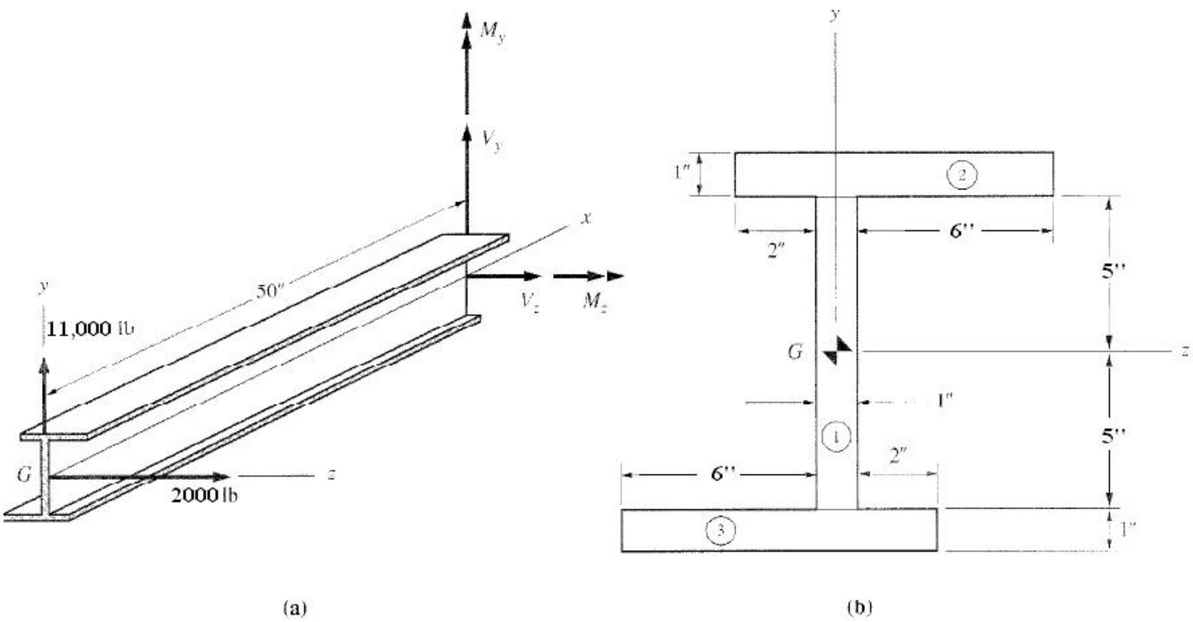
$$p_z(x) = -25\sqrt{1 - \left(\frac{x}{L}\right)^2} \text{ lb/in}$$

where $L = 180$ inches. Write a computer code using a programming language to determine all the load profiles about the aerodynamic center (assume quarter cord from the leading edge). Solve exact solution by hand and using the computer code. Also, solve the problem using Trapezoid Integration Rule (use 5, 10, and 20 intervals). On the same plot include exact solution, and the 5, 10, 20 interval numerical solution. Give a printout of your well-explained computer code. The code should provide the plots, which must be well-label plots for each load profile.

□

Problem 4.4.

Determine all internal loads for the following cantilevered beam:



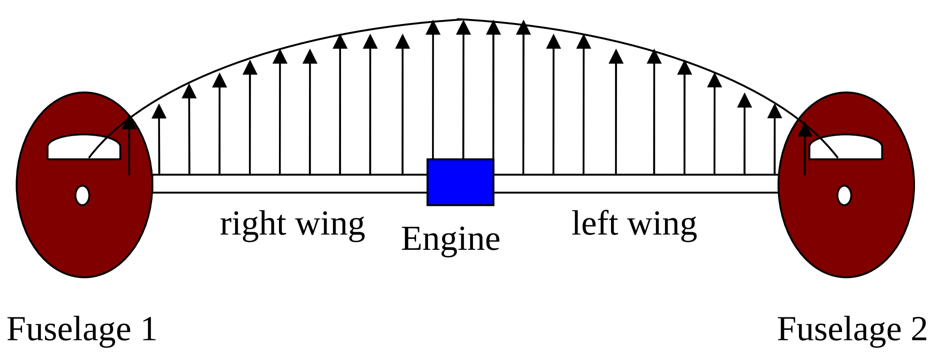
□

Problem 4.5.

One of the advanced airplanes that NASA is considering for carrying a large number of passengers is the Inboard-Wing Airplane. As shown in the figure the concept calls for the airplane to have two fuselages at the wing tips and the engine in the middle. Assuming the aerodynamic lift acting on the wing, per unit length, to be given by

$$p(x) = p_o\left(1 - \left(\frac{x}{b}\right)^2\right) \tag{4.25}$$

Here x is measured from the middle of the wing. The wing span is $2\,b$. Assume, the weight of the engine is $0.06W$ and the weight of the each of the fuselages to be $0.47W$; W being the total weight of the aircraft.



Determine:

1. Value of p_o such that the aircraft is in level flight.
2. Dimensionless shear force and moment diagrams neglecting wing weight.
3. Dimensionless Shear Force and Moment diagrams including wing weight.

□

Chapter 5
Analysis and Design of Beams

Instructional Objectives of Chapter 5

After completing this chapter, the student should be able to:

1. Discuss the consequences and apply the elementary beam theory: Euler-Bernoulli Beam Theory.
2. Identify the critical location and point in the beam.
3. Design beam under axial, bending, shear and torsional loads.

For many structural problems, it can be cumbersome to work with the three-dimensional model or even challenging to obtain an analytical solution. However, we can model most structural components using one- and two-dimensional model by using beams, plates or shells. In fact, we can often idealize many aerospace structural components as using *beams*. A beam can be defined as a structure having one of its dimensions much larger than the other two. The axis of the beam is defined along that longer dimension and the cross-section normal to this axis is assumed to smoothly vary along the span of the beam. Some examples of aeronautical structures modeled as thin-walled beams are wings and fuselages.

In this chapter we discuss the beam theory to analyze various types of beams. Beam theory is the solid mechanics theory describing beams and it plays an important role in structural analysis as it is a simple tool to analyze numerous structures. Although nowadays we have three-dimensional finite element computer codes to analyze loads and deflections, beam models help us in the pre-design stage as they provide valuable insight into the behavior of the structure. Such calculations are also very useful in validating computational solutions.

5.1 Properties of Plane Areas

5.1.1 Area

The total area bound by an area, A, of a bounded plane object is defined as the integral over the area of an element dA,

$$A = \iint_A dA$$

5.1.2 First Moments of Area

The first moment of area bound by an area, A, of a bounded plane object is defined as the integral of the distance parallel over the area of an element dA,

$$Q_{\mathrm{y}} = \iint_A z\,dA \qquad Q_{\mathrm{z}} = \iint_A y\,dA$$

5.1.3 Centroid of an Area

The **centroid** of an area is a geometric center of the cross section. The point in a member at the intersection of two perpendicular axes so located that the moments of the areas on opposite sides of an axis about that axis is zero.

$$y_c = \frac{Q_{\mathrm{z}}}{A} \qquad z_c = \frac{Q_{\mathrm{y}}}{A}$$

5.1.4 Second Moments of Area

The second moment of area bound by an area, A, of a bounded plane object is defined as the integral of the distance parallel squared over the area of an element dA,

$$I_{\mathrm{yy}} = \iint_A z^2\,dA \qquad I_{\mathrm{zz}} = \iint_A y^2\,dA \qquad I_{\mathrm{yz}} = \iint_A y\,z\,dA$$

The above moments of inertia are defined about the centroid. To define the moment of inertia about the another coordinate system we can use the parallel axis theorem:

$$\bar{I}_{\mathrm{yy}} = I_{\mathrm{yy}} + z_c^2\,A$$

$$\bar{I}_{\mathrm{zz}} = I_{\mathrm{zz}} + y_c^2\,A$$

$$\bar{I}_{\mathrm{yz}} = I_{\mathrm{yz}} + y_c\,z_c\,A$$

The polar moment of inertia is defined as

$$I_{xx} = I_{yy} + I_{zz}$$

5.1.5 Radius of Gyration

A distance known as the radius of gyration is occasionally encountered in mechanics. Radius of gyration of a plane area is defined as the square root of the moment of inertia of the area divided by the area itself:

$$r_y = \sqrt{\frac{I_{yy}}{A}} \qquad r_z = \sqrt{\frac{I_{zz}}{A}}$$

in which r_y and r_z denote the radius of gyration with respect to y and z axes, respectively. Although the radius of gyration of an area does not have an obvious physical meaning, we may consider it to be the distance (from the reference axis) at which the entire area could be concentrated and still have the same moment of inertia as the original area. The radius of gyration about the another coordinate system is be defined as

$$\bar{r}_y^2 = z^2 + r_y^2$$

$$\bar{r}_z^2 = y^2 + r_z^2$$

5.2 Beam Theory

In structural analysis, we have several beam theories based on various assumptions, and they lead to different levels of accuracy. One of the simplest and most useful of these theories was first described by Euler and Bernoulli and it is known as the Euler-Bernoulli beam theory. Before we proceed, let us define our sign convection and then go into the details of the Euler-Bernoulli beam theory assumptions and consequences. Afterwards we will discuss a more general beam theory known as the Timoshenko Beam Theory and Elasticity solution via Airy's Stress function.

5.2.1 Basic Considerations

In the previous chapter, we showed that to solve any problem involving an elastic body we are indeed solving the elasticity field (15 unknowns and 15 equations). When we study beams, this is not an exception: we are indeed solving the 15 unknowns by using 15 equations described by the elasticity field:

1. First, we need to ensure that the stresses and forces on any element of the beam satisfies the **equations of equilibrium**.

2. Second, we use the proper **constitutive law** that best describes the material behavior under a given state of stress.

3. Third, we need to ensure that the obtained strains are such that the resulting deflections of the members are compatible with each other.

4. Finally, if the system is subject to a temperature change, we may have to account for thermal expansions or contractions that can give rise to significant stresses and strains, so-called thermal effects.

As we will see later, many of the 15 unknown are zero as a consequence of beam theory assumptions.

5.2.2 Principle of Saint-Venant

Before we proceed, let us present a very useful assumption in our beam analysis: The Saint-Venant's principle. Saint-Venants principle is used to justify approximate solutions to boundary value problems in linear elasticity. For an example, when solving problems involving bending or axial deformation of slender beams and rods, one does not prescribe loads in detail. Instead, the resultant forces acting on the ends of a rod is specified, or the magnitudes of point forces acting on a beam. Saint Venants principle used to justify this approach.

Definition

Note that most solutions only provide average stresses at a section. Since, at concentrated forces and abrupt changes in cross section, irregular local stresses (and strains) arise, only at distance about equal to the depth of the member from such disturbances are the stresses in agreement with the mechanics of materials. This is due to SaintVenant's Principle: *The stress of a member at points away from points of load application may be obtained on the basis of a statically equivalent loading system; that is, the manner of force application on stresses is significant only in the vicinity of the region where the force is applied.* In other words, *it is to say that the manner in which the forces are distributed over a region is important only in the vicinity of the region.* This is also valid for the disturbances caused by the changes in the cross section. The mechanics of materials approach is therefore best suited for relatively slender members.

Applications and Limitations

The solution to the beam problem is quite complex when considering some definite distribution of surface forces on the end sections of the beam. Hence, we use Saint-Venants principle to justify the approximate solutions to boundary value problems in linear elasticity. It allows us to simplify the solution of many problems by altering the boundary conditions while keeping the systems of applied forces statically equivalent. We can obtain a satisfactory approximate solution to our problem.

We should keep in mind that the Saint Venants Principle does not yield any details about individual stress components at any specific point in an elastic body although we often want such information. Saint-Venant himself limited his principle to the problem of extension, torsion and flexure of prismatic and cylindrical bodies.

It should be pointed out that Saint-Venants principle should be handled with care when working with composites structures, which are strongly anisotropic and inhomogeneous materials. The main reason is that the end effects decay more slowly in anisotropic and inhomogeneous structures than in isotropic and homogeneous ones. The knowledge of characteristic decay length for end effects or local loadings is also important in numerical (i.e., finite elements) modeling of complex composite structures. In order to explain how the stress-strain fields depend on the distribution of tractions and anisotropy of material, we may examine an arbitrary distribution of tractions acting in a small region on the surface of an orthotropic elastic half-plane and analyze the generated stress and displacement fields. However, this is beyond the topic of this book.

5.2.3 Internal Force Sign Convention

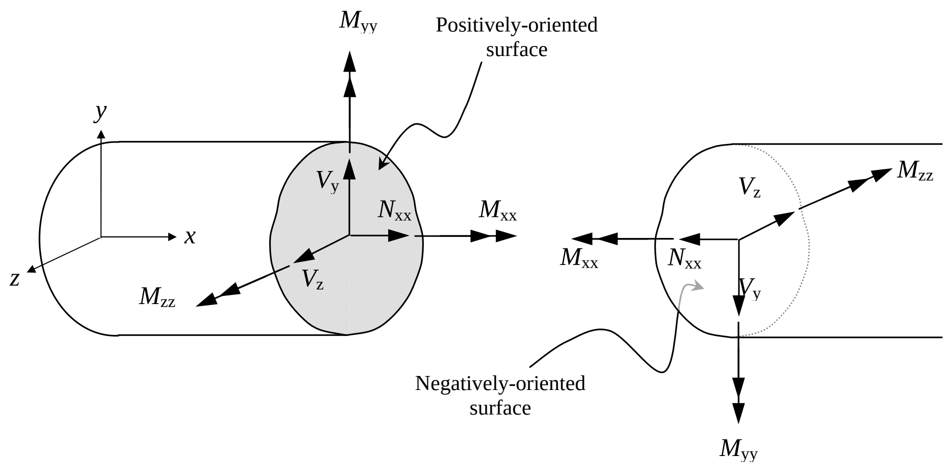


Figure 5.1: Sign convention for stress resultants on a beam cross section.

The sign convention will be consistent with the sign convention established for stress components, in chapter 2. We will use a rectangular coordinate system whose x-axis always coincides with, or is parallel to, the beam's axis. The y-axis and z-axis are the cross-sectional axis. The y-axis will be the transverse axis and the z-axis the lateral axis. On any cross section of the beam, the internal stress distribution generally gives rise to a resultant force and a couple, each being vectors with three components. These components are shown in Fig. 5.1.

5.2.4 Resultant Forces and Moments

The goal of the beam theory is to obtain a one-dimensional model of the three-dimensional structure which involves only sectional quantities, i.e. quantities solely dependent on the span-wise variable x. Now, we will describe the three-dimensional stress field, of a one-dimensional beam model, in terms of cross-sectional stresses called stress resultants.

Before we proceed, we should know that most *general* beams include: multiaxial bending, combined

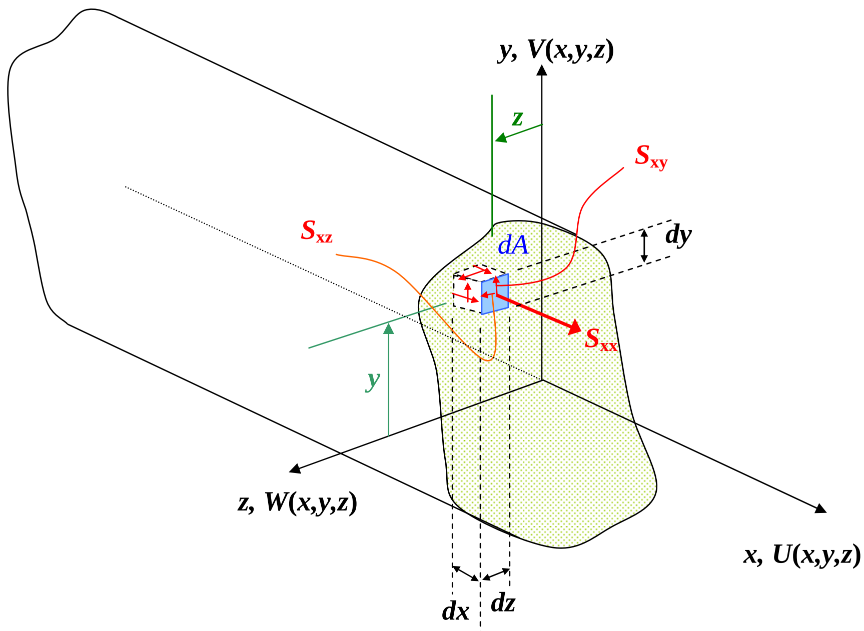


Figure 5.2: Stresses acting on a beam's cross-sectional differential volume.

axial and bending load, nonhomogeneous material makeup, thermal loads, and response to shear and torsion. With this in mind, consider a general beam's cross-section, as shown in Fig. 5.2. At the intersection of the x-axis with the cross-section, we obtain the three **internal stretching** resultants defined by the cutting plane as follow:

$$N_{xx} = N_{xx}(x) = \int_y \int_z S_{xx}\, dy\, dz = \iint_A S_{xx}\, dA$$

$$V_y = V_y(x) = \int_y \int_z S_{xy}\, dy\, dz = \iint_A S_{xy}\, dA$$

$$V_z = V_z(x) = \int_y \int_z S_{xz}\, dy\, dz = \iint_A S_{xz}\, dA$$

Also, at the intersection of the x-axis with the cross-section, we obtain the three **internal coupling** resultants defined by the cutting plane as follow:

$$M_{xx} = M_{xx}(x) = \int_y \int_z \left(S_{xz}\, y - S_{xy}\, z\right)\, dy\, dz = \iint_A \left(S_{xz}\, y - S_{xy}\, z\right)\, dA$$

$$M_{yy} = M_{yy}(x) = \int_y \int_z S_{xx}\, z\, dy\, dz = \iint_A S_{xx}\, z\, dA$$

$$M_{zz} = M_{zz}(x) = -\int_y \int_z S_{xx}\, y\, dy\, dz = -\iint_A S_{xx}\, y\, dA$$

Note that these internal resultants act on the positive face because the x-axis is coming out of it. Therefore, they must also be positive when they act in their respective negative coordinate axis directions on a negative face. In addition, these resultant loads are applied, generally, at the centroid and functions of the beams major axis, i.e., x-axis. The stresses are applied at a point but the load are applied over the entire cross-section.

5.3 Euler-Bernoulli Beam Theory

The bending elastic theory of beams is based on the following assumptions:

1. The beam has a constant and prismatic cross-section. It is made of a flexible and homogenous material that has the same modulus of elasticity in both tension and compression. In other words, it shortens or elongates equally for the same state of stress.
2. The material is isotropic and linearly elastic. Thus, Hooke's law is applicable. The relationship between the stress and strain is directly proportional. The beam is not stressed beyond its proportional limit.
3. Plane sections within the beam before bending remain plane after bending. This assumption ensures that the axial strain e_{xx} is a linear function of the cross-sectional coordinates y and z.
4. The neutral plane of a beam is a plane whose length is unchanged by the beam's deformation. This ensures that there will be no shear strain: $\gamma_{\mathrm{xy}} = \gamma_{\mathrm{xz}} = 0$. Since the material is isotropic: $S_{\mathrm{xy}} = S_{\mathrm{xz}} = 0$
5. Normal stresses S_{yy} and S_{zz} are negligible. Also, the shear stress S_{yz} is negligible.

These assumptions are known as the Euler-Bernoulli assumptions for beams. Experimental measurements show that these assumptions are valid for long, slender beams[1] made of isotropic materials with solid cross-sections. We should keep in mind that the Euler-Bernoulli beam theory does not apply when any of the above conditions is not met.

[1]Slender beams are those beams with their length typically greater than 15-20 times the largest cross-sectional dimension

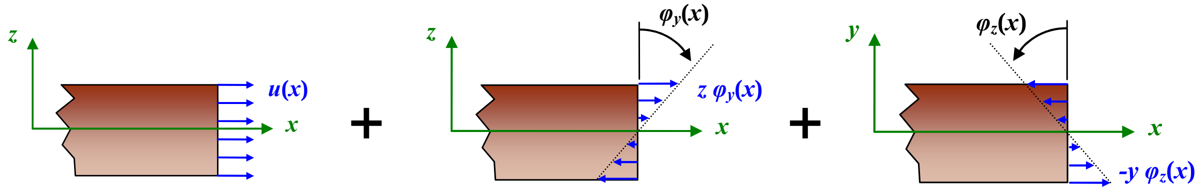


Figure 5.3: Decomposition of the axial displacement field.

5.3.1 Displacement Field

Consider a set of unit vectors $\hat{\underline{\mathbf{i}}}$, $\hat{\underline{\mathbf{j}}}$, and $\hat{\underline{\mathbf{k}}}$ with coordinates x, y, and z. This set of axes is attached at a point of the beam cross-section, x is along the axis of the beam, and y-z define the plane of the cross-section. Then, let $U(x, y, z)$, $V(x, y, z)$ and $W(x, y, z)$ be the displacement of an arbitrary point of the beam in the x, y and z directions, respectively. The displacement vector $\underline{\mathbf{R}}$ of a point is defined as

$$\underline{\mathbf{R}} = U(x, y, z)\,\hat{\underline{\mathbf{i}}} + V(x, y, z)\,\hat{\underline{\mathbf{j}}} + W(x, y, z)\,\hat{\underline{\mathbf{k}}} \tag{5.1}$$

Now we proceed to determine the directional displacements $U(x, y, z)$, $V(x, y, z)$ and $W(x, y, z)$ that satisfy Euler-Bernoulli beam theory:

1. First, Euler-Bernoulli assumption states that the cross-section is undeformable in its own plane. Hence, the displacement field in the plane of the cross-section consists solely of two rigid body translations $v(x)$ and $w(x)$:
$$V(x, y, z) = v(x)$$
$$W(x, y, z) = w(x)$$
2. Second, Euler-Bernoulli assumption states that the cross-section remains plane after deformation. This implies an axial displacement field consisting of a rigid body translation $u(x)$, and two rigid body rotations $\varphi_y(x)$ and $\varphi_z(x)$, as shown in Fig. 5.3.

As a consequence, the one-dimensional displacement field reduces to

$$U(x, y, z) = u(x) - y\,\varphi_z(x) + z\,\varphi_y(x) \tag{5.2a}$$

$$V(x, y, z) = v(x) \tag{5.2b}$$

$$W(x, y, z) = w(x) \tag{5.2c}$$

where $u(x)$ is the axial displacement, $v(x)$ the transverse displacement, $w(x)$ the lateral displacement, $\varphi_y(x)$ the rotation of the transverse normals with respect to x, and $\varphi_z(x)$ the in-plane rotation. All these displacements and rotations are measured at the midsurface.

5.3.2 Curvatures

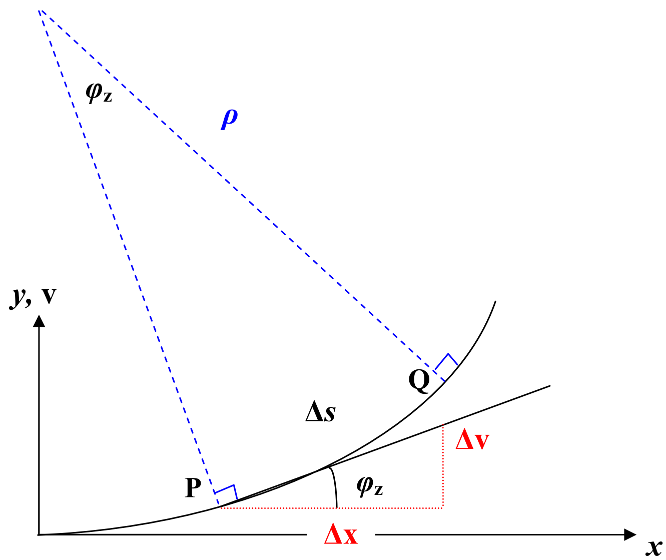


Figure 5.4: Definition of curvature

During bending, a beam bends in the form of a curve. The curvature κ_{zz} (more precisely, its inverse, the radius of curvature) helps us to determine the strain in any fiber of the beam. To understand the concept of curvature of a curve, let us consider the displacement v, in the y-direction, as a function of x. We can define the curvature κ_{zz} at a point **P** of a curve in the v-x plane with s, the length of the arc measured along the curve, as follows:

$$\kappa_{zz} = \frac{1}{\rho_z} = \frac{d\varphi_z}{ds}$$

where φ_z is the angle between the line tangent to the curve and the x-axis, and ρ_z is the radius of curvature at **P**. Now let us consider another point **Q** in the neighborhood of **P** at a distance Δs. As the point **Q** approaches **P**, we can write

$$\cos\varphi_z = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \frac{dx}{ds}$$

$$\sin\varphi_z = \lim_{\Delta s \to 0} \frac{\Delta v}{\Delta s} = \frac{dv}{ds}$$

Note that

$$ds = \sqrt{dv^2 + dx^2} = dx\sqrt{1 + \left(\frac{dv}{dx}\right)^2}$$

where

$$v' = \frac{dv}{dx} = \text{slope of the curve at P}$$

Thus

$$\cos\varphi_z = \frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}} = \sqrt{\frac{1}{1+\left(\frac{dv}{dx}\right)^2}}$$

$$\sin\varphi_z = \frac{dv}{ds} = \frac{dv}{dx\sqrt{1+\left(\frac{dv}{dx}\right)^2}} = \frac{dv}{dx}\sqrt{\frac{1}{1+\left(\frac{dv}{dx}\right)^2}}$$

Thus,

$$\tan\varphi_z = \frac{dv}{dx}$$

Taking the derivative of $\tan\varphi_z$ with respect to ds:

$$\frac{d\tan\varphi_z}{ds} = \frac{d}{ds}\left(\frac{dv}{dx}\right) = \frac{dx}{ds}\frac{d}{dx}\left(\frac{dv}{dx}\right) = \cos\varphi_z\left(\frac{d^2v}{dx^2}\right)$$
$$\sec^2\varphi_z\frac{d\varphi_z}{ds} = \cos\varphi_z\left(\frac{d^2v}{dx^2}\right) \rightarrow \frac{d\varphi_z}{ds} = \cos^3\varphi_z\left(\frac{d^2v}{dx^2}\right)$$

Thus

$$\frac{d\varphi_z}{ds} = \frac{d^2v}{dx^2}\left(\frac{1}{1+\left(\frac{dv}{dx}\right)^2}\right)^{3/2}$$

For beam analysis, it is safe to assume that the rotation of the beam is very small:

$$\frac{dv}{dx} \ll 1$$

As a consequence

$$ds \approx dx$$

and

$$\frac{d\varphi_z}{ds} \approx \frac{d^2v}{dx^2} = v''$$

Thus the curvature and radius of curvature for small beam rotations in the x-y plane is given by

$$\kappa_{zz} = \frac{1}{\rho_z} \approx v''$$

Similarly, the curvature and radius of curvature for small beam rotations in the x-z plane is given by

$$\kappa_{yy} = \frac{1}{\rho_y} \approx -w''$$

5.3.3 Strains-displacement Equations

Now let us proceed to evaluate the displacement gradients for the Euler-Bernoulli Beam Theory:

$$g_1 = \frac{\partial U}{\partial x} = u' - y\,\varphi_z' + z\,\varphi_y' \qquad g_4 = \frac{\partial U}{\partial y} = -\varphi_z \qquad g_7 = \frac{\partial U}{\partial z} = \varphi_y$$

$$g_2 = \frac{\partial V}{\partial x} = \frac{\partial v}{\partial x} \qquad g_5 = \frac{\partial V}{\partial y} = 0 \qquad g_8 = \frac{\partial V}{\partial z} = 0$$

$$g_3 = \frac{\partial W}{\partial x} = \frac{\partial w}{\partial x} \qquad g_6 = \frac{\partial W}{\partial y} = 0 \qquad g_9 = \frac{\partial W}{\partial z} = 0$$

Thus, the strain-displacement equations for normal strains are

$$e_{xx} = \frac{\partial U}{\partial x} = \frac{du}{dx} - y\,\frac{d\varphi_z}{dx} + z\,\frac{d\varphi_y}{dx}$$

$$e_{yy} = \frac{\partial V}{\partial y} = 0$$

$$e_{zz} = \frac{\partial W}{\partial z} = 0$$

The strain-displacement equations for shear strains are

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = -\varphi_z + \frac{dv}{dx}$$

$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \varphi_y + \frac{dw}{dx}$$

$$\gamma_{yz} = \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} = 0 + 0 = 0$$

Recall that the Euler-Bernoulli hypothesis states that plane sections remain plane after deformation, thus no shear strain will be present:

$$\gamma_{xy} = 0 = -\varphi_z + \frac{dv}{dx} \quad \rightarrow \varphi_z = \frac{dv}{dx}$$

$$\gamma_{xz} = 0 = \varphi_y + \frac{dw}{dx} \quad \rightarrow \varphi_y = -\frac{dw}{dx}$$

Thus

$$e_{xx} = \frac{\partial U}{\partial x} = \frac{du}{dx} - y\,\frac{d^2v}{dx^2} - z\,\frac{d^2w}{dx^2}$$

Let us proceed to physically understand the above strain equations. In order to do so, consider Figure 5.5. Based on the Euler-Bernoulli hypothesis, the deformed shape in the x-y plane is shown by $\mathbf{A'B'C'D'}$ of a beam differential element **ABCD**, of length dx. Note that the fibers towards the top of the element are under compression and those towards the bottom of the differential element are under tension. This means, there is a fiber in the differential element which will stay unstrained, say the fiber **PQ**. The complete deformation of the beam can then be expressed by the displacement v of this unstretched fiber (called neutral axis) in the y-direction.

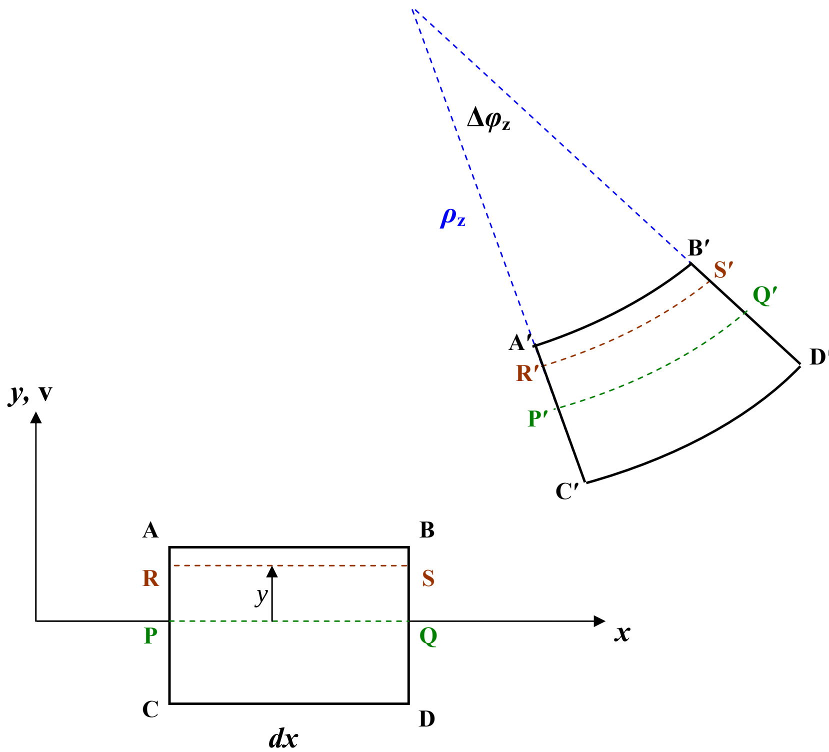


Figure 5.5: Bending of a beam element of length dx in the x-y plane.

Note that the transverse deflection v is positive if it is in the positive y-direction. As the length of the differential element approaches zero, the deformed shape of the beam approaches a circle of radius ρ_z, given as

$$\frac{1}{\rho_z} \approx \frac{d^2v}{dx^2}$$

The axial strain in any fiber, say **RS** before deformation and **R′S′** after deformation, lying at a distance y from the neutral axis, can be written as:

$$e_{xx}(y) = \frac{\mathbf{R'S'} - \mathbf{RS}}{\mathbf{RS}}$$

Note that

$$\mathbf{RS} = \mathbf{PQ} = \mathbf{P'Q'}$$

Thus

$$e_{xx}(y) = \frac{\mathbf{R'S'} - \mathbf{PQ}}{\mathbf{PQ}} = \frac{\mathbf{R'S'} - \mathbf{P'Q'}}{\mathbf{P'Q'}}$$

$$= \frac{(\rho_z - y)\,\Delta_z - \rho_z\,\Delta_z}{\rho_z\,\Delta_z} = -\frac{y}{\rho_z} \approx -y\,\frac{d^2 v}{dx^2}$$

Note that the axial strain e_{xx} due to a displacement in the x-y plane is a linear function of y. Using similar arguments, the axial strain e_{zz} for a fiber at a distance z from the unstretched fiber can be written as

$$e_{xx}(z) = -\frac{z}{\rho_y} \approx -z\,\frac{d^2 w}{dx^2}$$

In addition, axial strain in the x-direction is given by

$$e_{xx}(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$$

Thermal axial strain in the x-direction is given by

$$e_{xx} = \alpha\,\Delta T$$

The total axial strain in the x-direction is:

$$e_{xx} = \frac{du}{dx} - y\,\frac{d^2 v}{dx^2} - z\,\frac{d^2 w}{dx^2} + \alpha\,\Delta T = u' - y\,v'' - z\,w'' + \alpha\,\Delta T \tag{5.3}$$

and the **mechanical** axial strain in the x-direction is:

$$e_{xx} = \frac{du}{dx} - y\,\frac{d^2 v}{dx^2} - z\,\frac{d^2 w}{dx^2} = u' - y\,v'' - z\,w'' \tag{5.4}$$

Note that this is true only if the bending moments are caused by pure couples and axial loads. Theses applied couples must have no axial or torsional component to produce twist about the beam's longitudinal axis, x.

In general, we write the strains in terms of the midsurface strains as follows:

$$e_{xx} = \varepsilon^{\circ}_{xx} - y\,\kappa^{\circ}_{zz} + z\,\kappa^{\circ}_{yy} + \alpha\,\Delta T \tag{5.5}$$

where the mid-plane strains are defined as follow:

$$\varepsilon^{\circ}_{xx} = \frac{du}{dx}$$

$$\kappa^{\circ}_{yy} = \frac{d\varphi_y}{dx}$$

$$\kappa^{\circ}_{zz} = \frac{d\varphi_z}{dx}$$

For the Euler-Bernoulli beam assumptions, we derived the midplane strains as:

$$\varepsilon^{\circ}_{xx} = \frac{du}{dx}$$

$$\kappa^{\circ}_{yy} = \frac{d\varphi_y}{dx} \approx -\frac{d^2 w}{dx^2}$$

$$\kappa^{\circ}_{zz} = \frac{d\varphi_z}{dx} \approx \frac{d^2 v}{dx^2}$$

5.3.4 Stress-Strain Equations

Note that since the deformations and rotations are small, the PK2 stresses become:

$$\underline{\mathbf{S}} \approx \underline{\sigma}$$

Hence, from this point forward we will use cauchy's stresses throughout our beam analysis.

The Euler-Bernoulli theory imposes a state of uniaxial strain in a beam, with e_{xx} being the only nonzero strain. The Poisson effect, requires that

$$e_{yy} = e_{zz} = -\nu\, e_{xx}$$

The assumed deformation field has obvious errors even for small deformation. For instance, in the case of a thin beam ($b \ll L$) and with no loading on the sides of the beam, we expect that the stresses S_{zz} would be very close to zero. However, for a Hookean material the stresses show that

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} e_{xx} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\,\nu)} \begin{Bmatrix} 1-\nu \\ \nu \\ \nu \end{Bmatrix} e_{xx}$$

Now, when we take S_{zz} into account, we should note that there are zero tractions on the bottom surface of the beam and that we can consider the direct stress S_{zz} at the surface of contact for load p_z to be negligibly small compared to bending and stretching stress S_{xx}. Accordingly, for $h \ll L$, the stress S_{zz} should be very small. Hooke's law, however, gives:

$$S_{zz} = -\frac{E\,\nu}{(1+\nu)(1-2\,\nu)}\, e_{xx}$$

If we use a constitutive law wherein $\nu = 0$, we will get the desired zero value for the stresses S_{yy} and S_{zz}. Therefore, in elementary beam theory (Euler-Bernoulli), the Poisson effect is ignored.

Hence, setting $\nu = 0$ and substituting the strains into the Hooke's Law we get:

$$\left\{\begin{array}{c} S_{xx} \\ S_{yy} \\ S_{zz} \end{array}\right\} = \frac{E}{(1+0)(1-2\,(0))}\begin{bmatrix} 1-0 & 0 & 0 \\ 0 & 1-0 & 0 \\ 0 & 0 & 1-0 \end{bmatrix}\left\{\begin{array}{c} e_{xx} \\ 0 \\ 0 \end{array}\right\} = E\left\{\begin{array}{c} e_{xx} \\ 0 \\ 0 \end{array}\right\}$$

and

$$S_{xy} = G\,(0) = 0 \qquad S_{xz} = G\,(0) = 0 \qquad S_{yz} = G\,(0) = 0$$

Using the Hookean stress-strain relationship we get

$$S_{xx} = E\,e_{xx} = E\,\left(\varepsilon^{\circ}_{xx} - y\,\kappa^{\circ}_{zz} + z\,\kappa^{\circ}_{yy} - \alpha\,\Delta T\right) \tag{5.6}$$

5.3.5 Neutral Axis

The **neutral axis** is a line in the cross section of a beam along which no bending stresses occur. Thus at the neutral axis we have zero strain and therefore zero stress in absence of axial loads. The neutral axis is perpendicular to the line of applied force. *The neutral axis passes through the centroid for uniform elastic beams with symmetric cross-sections.*

5.3.6 Axial Stresses for Linear Thermoelastic Homogeneous Beams

To continue with our discussion of the elementary beam theory, let us consider a beam with homogeneous material properties; i.e., assume that they can be functions of x, y, and z coordinate location. Let us also assume the following:

1. Suppose also that the beam can be heated or cooled such that the change in temperature ΔT is a function of location in the beam.
2. The transverse components of normal stresses S_{yy} and S_{zz} are negligible compared to axial stress S_{xx}.
3. Cross-sections remain planar and normal to the centroid axis of deformation.

Therefore, strain-displacement equation, Eq. (5.4) is still valid. Thus, the linear thermoelastic stress-strain relationship becomes

$$S_{xx} = E\,e_{xx} = E\,\left(\varepsilon^{\circ}_{xx} - y\,\kappa^{\circ}_{zz} + z\,\kappa^{\circ}_{yy} - \alpha\,\Delta T\right) \tag{5.7}$$

where it should be noted that the modulus E and thermal coefficient may be a function of x, y, and z:

$$E = E(x, y, z) \qquad \alpha = \alpha(x, y, z)$$

Substituting into the stress resultant equations we get:

$$N_{\rm xx} = \iint_A S_{\rm xx}\, dA = \iint_A E\,\left(\varepsilon^\circ_{\rm xx} - y\,\kappa^\circ_{\rm zz} + z\,\kappa^\circ_{\rm yy} - \alpha\,\Delta T\right)\, dA$$

$$V_{\rm y} = \iint_A S_{\rm xy}\, dA = 0$$

$$V_{\rm z} = \iint_A S_{\rm xz}\, dA = 0$$

Substituting into the couple resultant equations we get:

$$M_{\rm xx} = \iint_A (S_{\rm xz}\, y - S_{\rm xy}\, z)\, dA = 0$$

$$M_{\rm yy} = \iint_A S_{\rm xx}\, z\, dA = \iint_A E\,\left(\varepsilon^\circ_{\rm xx} - y\,\kappa^\circ_{\rm zz} + z\,\kappa^\circ_{\rm yy} - \alpha\,\Delta T\right)\, z\, dA$$

$$M_{\rm zz} = -\iint_A S_{\rm xx}\, y\, dA = -\iint_A E\,\left(\varepsilon^\circ_{\rm xx} - y\,\kappa^\circ_{\rm zz} + z\,\kappa^\circ_{\rm yy} - \alpha\,\Delta T\right)\, y\, dA$$

where $dA = dy\, dz$. Because the modulus E can be dependent on location in the cross section, it cannot be taken outside the integral as it was in the theory of homogeneous beams[2]. Moreover, we can obtain can express the above internal resultants as functions of the cross-sectional properties. In order to accomplish this, let us factor our from the internal resultant equations an arbitrary constant E, with the same units as modulus and called the reference modulus, as follows:

$$\begin{aligned} N_{\rm xx} &= E \iint_A \left(\varepsilon^\circ_{\rm xx} - y\,\kappa^\circ_{\rm zz} + z\,\kappa^\circ_{\rm yy} - \alpha\,\Delta T\right)\, dA \\ &= E\,\varepsilon^\circ_{\rm xx} \iint_A dA - E\,\kappa^\circ_{\rm zz} \iint_A y\, dA + E\,\kappa^\circ_{\rm yy} \iint_A z\, dA - E \iint_A \xi\,\alpha\,\Delta T\, dA \end{aligned}$$

$$\begin{aligned} M_{\rm yy} &= E \iint_A \left(\varepsilon^\circ_{\rm xx} - y\,\kappa^\circ_{\rm zz} + z\,\kappa^\circ_{\rm yy} - \alpha\,\Delta T\right)\, z\, dA \\ &= E\,\varepsilon^\circ_{\rm xx} \iint_A z\, dA - E\,\kappa^\circ_{\rm zz} \iint_A y\, z\, dA + E\,\kappa^\circ_{\rm yy} \iint_A z^2\, dA - E \iint_A \xi\,\alpha\,\Delta T\, z\, dA \end{aligned}$$

[2] Although the Euler-Bernoulli theory assumes that the beam is homogeneous, it can be shown that this is acceptable

$$M_{zz} = -E \iint_A \left(\varepsilon^\circ_{xx} - y\,\kappa^\circ_{zz} + z\,\kappa^\circ_{yy} - \alpha\,\Delta T\right) y\,dA$$

$$= -E\,\varepsilon^\circ_{xx} \iint_A y\,dA + E\,\kappa^\circ_{zz} \iint_A y^2\,dA - E\,\kappa^\circ_{yy} \iint_A y\,z\,dA + E \iint_A \xi\,\alpha\,\Delta T\,y\,dA$$

Now we use the plane area definitions, to determine all modulus weighted sectional properties:

$$Q_y = \iint_A z\,dA \qquad 1^{st} \text{ moment of area of the cross section about the } y\text{-axis}$$

$$Q_z = \iint_A y\,dA \qquad 1^{st} \text{ moment of area of the cross section about the } z\text{-axis}$$

$$I_{yy} = \iint_A z^2\,dA \qquad \text{moment of inertia of the cross section about the } y\text{-axis}$$

$$I_{zz} = \iint_A y^2\,dA \qquad \text{moment of inertia of the cross section about the } z\text{-axis}$$

$$I_{yz} = \iint_A y\,z\,dA \qquad \text{moment of inertia of the cross section relative to the centroid}$$

Furthermore, let the thermal loads be define as,

$$N^t_{xx} = E \iint_A \xi\,\alpha\,\Delta T\,dA$$

$$M^t_{yy} = E \iint_A \xi\,\alpha\,\Delta T\,z\,dA$$

$$M^t_{zz} = E \iint_A \xi\,\alpha\,\Delta T\,y\,dA$$

Thus the internal force and moments are

$$N_{xx} + N^t_{xx} = E\,A\,\varepsilon^\circ_{xx} - E\,Q_y\,\kappa^\circ_{zz} + E\,Q_z\,\kappa^\circ_{yy}$$

$$M_{yy} + M^t_{yy} = E\,Q_y\,\varepsilon^\circ_{xx} - E\,I_{yz}\,\kappa^\circ_{zz} + E\,I_{yy}\,\kappa^\circ_{yy}$$

$$M_{zz} - M^t_{yy} = E\,Q_z\,\varepsilon^\circ_{xx} + E\,I_{zz}\,\kappa^\circ_{zz} - E\,I_{yz}\,\kappa^\circ_{yy}$$

or in matrix form

$$\begin{Bmatrix} N_{xx} + N_{xx}^t \\ M_{yy} + M_{yy}^t \\ M_{zz} - M_{zz}^t \end{Bmatrix} = E \begin{bmatrix} A & Q_y & Q_z \\ Q_y & I_{yz} & I_{yy} \\ Q_z & -I_{zz} & -I_{yz} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^\circ \\ -\kappa_{zz}^\circ \\ \kappa_{yy}^\circ \end{Bmatrix} \quad \text{for actual coordinates} \tag{5.8}$$

These equations express a general linear relationship between the sectional stress resultants and the cross-sectional strains. Thus, they are the constitutive laws for the cross-section of the beam, and the matrix on the right hand side of Eq. 5.8 is called the sectional stiffness matrix. Clearly, these equations are fully coupled: all of the sectional strains affect the values of each of the sectional stress and couple resultants.

For homogeneous beams we use the modulus weighted x centroidal axis; i.e., the x-axis is passed through the modulus weighted centroid:

$$y_c = \frac{Q_z}{A} = \frac{1}{A} \iint_A y \, dA \qquad z_c = \frac{Q_y}{A} = \frac{1}{A} \iint_A z \, dA$$

By transforming the x-axis in this way, it will follow that $\bar{y}_c = \bar{z}_c = 0$ and the internal resultant equations reduce to

$$\begin{Bmatrix} N_{xx} + N_{xx}^t \\ M_{yy} + M_{yy}^t \\ M_{zz} - M_{zz}^t \end{Bmatrix} = E \begin{bmatrix} A & 0 & 0 \\ 0 & I_{yz} & I_{yy} \\ 0 & -I_{zz} & -I_{yz} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^\circ \\ -\kappa_{zz}^\circ \\ \kappa_{yy}^\circ \end{Bmatrix} \quad \text{for centroidal coordinates} \tag{5.9}$$

In order to obtain the strain,

$$e_{xx} = \varepsilon_{xx}^\circ - y \, \kappa_{zz}^\circ + z \, \kappa_{yy}^\circ = \begin{bmatrix} 1 & y & z \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^\circ \\ -\kappa_{zz}^\circ \\ \kappa_{yy}^\circ \end{Bmatrix}$$

we need to obtain the inverse of Eq. (5.9). For simplicity let us define

$$\Delta_I = \begin{vmatrix} I_{yz} & I_{yy} \\ -I_{zz} & -I_{yz} \end{vmatrix} = I_{yy} \, I_{zz} - (I_{yz})^2$$

and the second moment of area ratios as:

$$R_{yy} = \frac{\Delta_I}{I_{yy}} \qquad R_{zz} = \frac{\Delta_I}{I_{zz}} \qquad R_{yz} = \frac{\Delta_I}{I_{yz}}$$

Then inverse of Eq. (5.9) can be easily shown to be

$$\left\{\begin{array}{c}\varepsilon^{\circ}_{xx}\\ -\kappa^{\circ}_{zz}\\ \kappa^{\circ}_{yy}\end{array}\right\} = \frac{1}{E}\begin{bmatrix}\frac{1}{A} & 0 & 0\\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}}\\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}}\end{bmatrix}\left\{\begin{array}{c}N_{xx}+N^t_{xx}\\ M_{yy}+M^t_{yy}\\ M_{zz}-M^t_{zz}\end{array}\right\} \tag{5.10}$$

Thus, the axial strain for general homogeneous beams with thermal loads is

$$e_{xx} = \left[\begin{array}{ccc}1 & y & z\end{array}\right]\left\{\begin{array}{c}\varepsilon^{\circ}_{xx}\\ -\kappa^{\circ}_{zz}\\ \kappa^{\circ}_{yy}\end{array}\right\} = \frac{1}{E}\left\{\begin{array}{ccc}1 & y & z\end{array}\right\}\begin{bmatrix}\frac{1}{A} & 0 & 0\\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}}\\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}}\end{bmatrix}\left\{\begin{array}{c}N_{xx}+N^t_{xx}\\ M_{yy}+M^t_{yy}\\ M_{zz}-M^t_{zz}\end{array}\right\} \tag{5.11}$$

Then axial stress for general homogeneous beams with thermal loads can be written as

$$S_{xx} = E\,e_{xx} = E\,\left(\varepsilon^{\circ}_{xx} - y\,\kappa^{\circ}_{zz} + z\,\kappa^{\circ}_{yy} - \alpha\,\Delta T\right)$$

$$S_{xx} = \left\{\begin{array}{ccc}1 & y & z\end{array}\right\}\begin{bmatrix}\frac{1}{A} & 0 & 0\\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}}\\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}}\end{bmatrix}\left\{\begin{array}{c}N_{xx}+N^t_{xx}\\ M_{yy}+M^t_{yy}\\ M_{zz}-M^t_{zz}\end{array}\right\} - E\,\alpha\,\Delta T \tag{5.12}$$

where $E = E(y, z)$ is the Young's modulus of the material where the stress value is desired. Recall that the origin of the y-z plane coincides with the centroid of the cross section. Furthermore, for cross-section with an axis of symmetry with respect to one of the cross-sectional axis:

$$I_{yz} = 0 \quad \rightarrow \quad \frac{1}{R_{zz}} = \frac{1}{I_{yy}}, \quad \frac{1}{R_{yy}} = \frac{1}{I_{zz}}, \quad \frac{1}{R_{yz}} = 0$$

Thus the axial stress and strain for homogeneous beams, with symmetric cross-sections, subject to thermal loads can be written as

$$e_{xx} = \frac{1}{E}\left\{\begin{array}{ccc}1 & y & z\end{array}\right\}\begin{bmatrix}\frac{1}{A} & 0 & 0\\ 0 & 0 & -\frac{1}{R_{yy}}\\ 0 & \frac{1}{R_{zz}} & 0\end{bmatrix}\left\{\begin{array}{c}N_{xx}+N^t_{xx}\\ M_{yy}+M^t_{yy}\\ M_{zz}-M^t_{zz}\end{array}\right\} \tag{5.13}$$

$$S_{xx} = \left\{ \begin{array}{ccc} 1 & y & z \end{array} \right\} \begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & 0 & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & 0 \end{bmatrix} \left\{ \begin{array}{c} N_{xx} + N^t_{xx} \\ M_{yy} + M^t_{yy} \\ M_{zz} - M^t_{zz} \end{array} \right\} - E\,\alpha\,\Delta T \tag{5.14}$$

where

$$\left\{ \begin{array}{c} \varepsilon^\circ_{xx} \\ -\kappa^\circ_{zz} \\ \kappa^\circ_{yy} \end{array} \right\} = \frac{1}{E} \begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & 0 & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & 0 \end{bmatrix} \left\{ \begin{array}{c} N_{xx} + N^t_{xx} \\ M_{yy} + M^t_{yy} \\ M_{zz} - M^t_{zz} \end{array} \right\} \tag{5.15}$$

5.3.7 Equations of Equilibrium

In section 4.4.2 we got the following expressions for the equations of equilibrium:

$$\begin{array}{lll} \frac{dN_{xx}}{dx} = -p_x(x) & \frac{dV_y}{dx} = -p_y(x) & \frac{dV_z}{dx} = -p_z(x) \\ \frac{dM_{xx}}{dx} = -m_x(x) & \frac{dM_{yy}}{dx} = -m_y(x) + V_z & \frac{dM_{zz}}{dx} = -m_z(x) - V_y \end{array} \tag{5.16}$$

where $p_x(x)$ is the distributed load in the axial direction (x-axis), $p_y(x)$ the distributed load in the transverse direction (y-axis), $p_z(x)$ the distributed load in the transverse direction (z-axis), $m_x(x)$ the distributed moments about the x-axis, $m_y(x)$ the distributed moments about the y-axis, and $m_z(x)$ the distributed moments about the z-axis.

These equations are the first order ordinary differential equations that may be solved by direct integration. The solution to these equations is:

$$N_{xx}(x) = N_{xx}(x_1) - \int_{x_1}^{x} p_x(\zeta)\,d\zeta \tag{5.17}$$

$$V_y(x) = V_y(x_1) - \int_{x_1}^{x} p_y(\zeta)\,d\zeta \tag{5.18}$$

$$V_z(x) = V_z(x_1) - \int_{x_1}^{x} p_z(\zeta)\,d\zeta \tag{5.19}$$

$$M_{xx}(x) = M_{xx}(x_1) - \int_{x_1}^{x} m_x(\zeta)\,d\zeta \tag{5.20}$$

$$M_{yy}(x) = M_{yy}(x_1) - \int_{x_1}^{x} \Big\{ m_y(\zeta) - V_z(\zeta) \Big\}\,d\zeta \tag{5.21}$$

$$M_{zz}(x) = M_{zz}(x_1) - \int_{x_1}^{x} \Big\{ m_z(\zeta) + V_y(\zeta) \Big\}\,d\zeta \tag{5.22}$$

The first term on the right-hand side of the above equations are known as the boundary conditions; i.e., if the beam is statically determinate there will exist some point along the x-axis $x = x_1$ at which the resultants are known. For the case of statically indeterminate, the boundary conditions may be found using compatibility equations.

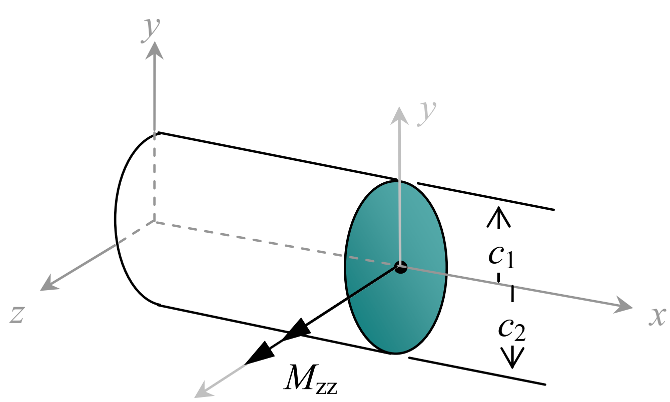
5.4 Symmetric cross-section about one axis

Suppose we have a symmetric cross-section of an isotropic beam and the beam can be analyzed using Euler-Bernoulli hypothesis. Consider a case of pure bending with $M_{yy} = 0$, thus,

$$\sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\, y \tag{5.23}$$

5.4.1 Maximum Bending Stresses

Consider the following figure:



Recall at the neutral axis there is no bending:

$$y = 0 \quad M_{zz} > 0 \quad \sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\,(0) = 0 \tag{5.24}$$

The maximum bending compression stress is:

$$y = c_1 \quad M_{zz} > 0 \quad \sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\,(c_1) = -\frac{M_{zz}}{I_{zz}}\, c_1 < 0 \qquad \therefore \sigma_{xx}\Big|_{\text{compression}} = \frac{M_{zz}}{I_{zz}}\, c_1$$

The maximum bending tension stress is:

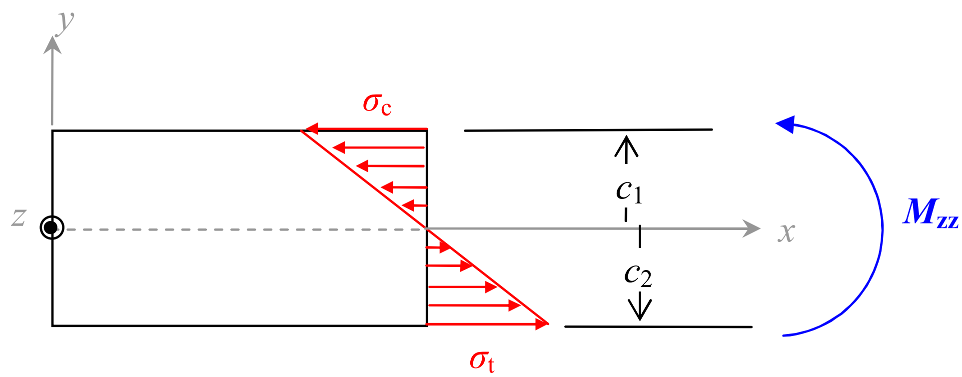
$$y = -c_2 \quad M_{zz} > 0 \quad \sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\,(-c_2) = \frac{M_{zz}}{I_{zz}}\, c_2 > 0 \qquad \therefore \sigma_{xx}\Big|_{\text{tension}} = \frac{M_{zz}}{I_{zz}}\, c_2$$

Note that the the maximum bending stress in tension and compression do not necessarily have the same magnitude:

$$\text{if} \quad c_1 = c_2 \quad \rightarrow \quad \sigma_{xx}\Big|_{\text{compression}} = \sigma_{xx}\Big|_{\text{tension}}$$

$$\text{if} \quad c_1 > c_2 \quad \rightarrow \quad \sigma_{xx}\Big|_{\text{compression}} < \sigma_{xx}\Big|_{\text{tension}}$$

$$\text{if} \quad c_1 < c_2 \quad \rightarrow \quad \sigma_{xx}\Big|_{\text{compression}} > \sigma_{xx}\Big|_{\text{tension}}$$



5.4.2 Summary: Pure Bending of Beams

The equation

$$\sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\, y \tag{5.25}$$

is limited to:

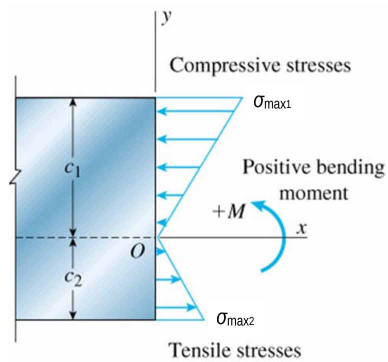
1. symmetric cross sections
2. moment acting in plane of symmetry
3. uniform bending
4. Euler-Bernoulli's assumptions:
   (a) plane sections remain plane
   (b) plane sections remain normal to reference surface
   (c) normal lines are inextensible
5. small deflections and slopes
6. no axial force
7. normal stresses in beams of linearly elastic material:

(a) normal stress distribution is linear

(b) z-axis is the neutral axis

(c) maximum normal stresses at

$$y = c_1 \rightarrow \sigma_{\max_1} = -\frac{M_{zz}}{I_{zz}}\, c_1 \qquad \text{and} \qquad y = -c_2 \rightarrow \sigma_{\max_2} = \frac{M_{zz}}{I_{zz}}\, c_2$$



It should be clear that the limitations of the present derivations consist in that the plane sections, in general, do not remain plane. We have shear forces associated with nonuniform bending (out-of-plane distortion, known as warping). However, normal stresses calculated by the flexure formula are not significantly altered by the presence of shear stresses and associated warping. Thus flexure formula gives results that are accurate in regions away from stress concentrations.

For design purposes we often define the stresses in terms of the section moduli as follows:

$$y = c_1 \qquad \sigma_{\max_1} = -\frac{M_{zz}}{I_{zz}}\, c_1 = -\frac{M_{zz}}{\frac{I_{zz}}{c_1}} = -\frac{M_{zz}}{Z_1}$$

$$y = -c_2 \quad \sigma_{\max_2} = -\frac{M_{zz}}{I_{zz}}\, (-c_2) = \frac{M_{zz}}{\frac{I_{zz}}{c_2}} = \frac{M_{zz}}{Z_2}$$

where the section moduli is defines as

$$Z_1 = \frac{I_{zz}}{c_1} \qquad Z_2 = \frac{I_{zz}}{c_2}$$

For doubly symmetric cross-sectional shapes:

$$c_1 = c_2 = c$$

and

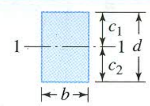
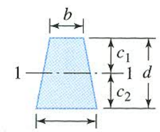
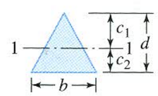
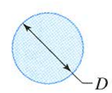
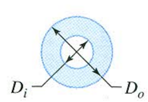
$$\sigma_{\max_o} = \sigma_{\max_2} = -\sigma_{\max_1} = \frac{M_{zz}}{I_{zz}}\, c = \frac{M_{zz}}{\frac{I_{zz}}{c}} = \frac{M_{zz}}{Z}$$

For efficient use of material:

1. place material as far away from neutral axis as possible
2. the large Z, the larger M_{zz} that can be resisted for a given allowable stress.

Table 5.1 was taken from Mechanical Design of Machine Elements and Machines by Jack A. Collins, 2003. John Wiley.

Table 5.1: Properties of Plane Cross-Sections

Shape		Area, A	Distances c_1 and c_2 to Outer Fibers	Moment of Inertia I About Centroidal Axis 1-1	Section Modulus $Z = I/c$ About Axis 1-1	Radius of Gyration, $\rho = \sqrt{I/A}$
1. Rectangle		bd	$c_1 = c_2 = \frac{d}{2}$	$\frac{bd^3}{12}$	$\frac{bd^2}{6}$	$\frac{d}{\sqrt{12}}$
2. Trapezoid		$\frac{(B+b)d}{2}$	$c_1 = \frac{b+2B}{3(b+B)}d$ $c_2 = \frac{2b+B}{3(b+B)}d$	$\frac{(B^2+4bB+b^2)d^3}{36(b+B)}$	—	$\frac{d}{6(b+B)}\sqrt{2(B^2+4bB+b^2)}$
3. Triangle		$\frac{bd}{2}$	$c_1 = \frac{2d}{3}$ $c_2 = \frac{d}{3}$	$\frac{bd^3}{36}$	$Z_1 = \frac{bd^2}{24}$ $Z_2 = \frac{bd^2}{12}$	$\frac{d}{\sqrt{18}}$
4. Solid circle		$\frac{\pi D^2}{4}$	$c = \frac{D}{2}$	$\frac{\pi D^4}{64}$	$\frac{\pi D^3}{32}$	$\frac{D}{4}$
5. Hollow circle		$\frac{\pi(D_o^2 - D_i^2)}{4}$	$c = \frac{D_o}{2}$	$\frac{\pi(D_o^4 - D_i^4)}{64}$	$\frac{\pi(D_o^4 - D_i^4)}{32D_o}$	$\frac{\sqrt{D_o^2 + D_i^2}}{4}$

Example 5.1.

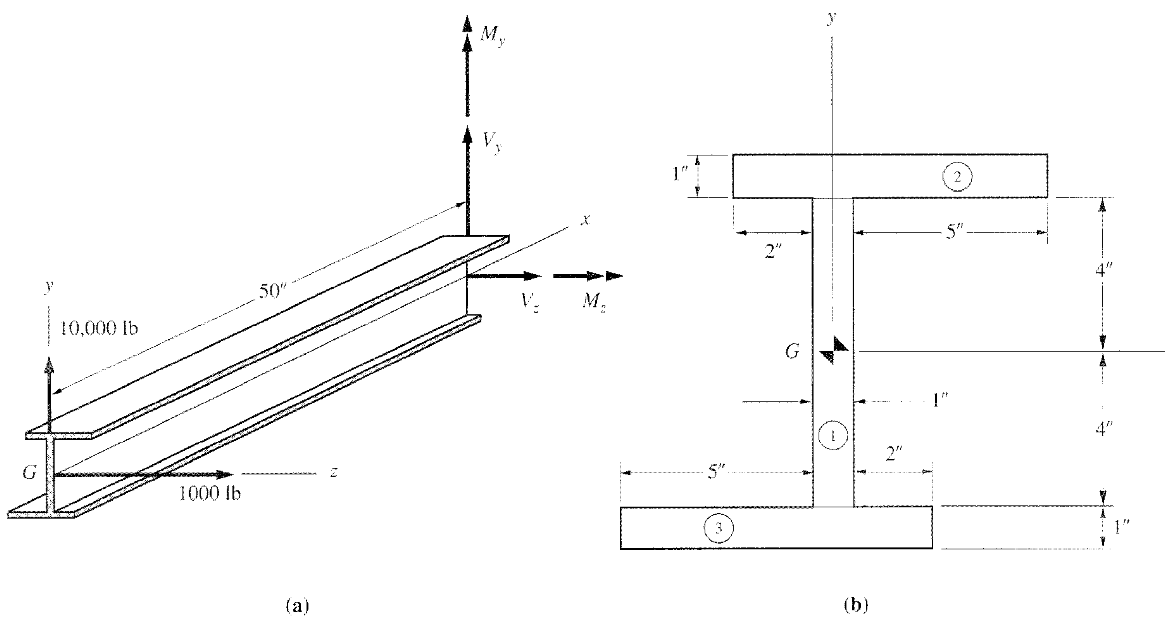


Figure 5.6: Beam's Cross Section

Figure 1a shows the first 50 inches of a 200-inch long cantilever beam. Applied loads act as shown through the centroid G (which in this case coincides with the shear center) of the free end. Calculate the bending stress distribution on the section at 50 inches from the free end.

We must calculate the area moments of inertia about the centroid G. To do so, we divide the cross section into three rectangles (Figure b) and use the parallel axis theorems. Thus, second moments of area are

$$
\begin{aligned}
I_{yy} &= I_{yy_1} + I_{yy_2} + I_{yy_3} \\
&= \left\{\frac{1}{12}(8)(1)^3\right\} + \left\{\frac{1}{12}(1)(8)^3 + (8\times 1)(1.5)^2\right\} + \left\{\frac{1}{12}(1)(8)^3 + (8\times 1)(-1.5)^2\right\} \\
&= 122.0 \text{ in}^4
\end{aligned}
$$

$$\begin{aligned} I_{zz} &= I_{zz_1} + I_{zz_2} + I_{zz_3} \\ &= \left\{ \frac{1}{12}\,(1)\,(8)^3 \right\} + \left\{ \frac{1}{12}\,(8)\,(1)^3 + (8\times 1)(4.5)^2 \right\} + \left\{ \frac{1}{12}\,(8)\,(1)^3 + (8\times 1)(-4.5)^2 \right\} \\ &= 368 \text{ in}^4 \end{aligned}$$

$$\begin{aligned} I_{yz} &= I_{yz_1} + I_{yz_2} + I_{yz_3} \\ &= \{(0)\} + \{(0) + (8)(4.5)(1.5)\} + \{(0) + (8)(-4.5)(-1.5)\} \\ &= 108 \text{ in}^4 \end{aligned}$$

The shear forces on the section can be calculated by taking sum of forces

$$+\uparrow \sum F_y = 0 = V_y + 10000 \text{ lb} \qquad \rightarrow \qquad V_y = -10000 \text{ lb}$$

$$+\rightarrow \sum F_z = 0 = V_z + 1000 \text{ lb} \qquad \rightarrow \qquad V_z = -1000 \text{ lb}$$

The bending moments on the section can be calculated by taking sum of moments at the free end

$$+\circlearrowleft \sum M_y = 0 = M_{yy} + (1000 \text{ lb})(50 \text{ in}) \qquad \rightarrow \qquad M_{yy} = -50000 \text{ lb–in}$$

$$+\circlearrowleft \sum M_z = 0 = M_{zz} - (10000 \text{ lb})(50 \text{ in}) \qquad \rightarrow \qquad M_{zz} = 500000 \text{ lb–in}$$

Substituting the bending moments and second moments of inertia into Eq. (5.12), we get

$$\sigma_{xx} = -1673\,y + 1071\,z \tag{5.26}$$

The maximum axial stress on the section is

$$\sigma_{xx}\Big|_{\max} = 11,040 \text{ psi at} \qquad y = -5 \text{ in}, \qquad z = 2.5 \text{ in} \tag{5.27}$$

The neutral axis of this section is defined by those points for which

$$\sigma_{xx} = 0 \qquad \rightarrow \qquad y = 0.6403\,z \tag{5.28}$$

End Example □

Example 5.2.

For example in Section 4.3: Students have approximated a machine component using a beam model as shown in Fig. 5.7. The cantilever beam's squared cross section is uniform. These engineers need your help to analyze this component and they have a five-day deadline to complete the analysis. Take $a = 25$ mm, $b = 5$ mm. Use the stress convention and show all your steps.

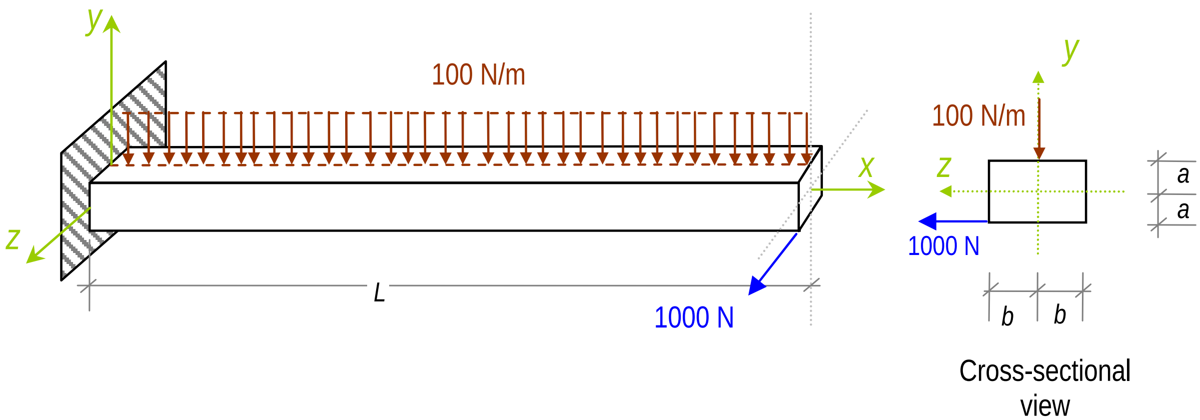


Figure 5.7: Machine component for example below.

What is the value for the maximum normal stress and where is it located at?

Using the flexure formula:

$$\sigma_{xx} = \left\{ \begin{array}{ccc} 1 & y & z \end{array} \right\} \begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}} \end{bmatrix} \left\{ \begin{array}{c} N_{xx} \\ M_{yy} \\ M_{zz} \end{array} \right\}$$

For our problem:

$$N_{xx} = 0 \qquad I_{yz} = 0 \qquad \Delta_{I} = \left(I_{yy}\, I_{zz} - I_{yz}^2\right) = \frac{16\, a^4\, b^4}{9}$$

Thus,

$$\begin{aligned}
\sigma_{\rm xx} &= -\frac{M_{\rm zz}}{I_{\rm zz}}\, y + \frac{M_{\rm yy}}{I_{\rm yy}}\, z \\
&= -\left\{ \frac{75\ (L-x)^2}{2\, a^3\, b} \right\} y - \left\{ \frac{750\ (L-x)}{a\, b^3} \right\} z \\
&= -4.8 \times 10^8\, (L-x)^2\, y - 2.4 \times 10^{11}\ (L-x)\ z
\end{aligned}$$

From the plots we see that the maximum moments occur at the fixed-end and thus the maximum normal stress will also occur at this point:

$$x = 0 \quad \rightarrow \quad \sigma_{\rm xx} = -4.8 \times 10^8\, L^2\, y - 2.4 \times 10^{11}\, L\, z$$

Maximum normal stress will occur at:

$$y = -a \qquad z = -b \qquad \rightarrow \sigma_{\rm xx} = 1200\, L + 12\, L^2 \text{ MPa}$$

End Example □

5.5 Transverse Shear Stress

Shear stress[3] is a stress state where the shape of a material tends to change (usually by *sliding* forces or torque by transversely-acting forces) without particular volume change. The shape change is evaluated by measuring the change of the angle's magnitude (shear strain). In laboratory testing, shear stress is achieved by torsion of a specimen. Direct shear of a specimen by a moment induces shear stress, as well as tensile and compressive stress. Structural members subjected in pure shear stress are the torsion bars and the driving shafts in automobiles. Riveted joints and some bolts are also subjected mainly to shear stress. Cantilevers, beams, consoles and column heads are subject in composite loading, consising of shear, tensile and compressive stress. Also constructions in soil can fail due to shear, e.g. the weight of an earth fill dam or dike may cause the subsoil to collapse, like a small landslide.

5.5.1 Bending of Symmetric Beams with Shear

For small members in shear, it is customary to assume shear is uniformly distributed over the entire area:

$$\tau = \frac{V_{\mathrm{y}}}{A_{\mathrm{ave}}}$$

However, this assumption is not permissible for many beam cross sections. The vertical shearing stress at any point in a beam may be determined from the horizontal shearing stress at that cross-section.

Shear Stress due to Shear Loads

In addition to the bending (axial) stress which develops in a loaded beam, there is also a shear stress which develops, including both a vertical shear stress, τ_{xy}, and a horizontal (longitudinal) shear stress, τ_{yx}. It was shown that at any given point in the beam, the values of vertical shear stress and the horizontal shear stress must be equal:

$$\tau_{\mathrm{xy}} = \tau_{\mathrm{yx}}$$

As a result it is usual to discuss and calculate the horizontal shear stress in a beam (and simply remember that the vertical shearing stress is equal in value to the horizontal shear stress at any given point).

Let us derive an expression for the horizontal shear stress, τ_{yx}. In order to do so, let us cut a section dx long out of the left end of the beam. The internal horizontal forces acting on the section are shown in Fig. 5.8. In the side view of section dx, the bending moment is larger on the right hand face of the section by an amount dM_{zz}.

Now let us take a top slice of section dx. Since the forces are different between the top of the section and the bottom of the section (less at the bottom) there is a differential (shearing) force which tries to shear the section, shown in Fig. 5.9, horizontally. This means there is a shear stress on the section, and in terms of the shear stress, the differential shearing force, F, can be written as F times the longitudinal

[3] According to Wikipedia

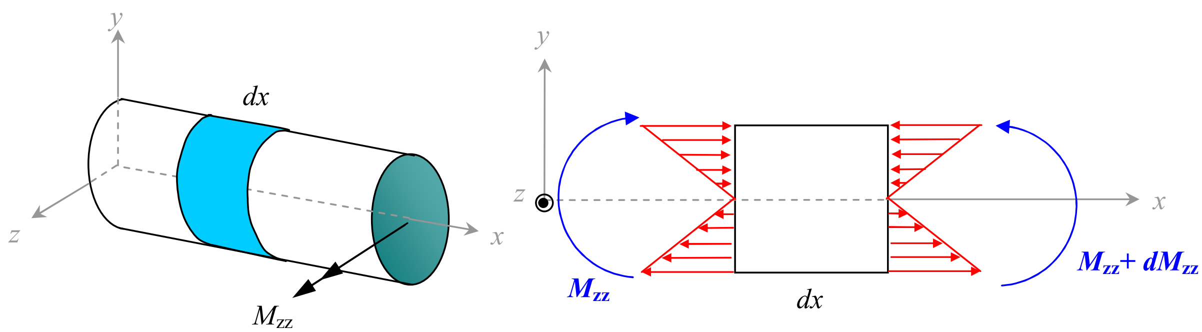


Figure 5.8: Side view of element dx of a beam.

area of the section $(t\,dx)$. Thus,

$$-F_{yx} + F_{xx} - dF_{xx} = 0$$

Using a two-term Taylor expansion:

$$dF_{xx} = F_{xx} + \frac{dF_{xx}}{dx}\,dx$$

The forces are

$$F_{xx} = \iint_A \sigma_{xx}\,dA$$

$$F_{yx} = \tau_{yx}\,t\,dx$$

and

$$dF_{xx} = F_{xx} + \frac{dF_{xx}}{dx}\,dx = \iint_A \sigma_{xx}\,dA + \left(\iint_A \frac{d\sigma_{xx}}{dx}\,dA\right)\,dx$$

Thus equilibrium in the x-direction gives

$$-F_{yx} + F_{xx} - dF_{xx} = 0$$

$$-\tau_{yx}\,t\,dx + \iint_A \sigma_{xx}\,dA - \left\{\iint_A \sigma_{xx}\,dA + \left(\iint_A \frac{d\sigma_{xx}}{dx}\,dA\right)\,dx\right\} = 0$$

$$-\tau_{yx}\,t\,dx - \left(\iint_A \frac{d\sigma_{xx}}{dx}\,dA\right)\,dx = 0$$

Now the average shear stress on the lower face of the element over the finite width t is

$$\tau_{yx}\,t\,dx = -\left(\iint_A \frac{d\sigma_{xx}}{dx}\,dA\right)\,dx \qquad \rightarrow \qquad \tau_{yx} = -\frac{1}{t}\iint_A \frac{d\sigma_{xx}}{dx}\,dA \tag{5.29}$$

Recall the normal bending stress, σ_{xx} for a symmetric cross-section of an isotropic beam under the

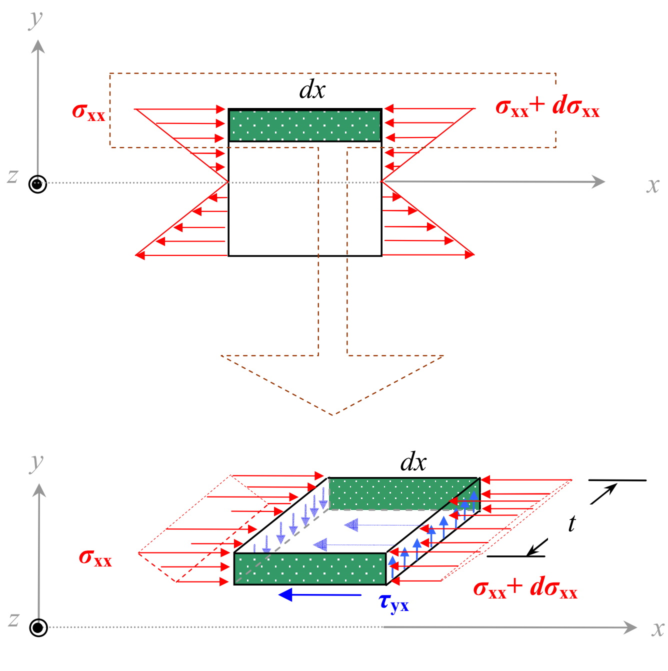


Figure 5.9: Element of a beam showing shear stress.

Euler-Bernoulli hypothesis and with $M_{yy} = 0$, is

$$\sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\, y$$

Thus

$$\tau_{yx} = -\frac{1}{t}\iint_A \frac{d\sigma_{xx}}{dx}\, dA = -\frac{1}{t}\iint_A \frac{d}{dx}\left(-\frac{M_{zz}}{I_{zz}}\, y\right) dA = \frac{1}{I_{zz}\, t}\iint_A \frac{d\, M_{zz}}{dx}\, y\, dA$$

To obtain the correct sign for the shear stress, let us examine the top horizontal cut taken above the neutral axis. The area formed from the horizontal cut is also above the neutral axis. We can then say the following:

$$V_y > 0 \quad \rightarrow \quad \frac{dM_{zz}}{dx} = -V_y \quad \rightarrow \quad \frac{dM_{zz}}{dx} < 0 \quad \rightarrow \quad dM_{zz} < 0 \tag{5.30}$$

Then

$$\tau_{yx} = \frac{1}{I_{zz}\, t}\iint_A \frac{-d\, M_{zz}}{dx}\, y\, dA$$

Also, recall

$$\frac{d\, M_{zz}(x)}{dx} = -V_y(x)$$

Thus

$$\tau_{yx} = \frac{1}{I_{zz}\, t}\iint_A V_y\, y\, dA = \frac{V_y}{I_{zz}\, t}\iint_A y\, dA = \frac{V_y\, Q_z}{I_{zz}\, t}$$

For static equilibrium,

$$\tau_{\text{xy}} = \tau_{\text{yx}} = \frac{V_{\text{y}}\, Q_{\text{z}}}{I_{\text{zz}}\, t}$$

Recall I_{zz} is the moment of inertia about the neutral axis of the entire undeformed cross section stemming from the use of the flexure formula, V_{y} the shear force at location along the beam where we wish to find from the horizontal shear stress, t the width of the beam at point where we wish to determine the shear stress, and Q_{z} the first moment of area about the y-axis. The first moment of area about the y-axis is defined as

$$Q_{\text{z}} = \iint_A y\, dA = \sum \bar{y}\, A$$

where A is the cross-sectional area of top portion of the neutral axis and $\bar{y}$ the distance to centroid of A, measured from neutral axis.

Usually we talk in terms of shear flow, which is defined as

$$q_{\text{xy}} = \frac{V_{\text{y}}\, Q_{\text{z}}}{I_{\text{zz}}} \tag{5.31}$$

The shear stress is then defined as

$$\tau_{\text{xy}} = \frac{q_{\text{xy}}}{t} = \frac{V_{\text{y}}\, Q_{\text{z}}}{I_{\text{zz}}\, t} \tag{5.32}$$

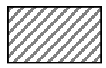
and the total shear force as

$$F_{\text{xy}} = q_{\text{xy}}\, s = \frac{V_{\text{y}}\, Q_{\text{z}}}{I_{\text{zz}}}\, s \tag{5.33}$$

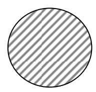
where s is the measure of the total distance to the location of the desired shear force. The minus sign in the shear stress indicates that the shear flow acts downwards (in the negative y-direction). However, to be consistent with our sign convention, we take the shear stress upwards and the that is the motivation we keep the sign.

Maximum Shear Stress for Several Cross-Sections

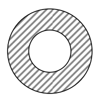
The maximum shear stress acts at the neutral axis. It can be shown that the maximum shear stresses for the following common cross-sections are:



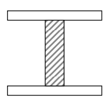
Solid rectangular: $$\tau_{\max} = \frac{3}{2}\frac{V_y}{A}$$



Solid circular: $$\tau_{\max} = \frac{4}{3}\frac{V_y}{A}$$



Hollow circular: $$\tau_{\max} = \frac{2\,V_y}{A}$$



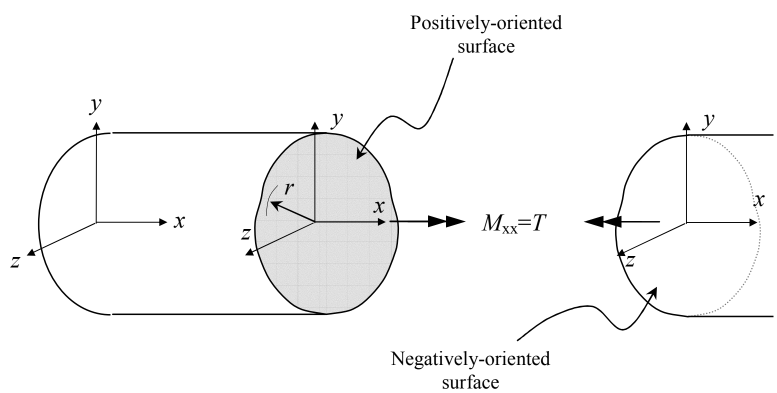
I-web: $$\tau_{\max} = \frac{V_y}{A_{\text{web}}}$$

5.5.2 Shear Stress due to Torsional Load

Theory

In most, mechanical engineering applications, torque applied to beam comes as a result of the beam been attached to a power source or an applied load. The beam in such a case is called shaft and the torque T applied is then the moment about the shaft's axis, i.e.,

$$M_{\text{xx}} = T$$



When it is associated to a power source, then the torque can be calculated using the following relationship:

$$T = \frac{396000\,\dot{W}_{\text{hp}}}{\omega} = \frac{63025\,\dot{W}_{\text{hp}}}{n} \qquad \text{for ips unit system}$$

where T is the torque in lb–in, n the shaft speed in rpm (rev/min), ω the angular velocity in rad/min, and $\dot{W}_{\text{hp}}$ the source's power measured in horsepower. In the SI units,

$$T = \frac{60000\,\dot{W}_{\text{kw}}}{\omega} = \frac{9549\,\dot{W}_{\text{kw}}}{n} \qquad \text{for SI unit system}$$

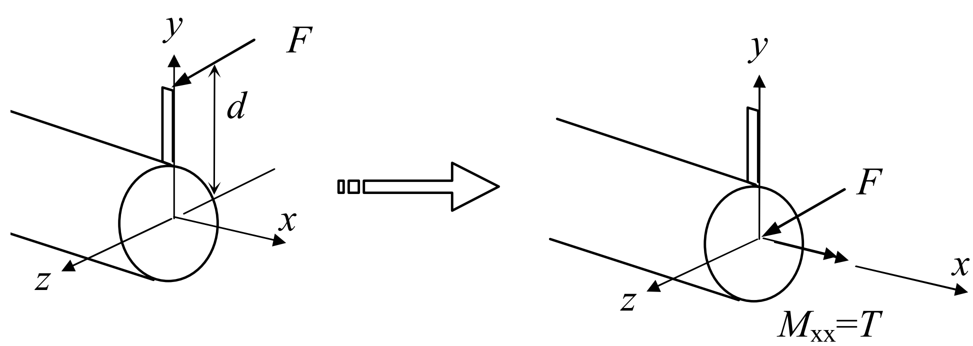
where T is the torque in N–m, n the shaft speed in rpm (rev/min), ω the angular velocity in rad/min, and $\dot{W}_{\text{kw}}$ the source's power measured in kilowatts.

When it is associated to an applied load, then the torque is

$$T = F\,d$$

where F is the applied force and d the distance from the force to the neutral axis.

The magnitude of the shear stress at distance r (distance from the neutral axis to the point the shear



will be calculated) is

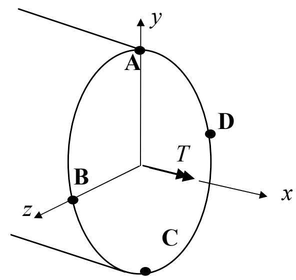
$$\tau_{\text{torsion}} = \frac{T\,r}{J_{\text{xx}}} \tag{5.34}$$

where J_{xx} is the polar moment of inertia of the cross-sectional area and is defined as:

$$J_{\text{xx}} = \iint_A r^2\,dA = \iint_A \left(y^2 + z^2\right)\,dA = \iint_A y^2\,dA + \iint_A z^2\,dA = I_{\text{yy}} + I_{\text{zz}} \tag{5.35}$$

The maximum shear stress for circular cross-sections is found at $r = c$, where c is the cross-sectional radius. The same is not true for noncircular cross-sections.

The sign and direction it determined depending on the desired location of the shear stress. As for an example,



The shear stress is

$$\text{at } \mathbf{A}: \quad \tau_{\text{xz}} = +\tau_{\text{torsion}} = \frac{T\,c}{J_{\text{xx}}} \qquad \text{at } \mathbf{B}: \quad \tau_{\text{xy}} = -\tau_{\text{torsion}} = -\frac{T\,c}{J_{\text{xx}}}$$

$$\text{at } \mathbf{C}: \quad \tau_{\text{xz}} = -\tau_{\text{torsion}} = -\frac{T\,c}{J_{\text{xx}}} \qquad \text{at } \mathbf{D}: \quad \tau_{\text{xy}} = +\tau_{\text{torsion}} = \frac{T\,c}{J_{\text{xx}}}$$

Maximum Shear Stress for Several Cross-Sections

The above equations are based under the assumptions used in the analysis are:

1. The bar is acted upon by a pure torque, and the sections under consideration are remote from the point of application of the load and from a change in diameter.
2. Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.
3. The material obeys Hooke's law.

For cases bars of non circular cross section, the above equations should not be used because cross-sectional planes distort significantly when bars of noncircular cross sections are twisted. Although the development of equations for torsional shearing stress in noncircular cross sections is complicated, one means of analysis in such cases is by utilizing the membrane analogy. Three important observations have been established for interpreting results from the membrane analogy:

1. The tangent to a contour line (line of constant elevation) at any point on the deflected membrane gives the direction of the torsional shearing stress vector at the corresponding point in the cross section of the twisted bar.
2. The maximum slope of the membrane at any point on the deflected membrane is proportional to the magnitude of the shearing stress at the corresponding point in the cross section of the twisted bar.

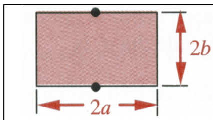
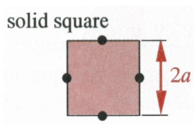
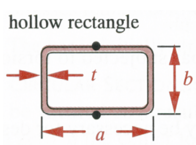
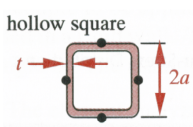
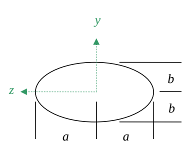
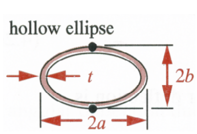
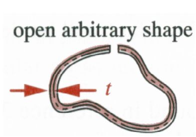
3. The volume enclosed between the datum base-plane and the contoured surface of the deflected membrane is proportional to the torque on the twisted bar.

All of these observations may be readily verified analytically for a twisted bar of circular cross section. Experimental agreement with the observations for noncircular shapes is excellent. Thus, typically it is possible to formulate an expression for maximum shearing stress in a noncircular bar subject to torque T as

$$\tau_{\max} = \frac{T\,c}{J_{\mathrm{xx}}} = \frac{T}{\frac{J_{\mathrm{xx}}}{c}} = \frac{T}{Q}$$

where Q is function of the cross-sectional geometry. These are given in Table 5.2.

Table 5.2: Geometric expressions for various cross-section shapes in torsion

Shape	J_{xx}	Q
	$a\,b^3\left\{\frac{16}{3}-3.36\left(\frac{b}{a}\right)\left(1-\frac{b^4}{12\,a^4}\right)\right\}$	$\frac{8\,a^2\,b^2}{3\,a+1.8\,b}$
	$2.25\,a^4$	$\frac{a^4}{0.6}$
	$\frac{2\,t\,(a-t)^2\,(b-t)^2}{a+b-2\,t}$	$2\,t\,(a-t)\,(b-t)$
	$t\,(a-t)^3$	$2\,t\,(a-t)^2$
	$\frac{\pi\,a^3\,b^3}{a^2+b^2}$	$\frac{\pi\,a\,b^2}{2}$
	$\frac{\pi\,a^3\,b^3}{a^2+b^2}\left\{1-\left(1-\frac{t}{a}\right)^4\right\}$	$\frac{\pi\,a\,b^2}{2}\left\{1-\left(1-\frac{t}{a}\right)^4\right\}$
	$\frac{1}{3}\,P\,t^3\ (t\ll P)$	$\frac{P^2\,t^2}{3\,P+1.8\,t}\ (t\ll P)$

P is the length of the median line. The back dots indicate the location of maximum shear stress

Example 5.3.

Design of Torsionally Loaded Shafts

Experimental power measurements made on a new-style rotary garden tiller indicate that under full load conditions the internal combustion engine must supply 4.3 horsepower, steadily, to the mechanical drive train. Power is transmitted through a solid 0.50-inchdiameter round steel shaft rotating at 1800 rpm. It is being proposed to replace the round steel shaft with a solid square shaft of the same material. Evaluate the proposal by determining the following information:

(a) What is the steady full-load torque being transmitted by the round steel shaft?

(b) What is the maximum stress in the round shaft, what type of stress is it, and where does it occur?

(c) Assuming that the measurements of power transmitted by the round shaft were made under full load, as specified, what design-allowable stress was probably used for the shaft?

(d) What design-allowable stress should be used in estimating the size required for the proposed square shaft?

(e) What size should the proposed square shaft be made to be "equivalent" to the existing round shaft in resisting failure?

Solution:

(a) What is the steady full-load torque being transmitted by the round steel shaft?

Using equation

$$T = \frac{63025\,\dot{W}_{\text{hp}}}{n} = \frac{(63025)(4.3)}{1800} = 150.6 \text{ lb–in}$$

(b) What is the maximum stress in the round shaft, what type of stress is it, and where does it occur?

The torque on the round shaft produces torsional shearing stress that reaches a maximum value all around the outer surface. The magnitude of the maximum shearing stress is

$$\tau_{\max} = \frac{T}{Q}$$

where

$$Q = \frac{\pi\, d^3}{16}$$

Thus

$$\tau_{\max} = \frac{T}{Q} = \frac{16\,T}{\pi\, d^3} = \frac{16\,(150.6)}{\pi\,(0.50)^3} = 6136 \text{ psi}$$

(c) Assuming that the measurements of power transmitted by the round shaft were made under full load, as specified, what design-allowable stress was probably used for the shaft?

Since the design objective is to size the part so that the maximum stress under "design conditions" is equal to the design-allowable stress τ_{all}:

$$\tau_{\text{all}} = \tau_{\max} = 6136 \text{ psi}$$

(d) What design-allowable stress should be used in estimating the size required for the proposed square shaft?

Since the material for the square shaft is the same as for the round shaft, the designallowable shearing stress should be the same. Hence, for the square shaft

$$\tau_{\text{all}} = 6136 \text{ psi}$$

(e) What size should the proposed square shaft be made to be "equivalent" to the existing round shaft in resisting failure?

The proposed square shaft is in the category of a noncircular bar subjected to torsion; hence,

$$Q_{\text{circular}} = \frac{T}{\tau_{\text{all}}} = \frac{150.6}{6136} = 0.025 \text{ in}^3$$

For the square shaft, case 3 of Table 4.4 may be utilized by setting $a = b$, giving

$$Q_{\text{squared}} = \frac{5}{3}\, a^3 = 1.67\, a^3$$

Equating $Q_{\text{circular}} = Q_{\text{squared}}$ gives

$$Q_{\text{circular}} = Q_{\text{squared}} \quad \rightarrow \quad 0.025 \text{ in}^3 = 1.67\, a^3 \quad \rightarrow \quad a = 0.25 \text{ in}$$

Hence the dimension.s of each side of the square should be

$$s = 2\, a = 0.50 \text{ in}$$

End Example □

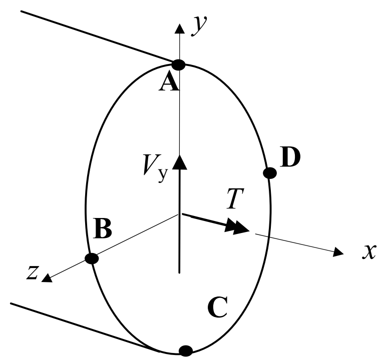
5.5.3 Total Shear Stress: Torsional Load and Shear Load

The total shear at a point is determined by superposition:

$$\tau = \tau_{\text{torsion}} + \tau_{\text{shear load}} \tag{5.36}$$

It is extremely important to watch out for signs: we must be consistent with our sign convention.

Thus in the presence of shear and torsional loads



The shear stress is

at **A**: $\tau_{xz} = \dfrac{T}{Q}$ at **B**: $\tau_{xy} = -\dfrac{T}{Q} + \dfrac{V_y Q_z}{I_{zz} t}$

at **C**: $\tau_{xz} = -\dfrac{T}{Q}$ at **D**: $\tau_{xy} = \dfrac{T}{Q} + \dfrac{V_y Q_z}{I_{zz} t}$

5.6 Design of Beams and shafts

- Draw the load diagrams whenever possible.
- Locate the critical points (points where the loads have maximum values along the beam or shaft's axis).
- Obtain the state of stress acting at critical points of the cross section.
- Obtain the allowable stress.

$$\sigma_{\text{allowable}} = \frac{S_{\text{yield}}}{n_{\text{SF}}} \qquad \tau_{\text{allowable}} = \frac{S_{\text{yield}}}{n_{\text{SF}}}$$

- Obtain the geometrical stress properties from tables.

$$Z = \frac{M_{\text{zz}}}{\sigma_{\text{all}}} \qquad Q = \frac{T}{\tau_{\text{all}}}$$

- Find the principal stress.
- Solve the problem.

Note that these equations are quite useful:

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right|, \qquad \sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = \sqrt{\sigma_{\text{xx}}^2 + 3\,\tau^2}$$

where τ is τ_{xy} or τ_{xz}.

Example 5.4.

DESIGN OF SHAFT FOR COIL SUTTER

The following example is obtained from Hamrock (2005). **Given:** Flat rolled sheets are produced in wide rolling mills, but many products are manufactured from strip stock. Figure bellow depicts a coil slitting line, where large sheets are cut into ribbons or strips.

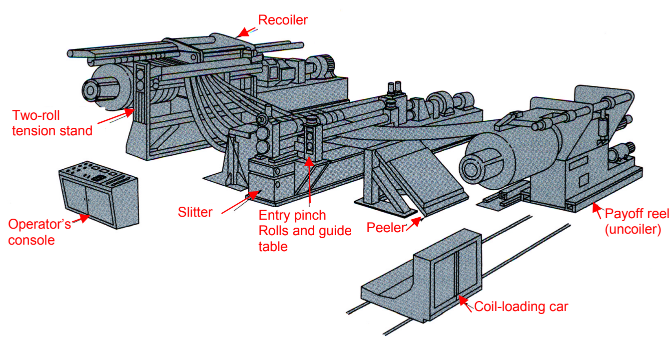


Figure below shows a shaft supporting the cutting blades. The rubber rollers ensure that the sheet does not wrinkle. For such slitting lines the shafts that support the slitting knives are a highly stressed and critical component.

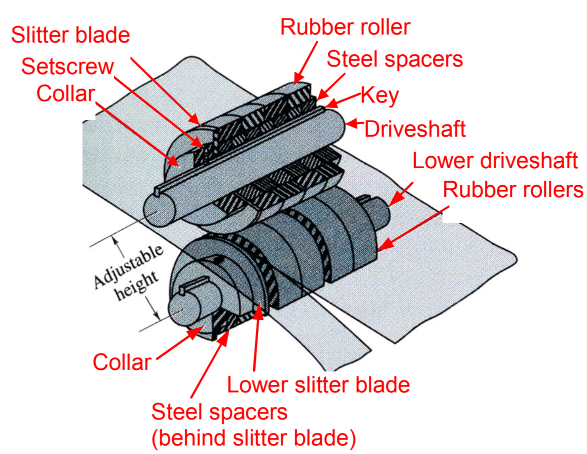
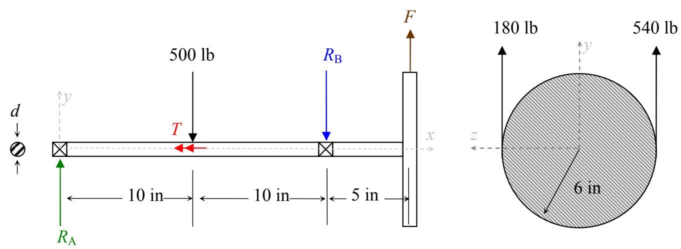


Figure below is a free-body diagram of a shaft for a short slitting line where a single blade is placed in the center of the shaft and a motor drives the shaft through a pulley at the far right end.



Find: If the maximum allowable shear stress is 6000 psi and the largest gage sheet causes a blade force of 500 lb, what shaft diameter d is needed?

First of all note that the goal is to obtain the maximum overall shear stress τ_{max} acting at the critical location of the shaft. This maximum overall shear stress will be a function of the shaft's diameter d. Thus

$$\tau_{\text{max}} = \tau_{\text{all}}$$

and we can solve for d. Let us think of a five-step solution:

(a) Obtain the reaction forces.

(b) Obtain and plot the shear, moment, and torque diagrams.

(c) Determine the critical location(s) of the shaft.

(d) Obtain the state of stress of the cross-section at each critical location and use the eigenvalue approach to determine the maximum allowable shear stress. Solve for the diameter.

(e) Choose the optimum diameter and discuss your results.

Solution:

(a) Obtain the reaction forces.

It should be clear that the total force exerted by the pulley is:

$$F = 180 \text{ lb} + 540 \text{ lb} = 720 \text{ lb}$$

The reactions are found through statics:

$$+\uparrow \sum F_y = 0 = R_A - 500 \text{ lb} - R_B + 720 \text{ lb}$$

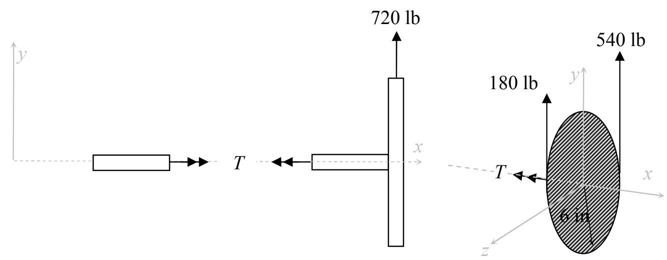
$$+\circlearrowleft \sum M_z\Big|_{\text{at reaction } R_A} = 0 = -(500 \text{ lb})(10 \text{ in}) - (R_B)(20 \text{ in}) + (720 \text{ lb})(15 \text{ in})$$

Solving the above system of equations, we find that the reaction forces are:

$$R_A = 430 \text{ lb} \qquad R_B = 650 \text{ lb}$$

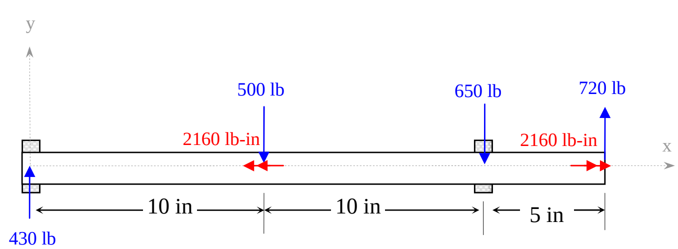
The positive sign indicated that the reaction forces are shown in Figure were assumed correctly.

In addition to the above, there is a constant torque along the shaft's axis. This torque is calculated using a free body diagram and our sign convention:



$$+\circlearrowleft \sum M_x\Big|_{\text{positive in the } x\text{-axis}} = 0 = -T(x) + (540 \text{ lb})(6 \text{ in}) - (180 \text{ lb})(6 \text{ in})$$

Thus there is a torque of 2160 lb–in between the pulley and the knife blade. Now the free body diagram is



(b) Obtain and plot the shear, moment, and torque diagrams.

Let us start with shear $V_y(x)$.
The equation for shear for $0 < x < 10$ (x measured from the reaction A) is

$$+\uparrow \sum F_{\text{y}} = 0 \qquad \Rightarrow \qquad V_{\text{y}_1}(x) + 430 \text{ lb} = 0$$

$$V_{\text{y}_1}(x) = -430 \text{ lb}$$

The equation for shear for $10 < x < 20$ (x measured from the reaction A) is

$$+\uparrow \sum F_{\text{y}} = 0 \qquad \Rightarrow \qquad V_{\text{y}_2}(x) + 430 \text{ lb} - 500 \text{ lb} = 0$$

$$V_{\text{y}_2}(x) = 70 \text{ lb}$$

The equation for shear for $20 < x < 25$ (x measured from the reaction A) is

$$+\uparrow \sum F_{\text{y}} = 0 \qquad \Rightarrow \qquad V_{\text{y}_3}(x) + 430 \text{ lb} - 500 \text{ lb} - 650 \text{ lb} = 0$$

$$V_{\text{y}_3}(x) = 720 \text{ lb}$$

Now find the moment equations $M_{\text{zz}}(x)$. Recall

$$\frac{d\,M_{\text{zz}}(x)}{dx} = -V_{\text{y}}(x) \quad \Rightarrow \quad M_{\text{zz}}(x) = -\int V_{\text{y}}(x)\,dx + M_{\text{zz}0}$$

The equation for moment for $0 < x < 10$ (x measured from the reaction A) is

$$M_{\text{zz}1}(x) = -\int V_{\text{y}_1}(x)\,dx + M_{\text{zz}1a} = -\int (-430)\,dx + M_{\text{zz}1a} = 430\,x + M_{\text{zz}1a}$$

Using boundary conditions:

$$M_{\text{zz}_1}(x)\big|_{x=0} = 0 = M_{\text{zz}1a}$$

Thus

$$M_{\text{zz}_1}(x) = 430\,x \ \ \text{[lb–in]}$$

The equation for moment for $10 < x < 20$ (x measured from the reaction A) is

$$M_{\text{zz}2}(x) = -\int V_{\text{y}_2}(x)\,dx + M_{\text{zz}2a} = -\int (70)\,dx + M_{\text{zz}2a} = -70\,x + M_{\text{zz}2a}$$

Using boundary conditions:

$$M_{\text{zz}_2}(x)\big|_{x=10} = M_{\text{zz}_1}(x)\big|_{x=10} \quad \rightarrow \quad -700 + M_{\text{zz}2a} = 4300 \quad \rightarrow \quad M_{\text{zz}2a} = 5000$$

Thus

$$M_{\text{zz}_2}(x) = -70\,x + 5000 \ \ \text{[lb–in]}$$

The equation for moment for $20 < x < 25$ (x measured from the reaction A) is

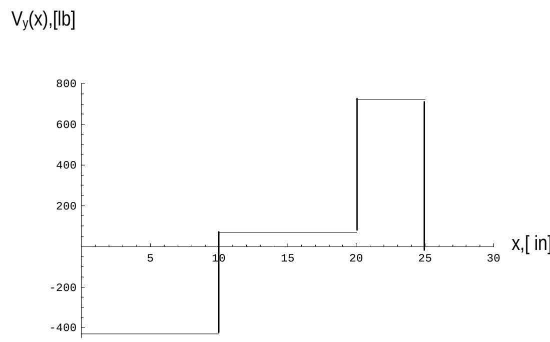
$$M_{\text{zz}3}(x) = -\int V_{\text{y}_3}(x)\,dx + M_{\text{zz}3a} = -\int (720)\,dx + M_{\text{zz}3a} = -720\,x + M_{\text{zz}3a}$$

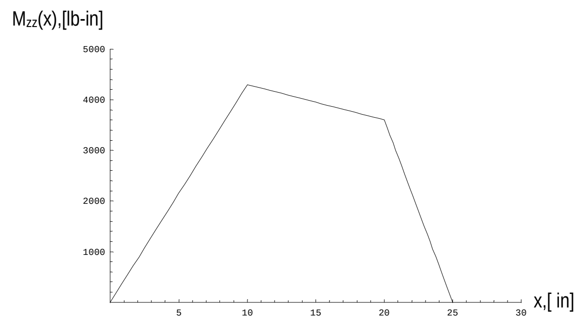
Using boundary conditions:

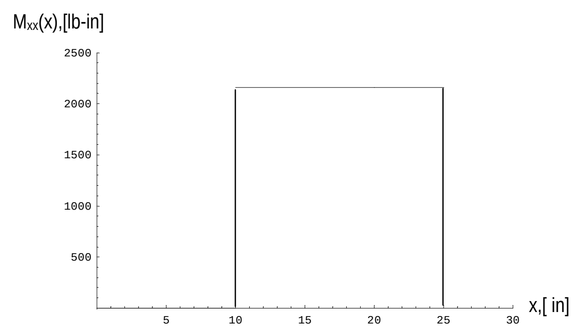
$$M_{zz_3}(x)\Big|_{x=20} = M_{zz_2}(x)\Big|_{x=20} \quad \rightarrow \quad -14400 + M_{zz_3 a} = 3600 \quad \rightarrow \quad M_{zz_3 a} = 18000$$

Thus

$$M_{zz_3}(x) = -720\,x + 18000 \;\; \text{[lb–in]}$$


Vy(x),[lb]
800
600
400
200
-200
-400
5
10
15
20
25
30
x,[in]


Mzz(x),[lb-in]
5000
4000
3000
2000
1000
5
10
15
20
25
30
x,[in]

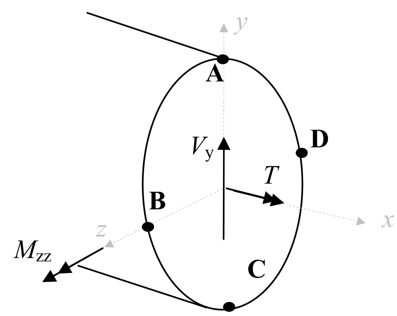

Mxx(x),[lb-in]
2500
2000
1500
1000
500
5
10
15
20
25
30
x,[in]

(c) Determine the critical location(s) of the shaft.

It is clear that the maximum shear occurs just to the left of the pulley and equals 720 lb ($x = 20^+$ in). The maximum bending moment is 4300 lb–in ($x = 10^+$ in). Thus the critical locations in the shaft are: the location in the shaft where the moment is largest ($x = 10^+$ in) and the location where the shear is largest ($x = 20^+$ in $\rightarrow$ 25 in).

(d) Obtain the state of stress of the cross-section at each critical location and use the eigenvalue approach to determine the maximum allowable shear stress. Solve for the diameter.

i) **Location of maximum bending moment:** $x = 10^+$ **in.**



At this cross-section, the loads are:

$$V_y = 70 \text{ lb} \qquad M_{zz} = 4300 \text{ lb–in} \qquad T = M_{xx} = 2160 \text{ lb–in}$$

First of all, note that:

✓ at points **A** and **C** the shear stress due to shear load (V_y) is zero

✓ at points **A** and **C** the normal stress due to bending is critical

✓ at points **B** and **D** the normal stress due to bending is zero

✓ at points **B** and **D** the shear stress due to shear load is maximum

✓ at points **A**, **B**, **C**, and **D** the shear stress due torsional load exits

✓ at all points the normal stress due axial load is zero (does not exit)

✓ $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = 0$

At the point A, we only have normal stress due to bending. At point A, there is no shear stress due to shear load. However, the shear stress is due to the torque exerted on the pulley. Thus

$$\sigma_{xx} = -\frac{M_{zz}}{I_{zz}}\, y \qquad \text{where} \qquad y = c = +\frac{d}{2} \qquad \rightarrow \qquad \sigma_{xx} = \frac{M_{zz}}{I_{zz}}\, c$$

$$\sigma_{xx} = -\frac{M_{zz}}{Z} \qquad \text{where} \qquad Z = \frac{\pi\, d^3}{32}$$

$$\sigma_{xx} = -\frac{(4300)}{\dfrac{\pi\, d^3}{32}} = -\frac{43800}{d^3}$$

and

$$\tau_{xz} = \frac{T\,r}{J_{xx}} \qquad \text{where} \qquad r = c = \frac{d}{2} \qquad \rightarrow \qquad \tau_{xz} = \frac{T\,c}{J_{xx}}$$

$$\tau_{xz} = \frac{T}{Q} \qquad \text{where} \qquad Q = \frac{\pi\,d^3}{16}$$

$$\tau_{xz} = \frac{(2160)}{\frac{\pi\,d^3}{16}} = \frac{11000}{d^3}$$

The state of stress is

$$\underline{\boldsymbol{\sigma}}_{A} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \frac{1}{d^3} \begin{bmatrix} -43800 & 0 & 11000 \\ 0 & 0 & 0 \\ 11000 & 0 & 0 \end{bmatrix} \text{ psi}$$

Now we proceed to evaluate the principal stresses. The stresses invariants are

$$\begin{aligned} I_{\sigma_1} &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -\frac{43800}{d^3} \\ I_{\sigma_2} &= \sigma_{xx}\,\sigma_{yy} + \sigma_{zz}\,\sigma_{xx} + \sigma_{yy}\,\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = -\left(\frac{11000}{d^3}\right)^2 \\ I_{\sigma_3} &= \sigma_{xx}\,\sigma_{yy}\,\sigma_{zz} + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_{xx}\,\tau_{yz}^2 - \sigma_{yy}\,\tau_{zx}^2 - \sigma_{zz}\,\tau_{xy}^2 = 0 \end{aligned}$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0$$

The principal stresses can be obtained analytically:

$$\begin{aligned} \lambda_1 &= \frac{I_{\sigma_1}}{2} + \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = \frac{2607}{d^3} \\ \lambda_2 &= \frac{I_{\sigma_1}}{2} - \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = -\frac{46407.34}{d^3} \\ \lambda_3 &= 0 \end{aligned}$$

Since $d > 0$, then

$$\sigma_1 = \frac{2607}{d^3}, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\frac{46407.34}{d^3}$$

The overall maximum shear stress is

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{24507.17}{d^3}$$

Thus,

$$\tau_{\max} = \tau_{\text{all}} \qquad \frac{24507.17}{d^3} = 6000 \qquad \rightarrow \qquad d = 1.60 \text{ in}$$

At the point C, we only have normal stress due to bending. At point C, there is no shear stress due to shear load. However, the shear stress is due to the torque exerted on the pulley. Thus

$$\sigma_{\text{xx}} = -\frac{M_{\text{zz}}}{I_{\text{zz}}}\, y \qquad \text{where} \qquad y = -c = -\frac{d}{2} \qquad \rightarrow \qquad \sigma_{\text{xx}} = \frac{M_{\text{zz}}}{I_{\text{zz}}}\, c$$

$$\sigma_{\text{xx}} = \frac{M_{\text{zz}}}{Z} \qquad \text{where} \qquad Z = \frac{\pi\, d^3}{32}$$

$$\sigma_{\text{xx}} = \frac{(4300)}{\frac{\pi\, d^3}{32}} = \frac{43800}{d^3}$$

and

$$\tau_{\text{xz}} = -\frac{T\, r}{J_{\text{xx}}} \qquad \text{where} \qquad r = c = \frac{d}{2} \qquad \rightarrow \qquad \tau_{\text{xz}} = -\frac{T\, c}{J_{\text{xx}}}$$

$$\tau_{\text{xz}} = -\frac{T}{Q} \qquad \text{where} \qquad Q = \frac{\pi\, d^3}{16}$$

$$\tau_{\text{xz}} = -\frac{(2160)}{\frac{\pi\, d^3}{16}} = -\frac{11000}{d^3}$$

The state of stress is

$$\underline{\boldsymbol{\sigma}}_{\text{C}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \frac{1}{d^3} \begin{bmatrix} 43800 & 0 & -11000 \\ 0 & 0 & 0 \\ -11000 & 0 & 0 \end{bmatrix} \text{ psi}$$

Now we proceed to evaluate the principal stresses. The stresses invariants are

$$\begin{aligned} I_{\sigma_1} &= \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = \frac{43800}{d^3} \\ I_{\sigma_2} &= \sigma_{\text{xx}}\, \sigma_{\text{yy}} + \sigma_{\text{zz}}\, \sigma_{\text{xx}} + \sigma_{\text{yy}}\, \sigma_{\text{zz}} - \tau_{\text{xy}}^2 - \tau_{\text{yz}}^2 - \tau_{\text{zx}}^2 = -\left(\frac{11000}{d^3}\right)^2 \\ I_{\sigma_3} &= \sigma_{\text{xx}}\, \sigma_{\text{yy}}\, \sigma_{\text{zz}} + 2\, \tau_{\text{xy}}\, \tau_{\text{yz}}\, \tau_{\text{zx}} - \sigma_{\text{xx}}\, \tau_{\text{yz}}^2 - \sigma_{\text{yy}}\, \tau_{\text{zx}}^2 - \sigma_{\text{zz}}\, \tau_{\text{xy}}^2 = 0 \end{aligned}$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\, \lambda^2 + I_{\sigma_2}\, \lambda = \lambda\, \left(\lambda^2 - I_{\sigma_1}\, \lambda + I_{\sigma_2}\right) = 0$$

The principal stresses can be obtained analytically:

$$\begin{aligned}
\lambda_1 &= \frac{I_{\sigma_1}}{2} + \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = \frac{46407.34}{d^3} \\
\lambda_2 &= \frac{I_{\sigma_1}}{2} - \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = -\frac{2607}{d^3} \\
\lambda_3 &= 0
\end{aligned}$$

Since $d > 0$, then

$$\sigma_1 = \frac{46407.34}{d^3}, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\frac{2607}{d^3}$$

The overall maximum shear stress is

$$\tau_{\text{max}} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{24507.17}{d^3}$$

Thus,

$$\tau_{\text{max}} = \tau_{\text{all}} \qquad \frac{24507.17}{d^3} = 6000 \qquad \rightarrow \qquad d = 1.60 \text{ in}$$

Note that at Point C gives the same result as point A.
At point B, the bending stress is zero and the stress is due to torsional and shear loads. The torsion-induced shear stress is subtracted from this shear stress. Thus, the total shear is

$$\tau_{\text{xy}} = -\frac{T\,c}{J_{\text{xx}}} + \frac{V_{\text{y}}\,Q_{\text{z}}}{I_{\text{zz}}\,t}$$

At point B the shear stress due to shear load is maximum and for a circular cross section is:

$$\tau_{\text{xy}}\Big|_{\text{max shear}} = \frac{4}{3}\frac{V_{\text{y}}}{A} = \frac{4\,(70)}{3\left(\dfrac{\pi\,d^2}{4}\right)} = \frac{119}{d^2}$$

Thus

$$\tau_{\text{xy}} = -\frac{T}{Q} + \frac{4}{3}\frac{V_{\text{y}}}{A} = -\frac{11000}{d^3} + \frac{119}{d^2}$$

However, the maximum shear stress will occur at D. At one end of the shaft, the torsion-induced shear stress is subtracted from this shear stress; at the other end, the effects are cumulative. Thus at point D, the total shear is

$$\tau_{\text{xy}} = \frac{T\,c}{J_{\text{xx}}} + \frac{V_{\text{y}}\,Q_{\text{z}}}{I_{\text{zz}}\,t}$$

At point D the shear stress due to shear load is maximum and for a circular cross section is:

$$\tau_{\text{xy}}\Big|_{\text{max shear}} = \frac{4}{3}\frac{V_{\text{y}}}{A} = \frac{4\,(70)}{3\left(\dfrac{\pi\,d^2}{4}\right)} = \frac{119}{d^2}$$

Thus

$$\tau_{xy} = \frac{T}{Q} + \frac{4}{3}\frac{V_y}{A} = \frac{11000}{d^3} + \frac{119}{d^2}$$

$$\underline{\boldsymbol{\sigma}}_D = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{119}{d^2} + \dfrac{11000}{d^3} & 0 \\ \dfrac{119}{d^2} + \dfrac{11000}{d^3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \qquad (5.37)$$

The principal stresses are

$$\sigma_1 = \frac{119}{d^2} + \frac{11000}{d^3}$$

$$\sigma_2 = 0$$

$$\sigma_3 = -\frac{119}{d^2} - \frac{11000}{d^3}$$

The overall maximum shear stress is

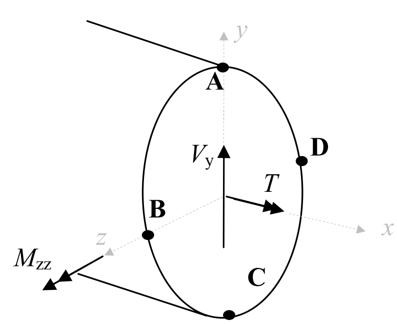
$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{119}{d^2} + \frac{11000}{d^3}$$

Thus,

$$\tau_{max} = \tau_{all} \qquad \frac{119}{d^2} + \frac{11000}{d^3} = 6000 \qquad \rightarrow \qquad d = 1.22 \text{ in}$$

It was not necessary to obtain the diameter for shear since, shear load is not maximum at $x = 10^+$. This is the motivation for verifying $x = 20^+$, where shear is maximum. However, the previous analysis was done for completeness.

ii) **Location of maximum shear:** $x = 20^+$ **in** $\rightarrow$ **25 in.** Let us choose $x = 25$ in because the bending moment is zero and the the bending stress vanishes, greatly simplifying our results. (However, one should also always evaluate at $x = 20^+$ in to ensure the accuracy of the results.)



At this cross-section, the loads are:

$$V_y = 720 \text{ lb} \qquad M_{zz} = 0 \text{ lb–in} \qquad T = M_{xx} = 2160 \text{ lb–in}$$

First of all, note that:

✓ at points **A** and **C** the shear stress due to shear load (V_y) is zero

✓ at points **A** and **C** the normal stress due to bending is zero (no bending moment)
✓ at points **B** and **D** the normal stress due to bending is zero
✓ at points **B** and **D** the shear stress due to shear load is maximum
✓ at points **A**, **B**, **C**, and **D** the shear stress due torsional load exits
✓ at all points the normal stress due axial load is zero (does not exit)
✓ $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = 0$

As stated previously, the maximum shear stress will occur at D. At one end of the shaft, the torsion-induced shear stress is subtracted from this shear stress; at the other end, the effects are cumulative. Thus, the total shear is

$$\tau_{xy} = \frac{T\,c}{J} + \frac{V_y\,Q_z}{I_{zz}\,t}$$

At point D the shear stress is due to shear load is maximum and for a circular cross section:

$$\tau_{xy}\Big|_{\text{max shear}} = \frac{4}{3}\frac{V_y}{A} = \frac{4\,(720)}{3\left(\dfrac{\pi\,d^2}{4}\right)} = \frac{1222}{d^2}$$

Thus

$$\tau_{xy} = \frac{T}{Q} + \frac{4}{3}\frac{V_y}{A} = \frac{11000}{d^3} + \frac{1222}{d^2}$$

$$\underline{\boldsymbol{\sigma}}_B = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1222}{d^2} + \dfrac{11000}{d^3} & 0 \\ \dfrac{1222}{d^2} + \dfrac{11000}{d^3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \qquad (5.38)$$

The principal stresses are

$$\begin{aligned} \sigma_1 &= \frac{1222}{d^2} + \frac{11000}{d^3} \\ \sigma_2 &= 0 \\ \sigma_3 &= -\frac{1222}{d^2} - \frac{11000}{d^3} \end{aligned}$$

The overall maximum shear stress is

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{1222}{d^2} + \frac{11000}{d^3}$$

Thus,

$$\tau_{max} = \tau_{all} \qquad \frac{1222}{d^2} + \frac{11000}{d^3} = 6000 \qquad \rightarrow \qquad d = 1.279 \text{ in}$$

(e) Choose the optimum diameter and discuss your results.

As in most shaft applications the normal stresses due to bending determine the shaft diameter, so that a shaft with a diameter not less than 1.60 in should be used. A number of points should be made regarding this analysis:

✓ When the normal stress due to bending was calculated, the shear stress due to shear was neglected, even though there was shear in the shaft at that location. However, the distribution of normal stress is such that it is extreme at the top and the bottom, where the shear stress is zero.

✓ When the maximum shear stress due to vertical shear was calculated, the effects of bending were ignored. The bending stress is zero at the neutral axis, the location of the maximum shear stress.

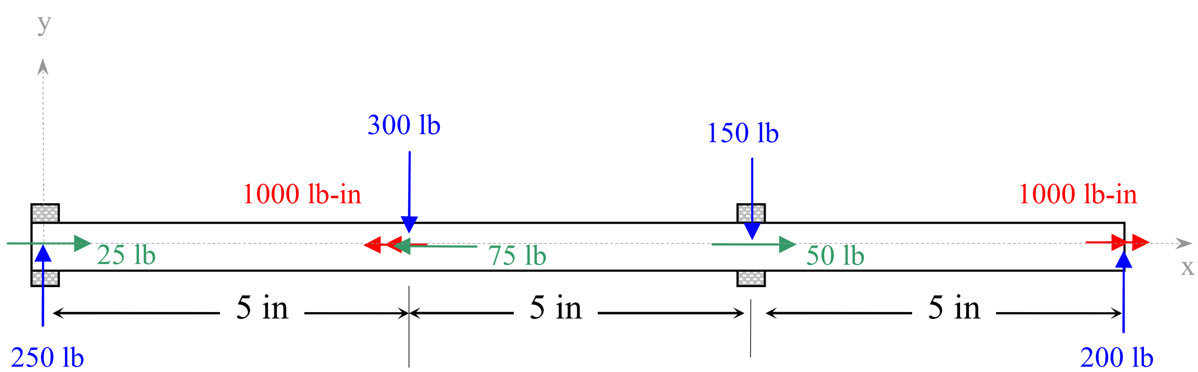
✓ There are two general shaft applications. Some shafts are extremely long, as in this problem, whereas others are made much shorter to obtain compact designs. A coil slitter can have shafting more than 20 ft long, but more supporting bearings would be needed for stiffness. This shaft was used only as an illustrative example; in actuality the supporting bearings would be placed much closer to the load application, and more than two bearing packs would probably be appropriate.

✓ The 10-in clearance between the leftmost bearing and the slitting knives is totally unnecessary and would lead to larger shaft deflections.

End Example □

Example 5.5.

DESIGN OF SHAFT A circular-steel solid one-inch diameter shaft is loaded by the four



vertical forces (an upward vertical force of 250 lb at $x = 0$ in, a downward vertical force of 300 lb at $x = 5$ in, a downward vertical force of 150 lb at $x = 10$ in, an upward vertical force of 200 lb at $x = 15$ in), three axial forces (an outward axial force of 25 lb at $x = 0$ in, an inward axial force of 75 lb at $x = 5$ in, an outward axial force of 50 lb at $x = 10$ in), and torques (an inward torque of 1000 lb–in at $x = 5$ in, an outward torque of 1000 lb–in at $x = 15$ in) as shown in Figure, which result from the actions of helical gears and shaft's rolling-element bearing supports. All loads are applied at the shafts neutral axis. Give dimensional units to all answers, including plots.

(a) Determine the cross sectional area A_x, the second moment of inertia I_{zz}, and the polar moment of inertia J_{xx}.

(b) Draw the shear force diagram: $V_y(x)$ for $0 < x < 15$.

(c) Draw the bending moment diagram: $M_{zz}(x)$ for $0 < x < 15$.

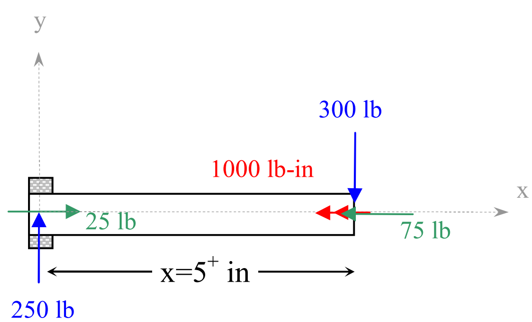
(d) Draw the normal force diagram: $N_{xx}(x)$ for $0 < x < 15$.

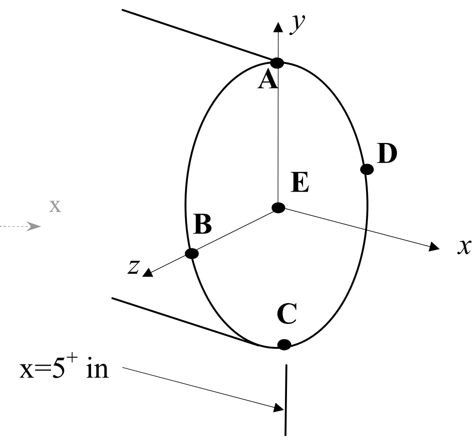
(e) Using the above information determine the location(s) in the shaft's axis (x-axis) where the stresses are critical. Justify your answer with a sentence or two.

(f) For the cross-section at $x = 5^+$, determine the state of stress at points **A**, **B**, **C**, **D**, and **E**.

(g) For each state of stress (points **A**, **B**, **C**, **D**, and **E**), determine the maximum normal stress and maximum shear stress.

(h) What is the critical cross-sectional stress location at $x = 5^+$?





(a) Determine the cross sectional area A_x, the second moment of inertia I_{zz}, and the polar moment of inertia J_{xx}.

Using Table 4.2 (page 171)

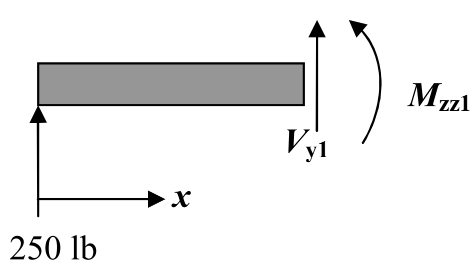
$$A_x = \frac{\pi\, d^2}{4} = \frac{\pi}{4} = 0.785 \quad \text{in}^2$$

$$I_{zz} = \frac{\pi\, d^4}{64} = \frac{\pi}{64} = 0.0491 \quad \text{in}^4$$

$$J_{xx} = I_{zz} + I_{yy} = \frac{\pi\, d^4}{32} = \frac{\pi}{32} = 0.0982 \quad \text{in}^4$$

(b) Draw the shear force diagram: $V_y(x)$ for $0 < x < 15$.

(c) Draw the bending moment diagram: $M_{zz}(x)$ for $0 < x < 15$.
We proceed to calculate the shear. Note that because of discontinuity in shear at $x = 5$ and $x = 10$, we calculate the shear in three sections.
The equation for shear is $(0 \le x \le 5)$



$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad V_{y_1}(x) + 250 = 0 \tag{5.39}$$

$$V_{y_1}(x) \quad = \quad -250 \quad \text{lb} \tag{5.40}$$

The equation for moment is

$$M_{zz_1}(x) = -\int V_{y_1}(x)dx + M_{z1_0}$$

$$= -\int(-250)dx + M_{z1_0} = 250\,x + M_{z1_0}$$

M_{z1_0} is found from boundary conditions: at $x = 0$ the value of the moment should be zero (There are no external moments):

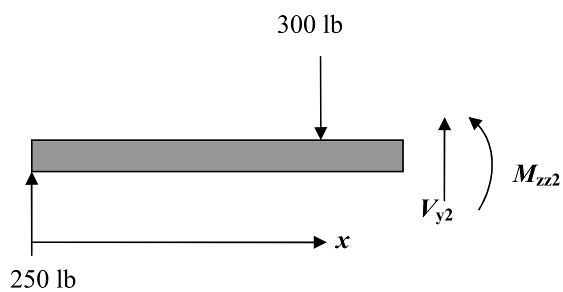
$$M_{zz_1}\big|_{x=0} + 0 = 0 \qquad \Rightarrow \qquad M_{zz_1}\big|_{x=0} = 0$$

Then

$$M_{zz_1}(0) = M_{z1_0} = 0$$

$$M_{zz_1}(x) = 250\,x \quad \text{lb-in} \tag{5.41}$$

The equation for shear is ($5 \leq x \leq 10$)



$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad V_{y_2}(x) + 250 - 300 = 0 \tag{5.42}$$

$$V_{y_2}(x) \quad = \quad 50 \quad \text{lb} \tag{5.43}$$

The equation for moment is

$$M_{zz_2}(x) = -\int V_{y_2}(x)dx + M_{z2_0}$$

$$= -\int(50)dx + M_{z2_0} = -50\,x + M_{z2_0}$$

M_{z2_0} is found from boundary conditions: at $x = 5$:

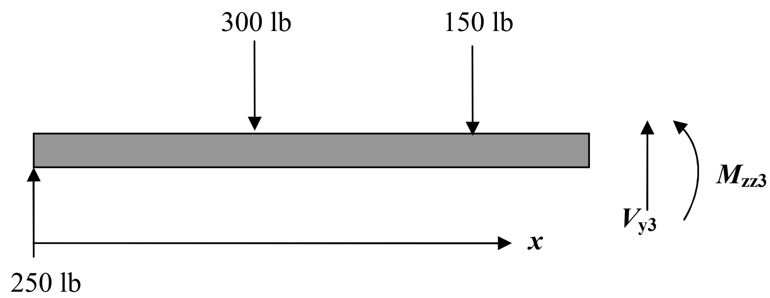
$$M_{zz_1}\big|_{x=5} = M_{zz_2}\big|_{x=5} \qquad \Rightarrow \qquad M_{zz_2}\big|_{x=5} = M_{zz_1}(5) = 1250$$

Then

$$M_{zz_2}(5) = 1250 \qquad \rightarrow \qquad -50\,(5) + M_{z2_0} = 1250 \qquad \rightarrow \qquad M_{z2_0} = 1500$$

$$M_{zz_2}(x) = -50\,x + 1500 \quad \text{lb-in} \tag{5.44}$$

The equation for shear is $(10 \leq x \leq 15)$



$$+\uparrow \sum F_y = 0 \qquad \Rightarrow \qquad V_{y_3}(x) + 250 - 300 - 150 = 0 \tag{5.45}$$

$$V_{y_3}(x) \quad = \quad 200 \quad \text{lb} \tag{5.46}$$

The equation for moment is

$$M_{zz_3}(x) = -\int V_{y_3}(x)dx + M_{z3_0}$$

$$= -\int (200)dx + M_{z3_0} = -200\,x + M_{z3_0}$$

M_{z3_0} is found from boundary conditions: at $x = 10$:

$$M_{zz_2}\big|_{x=10} = M_{zz_3}\big|_{x=10} \qquad \Rightarrow \qquad M_{zz_2}\big|_{x=10} = M_{zz_2}(10) = 1000$$

Then

$$M_{zz_3}(10) = 1000 \qquad \rightarrow \qquad -200\,(10) + M_{z3_0} = 1000 \qquad \rightarrow \qquad M_{z3_0} = 3000$$

$$M_{zz_3}(x) = -200\,x + 3000 \quad \text{lb-in} \tag{5.47}$$

The moment diagram is

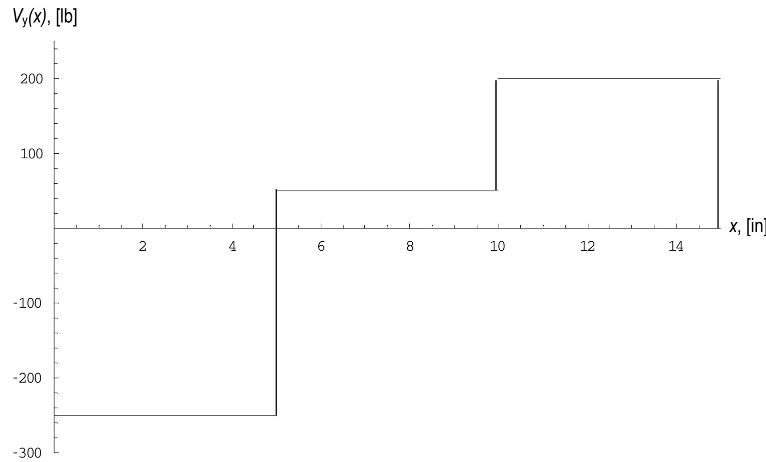
(d) Draw the normal force diagram: $N_{xx}(x)$ for $0 < x < 15$.

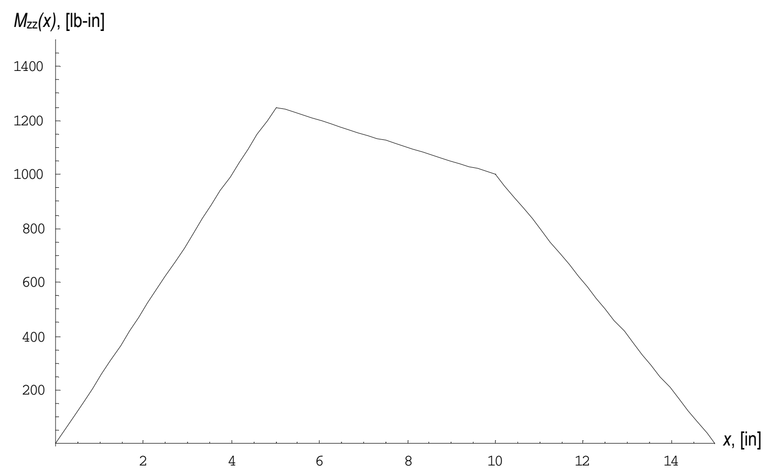
We proceed to calculate the axial force. Note that because of discontinuity in axial at $x = 5$ and $x = 10$, we calculate the axial force in three sections.

The equation for axial force is $(0 \leq x \leq 5)$

$$+\rightarrow \sum F_x = 0 \qquad \Rightarrow \qquad V_{x_1}(x) + 25 = 0 \tag{5.48}$$

The shear diagram is





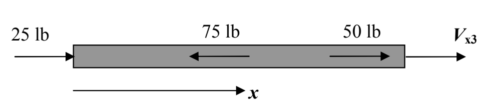
$$V_{x_1}(x) \quad = \quad -25 \quad \text{lb} \tag{5.49}$$

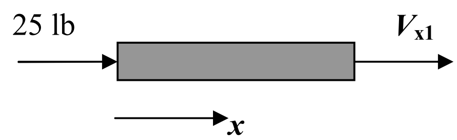
The equation for axial force is $(5 \leq x \leq 10)$

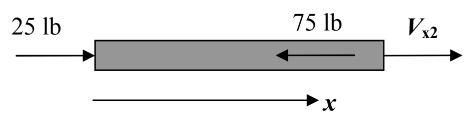
$$+\rightarrow \sum F_x = 0 \qquad \Rightarrow \qquad V_{x_2}(x) + 25 - 75 = 0 \tag{5.50}$$

$$V_{x_2}(x) \quad = \quad 50 \quad \text{lb} \tag{5.51}$$

The equation for axial force is $(10 \leq x \leq 15)$



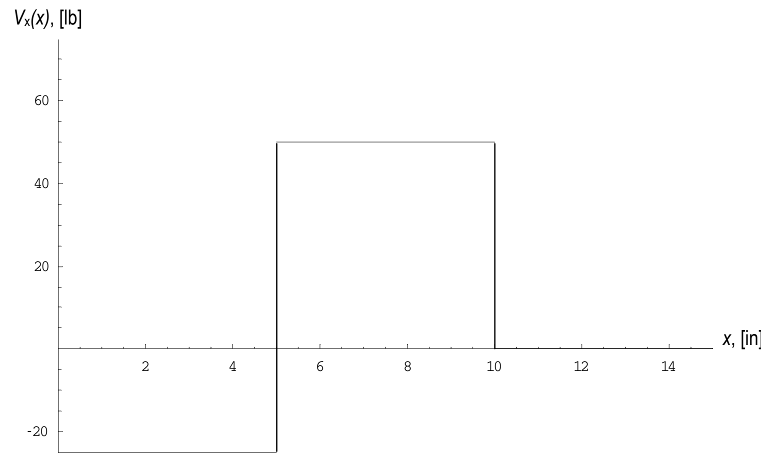




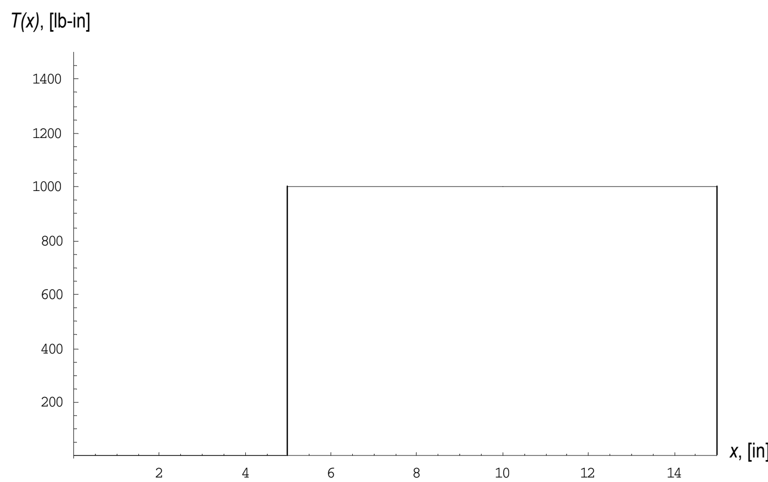
$$+\rightarrow \sum F_x = 0 \qquad \Rightarrow \qquad V_{x_3}(x) + 25 - 75 + 50 = 0 \tag{5.52}$$

$$V_{x_3}(x) \;=\; 0 \quad \text{lb} \tag{5.53}$$

The axial load diagram is



Although not asked, the torque diagram is

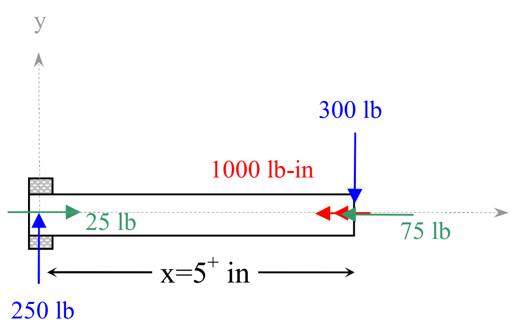


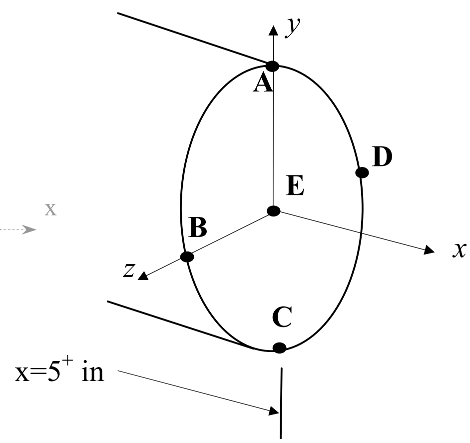
(e) Using the above information determine the location(s) in the shaft's axis (x-axis) where the stresses are critical. Justify your answer with a sentence or two.

At first it is not clear where the critical location is; the shear force is highest between $0 < x < 5$, but the moment is highest at $x = 5$. Also, the torque is highest in the range $5 < x < 10$, and the axial load in tensile between $5 < x < 10$ but compressive for $0 < x < 5$. It seems like $5^- < x < 5^+$ is the critical point the shaft.

In practice, a design engineer must analyze all potential critical locations to determine the most critical one. That is the motivation to choose $x = 5^+$ in the next question. However, one should check each critical location to determine whether the structure is safe or not.

(f) For the cross-section at $x = 5^+$, determine the state of stress at points **A**, **B**, **C**, **D**, and **E**.





Hint: Recall

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

Thus find

$$\underline{\sigma}_{\mathbf{A}}, \quad \underline{\sigma}_{\mathbf{B}}, \quad \underline{\sigma}_{\mathbf{C}}, \quad \underline{\sigma}_{\mathbf{D}}, \quad \underline{\sigma}_{\mathbf{E}}$$

Before we proceed note that at $x = 5^+$ the loads are:

$$V_y = 50 \text{ lb} \quad M_{zz} = 1250 \text{ lb–in} \quad T = 1000 \text{ lb–in} \quad P = 50 \text{ lb}$$

and since points **A**, **B**, **C**, **D**, and **E** are point where either the shear or normal stresses are maximum, let us proceed to obtain the absolute value of these stresses and then we will study each point.

First of all, note that:

✓ at points **A** and **C** the shear stress due to shear load (V_y) is zero

✓ at points **A** and **C** the normal stress due to bending is critical

✓ at points **B**, **E** and **D** the normal stress due to bending is zero, but not the normal stress due to axial load.

✓ at points **B**, **E** and **D** the shear stress due to shear load is maximum

✓ at point **E** the shear stress due to torsional load is zero

✓ at points **A**, **B**, **C**, and **D** the shear stress due torsional load exits

✓ at all points the normal stress due axial load exits

✓ $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = 0$

For magnitude of normal stress due to normal force is

$$\sigma_{xx}\big|_{axial} = \frac{P}{A_x} = 63.66 \text{ psi}$$

For magnitude of normal stress due to bending is

$$\sigma_{xx}\big|_{bending} = \frac{M_{zz}\,c}{I_{zz}} = \frac{M_{zz}}{Z} = 12732 \text{ psi}$$

For magnitude of shear stress due to vertical shear load is

$$\tau_{xy}\big|_{shear} = \frac{4}{3}\frac{V_y}{A_x} = 84.88 \text{ psi}$$

For magnitude of shear stress due to torsion is

$$\tau\big|_{torsion} = \frac{T\,c}{J_{xx}} = \frac{T}{Q} = 5093 \text{ psi}$$

At point **A**:

$$y = c = +\frac{d}{2} \qquad r = c = \frac{d}{2}$$

$$\tau\big|_{torsion} = \frac{T\,r}{J_{xx}} = \frac{T}{Q} \qquad \sigma_{xx}\big|_{bending} = -\frac{M_{zz}\,y}{I_{zz}} = -\frac{M_{zz}}{Z}$$

$$\sigma_{xx} = \sigma_{xx}\big|_{bending} + \sigma_{xx}\big|_{axial}$$

$$= -\frac{M_{zz}}{Z} + \frac{P}{A_x} = -12732 + 63.66 = -12668.7 \text{ psi}$$

and

$$\tau_{xz} = \tau\big|_{torsion} = \frac{T}{Q} = 5093 \text{ psi}$$

$$\underline{\boldsymbol{\sigma}}_A = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -12668.7 & 0 & 5093 \\ 0 & 0 & 0 \\ 5093 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.54}$$

At point **C**:

$$y = -c = -\frac{d}{2} \qquad r = c = \frac{d}{2}$$

$$\tau\big|_{torsion} = \frac{T\,r}{J_{xx}} = -\frac{T}{Q} \qquad \sigma_{xx}\big|_{bending} = \frac{M_{zz}\,y}{I_{zz}} = \frac{M_{zz}}{Z}$$

$$\sigma_{xx} = \sigma_{xx}\Big|_{\text{bending}} + \sigma_{xx}\Big|_{\text{axial}}$$

$$= \frac{M_{zz}}{Z} + \frac{P}{A_x} = 12732 + 63.66 = 12796.1 \text{ psi}$$

and

$$\tau_{xz} = \tau\Big|_{\text{torsion}} = -\frac{T}{Q} = -5093 \text{ psi}$$

$$\underline{\boldsymbol{\sigma}}_C = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 12796.1 & 0 & -5093 \\ 0 & 0 & 0 \\ -5093 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.55}$$

At point **B**:

$$y = 0 \qquad r = c = \frac{d}{2}$$

$$\tau\Big|_{\text{torsion}} = \frac{T\,r}{J_{xx}} = \frac{T}{Q} \qquad \sigma_{xx}\Big|_{\text{bending}} = \frac{M_{zz}\,y}{I_{zz}} = 0$$

$$\sigma_{xx} = \sigma_{xx}\Big|_{\text{bending}} + \sigma_{xx}\Big|_{\text{axial}}$$

$$= 0 + \frac{P}{A_x} = 0 + 63.66 = 63.66 \text{ psi}$$

and

$$\tau_{xy} = \tau\Big|_{\text{torsion}} + \tau_{xy}\Big|_{\text{shear}} = -\frac{T}{Q} + \frac{4}{3}\frac{V_y}{A_x} = -5093 + 84.88 = -5008.08 \text{ psi}$$

$$\underline{\boldsymbol{\sigma}}_B = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 63.66 & -5008.08 & 0 \\ -5008.08 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.56}$$

At point **D**:

$$y = 0 \qquad r = \frac{d}{2}$$

$$\tau\Big|_{\text{torsion}} = \frac{T\,r}{J_{xx}} = \frac{T}{Q} \qquad \sigma_{xx}\Big|_{\text{bending}} = \frac{M_{zz}\,y}{I_{zz}} = 0$$

$$\sigma_{xx} = \sigma_{xx}\Big|_{\text{bending}} + \sigma_{xx}\Big|_{\text{axial}}$$

$$= 0 + \frac{P}{A_x} = 0 + 63.66 = 63.66 \text{ psi}$$

and

$$\tau_{xy} = \tau\Big|_{\text{torsion}} + \tau_{xy}\Big|_{\text{shear}} = \frac{T}{Q} + \frac{4}{3}\frac{V_y}{A_x} = 5093 + 84.88 = 5177.84 \text{ psi}$$

$$\underline{\boldsymbol{\sigma}}_D = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 63.66 & 5177.84 & 0 \\ 5177.84 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.57}$$

At point **E**:

$$y = 0 \qquad r = 0$$

$$\tau\Big|_{\text{torsion}} = \frac{T\,r}{J_{\text{xx}}} = 0 \qquad \sigma_{\text{xx}}\Big|_{\text{bending}} = \frac{M_{\text{zz}}\,y}{I_{\text{zz}}} = 0$$

$$\sigma_{\text{xx}} = \sigma_{\text{xx}}\Big|_{\text{bending}} + \sigma_{\text{xx}}\Big|_{\text{axial}}$$

$$= 0 + \frac{P}{A_{\text{x}}} = 0 + 63.66 = 63.66 \text{ psi}$$

and

$$\tau_{\text{xy}} = \tau\Big|_{\text{torsion}} + \tau_{\text{xy}}\Big|_{\text{shear}} = 0 + \frac{4}{3}\frac{V_{\text{y}}}{A_{\text{x}}} = 0 + 84.88 = 84.88 \text{ psi}$$

$$\underline{\boldsymbol{\sigma}}_{\text{E}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 63.66 & 84.88 & 0 \\ 84.88 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.58}$$

(g) For each state of stress (points **A**, **B**, **C**, **D**, and **E**), determine the maximum normal stress and maximum shear stress. (Hint: Using eigenvalue approach to determine the principal stresses.)

At point **A**:

$$\underline{\boldsymbol{\sigma}}_{\text{A}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} -12668.7 & 0 & 5093 \\ 0 & 0 & 0 \\ 5093 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.59}$$

The stress invariants are

$$I_{\sigma_1} = \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = -12668.7 \text{ psi}$$

$$I_{\sigma_2} = \sigma_{\text{xx}}\,\sigma_{\text{yy}} + \sigma_{\text{zz}}\,\sigma_{\text{xx}} + \sigma_{\text{yy}}\,\sigma_{\text{zz}} - \tau_{\text{xy}}^2 - \tau_{\text{yz}}^2 - \tau_{\text{zx}}^2 = -2.59382 \times 10^7 \text{ psi}^2$$

$$I_{\sigma_3} = \sigma_{\text{xx}}\,\sigma_{\text{yy}}\,\sigma_{\text{zz}} + 2\,\tau_{\text{xy}}\,\tau_{\text{yz}}\,\tau_{\text{zx}} - \sigma_{\text{xx}}\,\tau_{\text{yz}}^2 - \sigma_{\text{yy}}\,\tau_{\text{zx}}^2 - \sigma_{\text{zz}}\,\tau_{\text{xy}}^2 = 0$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0$$

The principal stresses are

$$\sigma_1 = 1793.51 \text{ psi} \quad \sigma_2 = 0 \quad \sigma_3 = -14462.2 \text{ psi}$$

The overall maximum shear stress is

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 8127.88 \text{ psi}$$

At point **B**:

$$\underline{\boldsymbol{\sigma}}_{\text{B}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 63.66 & -5008.08 & 0 \\ -5008.08 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.60}$$

The stress invariants are

$$
\begin{aligned}
I_{\sigma_1} &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 63.66 \text{ psi} \\
I_{\sigma_2} &= \sigma_{xx}\,\sigma_{yy} + \sigma_{zz}\,\sigma_{xx} + \sigma_{yy}\,\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = -2.50808 \times 10^7 \text{ psi}^2 \\
I_{\sigma_3} &= \sigma_{xx}\,\sigma_{yy}\,\sigma_{zz} + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_{xx}\,\tau_{yz}^2 - \sigma_{yy}\,\tau_{zx}^2 - \sigma_{zz}\,\tau_{xy}^2 = 0
\end{aligned}
$$

The characteristic equation is

$$
\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0
$$

The principal stresses are

$$
\sigma_1 = 5040.01 \text{ psi} \quad \sigma_2 = 0 \quad \sigma_3 = -4976.35 \text{ psi}
$$

The overall maximum shear stress is

$$
\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 5008.18 \text{ psi}
$$

At point **C**:

$$
\underline{\boldsymbol{\sigma}}_C = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 12796.1 & 0 & -5093 \\ 0 & 0 & 0 \\ -5093 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.61}
$$

The stress invariants are

$$
\begin{aligned}
I_{\sigma_1} &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 12796.1 \text{ psi} \\
I_{\sigma_2} &= \sigma_{xx}\,\sigma_{yy} + \sigma_{zz}\,\sigma_{xx} + \sigma_{yy}\,\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = -2.29382 \times 10^7 \text{ psi}^2 \\
I_{\sigma_3} &= \sigma_{xx}\,\sigma_{yy}\,\sigma_{zz} + 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_{xx}\,\tau_{yz}^2 - \sigma_{yy}\,\tau_{zx}^2 - \sigma_{zz}\,\tau_{xy}^2 = 0
\end{aligned}
$$

The characteristic equation is

$$
\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0
$$

The principal stresses are

$$
\sigma_1 = 14475.6 \text{ psi} \quad \sigma_2 = 0 \quad \sigma_3 = -1779.56 \text{ psi}
$$

The overall maximum shear stress is

$$
\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 8177.59 \text{ psi}
$$

At point **D**:

$$\underline{\boldsymbol{\sigma}}_{\text{D}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 63.66 & 5177.84 & 0 \\ 5177.84 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.62}$$

The stress invariants are

$$\begin{aligned} I_{\sigma_1} &= \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = 63.66 \text{ psi} \\ I_{\sigma_2} &= \sigma_{\text{xx}}\,\sigma_{\text{yy}} + \sigma_{\text{zz}}\,\sigma_{\text{xx}} + \sigma_{\text{yy}}\,\sigma_{\text{zz}} - \tau_{\text{xy}}^2 - \tau_{\text{yz}}^2 - \tau_{\text{zx}}^2 = -2.681 \times 10^7 \text{ psi}^2 \\ I_{\sigma_3} &= \sigma_{\text{xx}}\,\sigma_{\text{yy}}\,\sigma_{\text{zz}} + 2\,\tau_{\text{xy}}\,\tau_{\text{yz}}\,\tau_{\text{zx}} - \sigma_{\text{xx}}\,\tau_{\text{yz}}^2 - \sigma_{\text{yy}}\,\tau_{\text{zx}}^2 - \sigma_{\text{zz}}\,\tau_{\text{xy}}^2 = 0 \end{aligned}$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0$$

The principal stresses are

$$\sigma_1 = 5209.77 \text{ psi} \quad \sigma_2 = 0 \quad \sigma_3 = -5146.11 \text{ psi}$$

The overall maximum shear stress is

$$\tau_{\text{max}} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 5177.94 \text{ psi}$$

At point **E**:

$$\underline{\boldsymbol{\sigma}}_{\text{E}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 63.66 & 84.88 & 0 \\ 84.88 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ psi} \tag{5.63}$$

The stress invariants are

$$\begin{aligned} I_{\sigma_1} &= \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = 63.66 \text{ psi} \\ I_{\sigma_2} &= \sigma_{\text{xx}}\,\sigma_{\text{yy}} + \sigma_{\text{zz}}\,\sigma_{\text{xx}} + \sigma_{\text{yy}}\,\sigma_{\text{zz}} - \tau_{\text{xy}}^2 - \tau_{\text{yz}}^2 - \tau_{\text{zx}}^2 = -7205.06 \text{ psi}^2 \\ I_{\sigma_3} &= \sigma_{\text{xx}}\,\sigma_{\text{yy}}\,\sigma_{\text{zz}} + 2\,\tau_{\text{xy}}\,\tau_{\text{yz}}\,\tau_{\text{zx}} - \sigma_{\text{xx}}\,\tau_{\text{yz}}^2 - \sigma_{\text{yy}}\,\tau_{\text{zx}}^2 - \sigma_{\text{zz}}\,\tau_{\text{xy}}^2 = 0 \end{aligned}$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = 0$$

The principal stresses are

$$\sigma_1 = 122.486 \text{ psi} \quad \sigma_2 = 0 \quad \sigma_3 = -58.82 \text{ psi}$$

The overall maximum shear stress is

$$\tau_{\max} = \left| \frac{\sigma_1 - \sigma_3}{2} \right| = 90.65 \text{ psi}$$

(h) What is the critical cross-sectional stress location at $x = 5^+$?

From consideration of these elements, we conclude that the element at location **C** is critical because it has the largest normal and maximum shear stress. One can also show that the von Mises stresses are highest at this point:

$$\sigma_{\mathrm{eA}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 15437.3 \text{ psi}$$

$$\sigma_{\mathrm{eB}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 8674.47 \text{ psi}$$

$$\sigma_{\mathrm{eC}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 15542 \text{ psi}$$

$$\sigma_{\mathrm{eD}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 8968.51 \text{ psi}$$

$$\sigma_{\mathrm{eE}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 160.212 \text{ psi}$$

The above is an alternative identifying the cross-sectional's most critical point.

End Example □

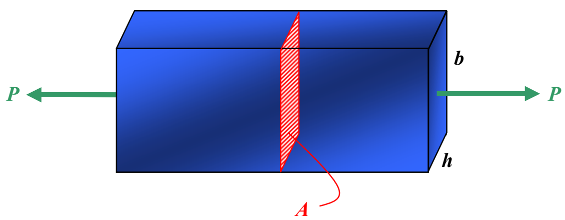
5.7 Stress Concentration

So far, the basic stress analysis calculations we have performed assume smooth components with uniform sections and no irregularities. However, in practice almost all engineering components have some changes in their sections and/or shape. As for an example, shoulders on shafts, oil holes, key ways and screw threads.

Any discontinuity changes the stress distribution in the vicinity of the discontinuity and as a consequence the basic stress analysis equations no longer apply. Stress as these *discontinuities* are sometimes called *stress raisers* and they cause local increase of stress, commonly referred to as *stress concentration*.

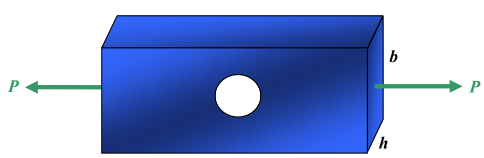
5.7.1 Stress Concentration Factor

Let the actual maximum stress at the discontinuity be $\sigma_{\rm t}$, and the stress without the discontinuity (called nominal stress) be $\sigma_{\rm N}$. As an example, consider a plate subject to axial load P and a uniform cross-sectional area A: The nominal stress will be given by



$$\sigma_{\rm N} = \frac{P}{A} = \frac{P}{b\,h} \tag{5.64}$$

Now consider the same plate subject to the same load P but with a hole of diameter d in the middle:

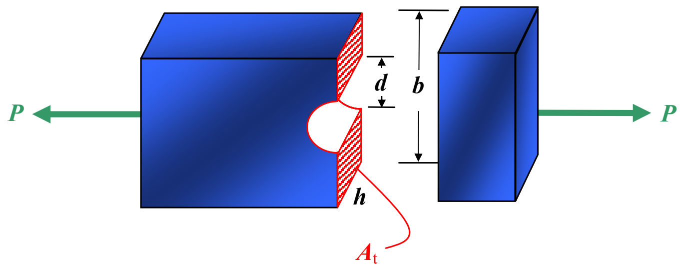


Then the actual stress will be given by

$$\sigma_{\rm t} = \frac{P}{A_{\rm t}} = \frac{P}{(b-d)\,h} \tag{5.65}$$

Thus from the above example it is clear that

$$\sigma_{\rm t} \geq \sigma_{\rm N} \qquad \rightarrow \qquad \frac{\sigma_{\rm t}}{\sigma_{\rm N}} \geq 1 \tag{5.66}$$



In general, the *theoretical* or *geometric* stress concentration factor K_{t} is used to relate the actual maximum stress σ_{t} at the discontinuity to the nominal stress σ_{N} as follows

$$K_{\mathrm{t}} = \frac{\sigma_{\mathrm{t}}}{\sigma_{\mathrm{N}}} \qquad \rightarrow \qquad K_{\mathrm{t}} \geq 1 \tag{5.67}$$

In published information relating to stress concentration values the nominal stress may be defined on either the original *gross cross section* or on the *reduced net cross section* and care needs to be taken that the correct nominal stress is used. The subscript "t" indicates that the stress concentration value is a theoretical calculation based only on the geometry of the component and discontinuity.

5.7.2 Stress Concentration Charts

As stated the stated the stress concentration factor is a function of the type of discontinuity (hole, fillet, groove), the geometry of the discontinuity, and the type of loading being experienced. The stress concentration factor will depend on the type of loading. In static loading, stress-concentration factors are applied as follows:

1. Ductile materials: the stress concentration factor is not usually applied to predict the critical stress, because plastic strain in the region of the stress is localized and has a strengthening effect. In these cases, we use the following expression:

$$\sigma_{\mathrm{t}}\Big|_{\mathrm{axial}} \simeq \sigma_{\mathrm{N}}\Big|_{\mathrm{axial}} \qquad \sigma_{\mathrm{t}}\Big|_{\mathrm{bending}} \simeq \sigma_{\mathrm{N}}\Big|_{\mathrm{bending}} \qquad \tau_{\mathrm{t}}\Big|_{\mathrm{torque}} \simeq \tau_{\mathrm{N}}\Big|_{\mathrm{torque}}$$

2. Brittle materials: The geometric stress concentration factor K_t is applied to the nominal stress before comparing it with strength. In these cases, we use the following expression:

$$\sigma_{\mathrm{t}}\Big|_{\mathrm{axial}} = K_{\mathrm{t}_a}\,\sigma_{\mathrm{N}}\Big|_{\mathrm{axial}} \qquad \sigma_{\mathrm{t}}\Big|_{\mathrm{bending}} = K_{\mathrm{t}_b}\,\sigma_{\mathrm{N}}\Big|_{\mathrm{bending}} \qquad \tau_{\mathrm{t}}\Big|_{\mathrm{torque}} = K_{\mathrm{t}_s}\,\tau_{\mathrm{N}}\Big|_{\mathrm{torque}}$$

   Figures of the course textbook help us determine the stress concentration factor.

From these charts a number of observations can be made about the stress concentration factor:

a) The stress concentration factor is independent of the part's material properties.

b) The stress concentration factor is significantly affected by geometry.

c) The stress concentration factor is affected by the type of discontinuity.

These observations are relevant in reducing stresses in a part.

In some problems the actual maximum stress may be taken as the allowable stress

$$\sigma_{\text{allowable}} = \frac{S_{\text{yield}}}{n_{\text{SF}}}$$

and for these cases the problem becomes a design oriented problem, where the exact dimensions need to be picked to avoid failure.

5.7.3 When to Use Stress Concentration Values

To apply stress concentration calculations, the part and notch geometry must be known. However where a part is known to contain cracks, the geometry of these may not be known and in any case as the notch radius tends to zero, as it does in a crack, then the stress concentration value tends to infinity and the stress concentration is no longer a helpful design tool. In these cases *Fracture Mechanics* techniques are used and these techniques will be discussed later in the course.

Where the geometry is known, then for brittle materials, stress concentration values should be used. In the case of ductile materials that are subject only to one load cycle during their lifetime (fairly unusual in Mechanical Engineering) it is not necessary to use stress concentration factors as local plastic flow and work hardening will prevent failure provided the average stress is below the yield stress.

Not all ductile materials are ductile under all conditions, many become brittle under some circumstances. The most common cause of brittle behavior in materials normally considered to be ductile is being exposed to low temperatures. For ductile materials subjected to cyclic loading the stress concentration factor has to be included in the factors that reduce the fatigue strength of a component.

Some materials are not as sensitive to notches as implied by the theoretical stress concentration factor. For these materials a reduced value of K_{t} is used: K_{f}. In these materials the maximum stress is:

$$\sigma_{\text{t}} = K_{\text{f}}\,\sigma_{\text{N}} \tag{5.68}$$

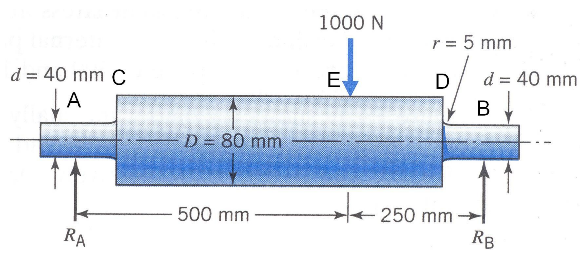
The notch sensitivity, q, is defined as:

$$q = \frac{K_{\text{f}} - 1}{K_{\text{t}} - 1} \tag{5.69}$$

where q takes values between 0 and 1. This will be discussed later when working with fatigue analysis.

Example 5.6.

Shaft analysis
A shaft is supported by bearings at locations A and B and is loaded with a downward 1000 N force at E, as shown in Figure. Fillets at C and D are identical. The distance between A and C is 70 mm and D and B is 70 mm. Identify the critical location and find the maximum stress at the most critical shaft fillet.



Solution:

First, let us assume: (i) the shaft remains straight; (ii) the material is homogeneous, isotropic and perfectly elastic. We need to plot the shear and moment diagrams to determine the critical location. Thus we proceed to obtain the reaction forces first:

$$+\circlearrowleft M_{\text{B}} = 0 = -R_A\,(750) + 1000\,(250) \qquad \rightarrow \qquad R_A = 333 \text{ N}$$

$$+\uparrow \sum F_{\text{y}} = 0 = R_A + R_B - 1000 = 333 + R_B - 1000 \qquad \rightarrow \qquad R_B = 667 \text{ N}$$

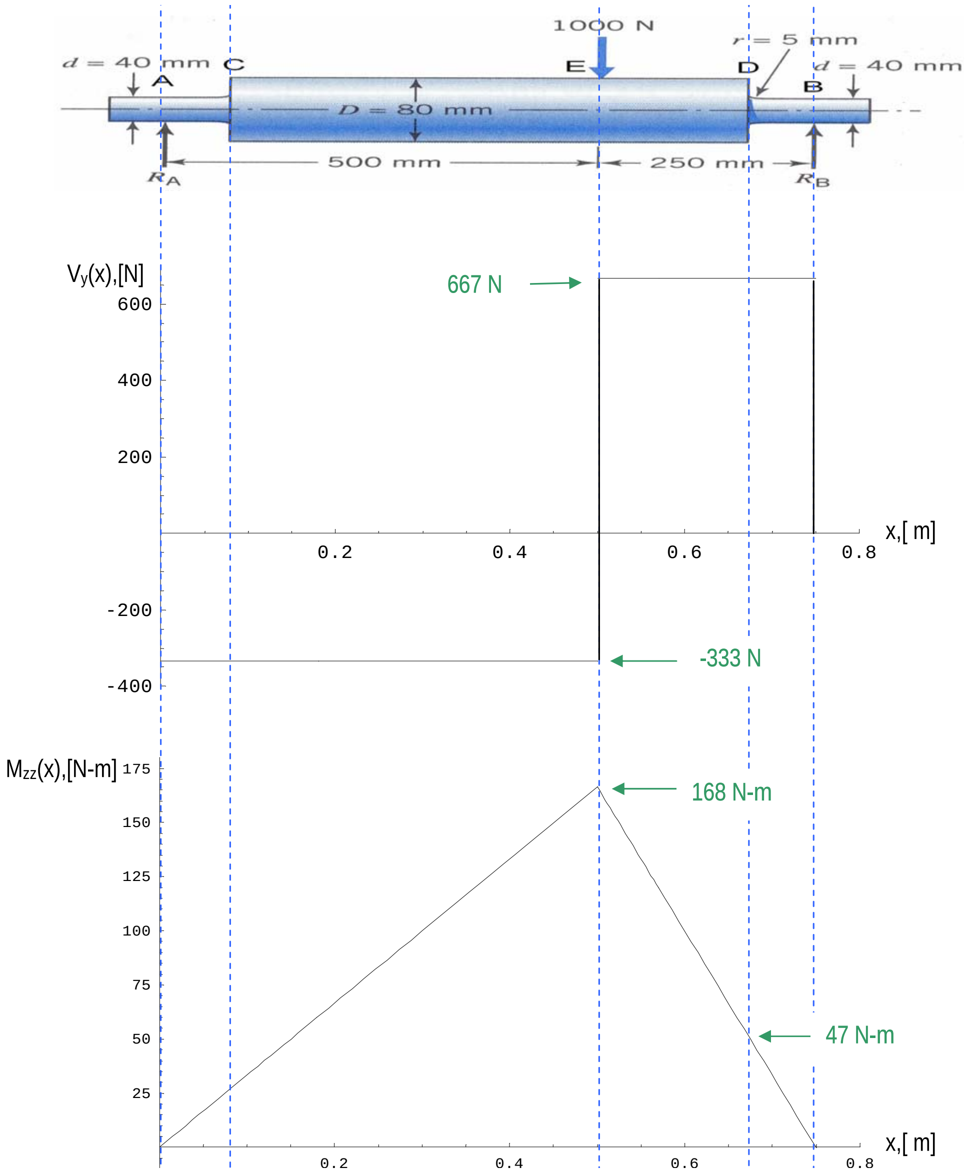
We should obtain the shear and moment equation for each section with discontinuities. This shaft has load and geometric discontinuities, thus:

$$\text{section } AC: \quad V_{\text{y}_1}(x) = -333 \qquad M_{\text{zz}_1}(x) = 333\,x$$

$$\text{section } CE: \quad V_{\text{y}_2}(x) = -333 \qquad M_{\text{zz}_2}(x) = 333\,x$$

$$\text{section } ED: \quad V_{\text{y}_3}(x) = 667 \qquad M_{\text{zz}_3}(x) = 500 - 667\,x$$

$$\text{section } DB: \quad V_{\text{y}_3}(x) = 667 \qquad M_{\text{zz}_3}(x) = 500 - 667\,x$$



1000 N
r = 5 mm
d = 40 mm
C
E
D
d = 40 mm
A
B
D = 80 mm
500 mm
250 mm
R_A
R_B
V_y(x),[N]
600
400
200
0.2
0.4
0.6
0.8
x,[m]
-200
-400
667 N
-333 N
M_{zz}(x),[N-m]
175
150
125
100
75
50
25
168 N-m
47 N-m
0.2
0.4
0.6
0.8
x,[m]

The critical location is at **E**. The critical fillet is at **D**. Only the following points will be affected by stress concentration effect due to fillet: **C** and **D**. Stress at all other points are not modified by stress concentration factors.

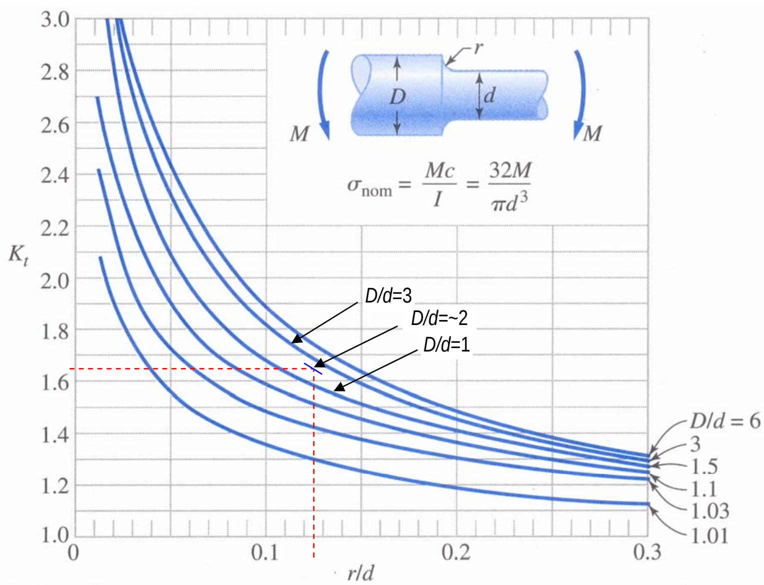
Thus at **D**, for an element at top:

$$\sigma_{xx} = -\frac{M_{zz}}{Z} \qquad Z = \frac{\pi\, d^3}{32} \qquad \rightarrow \quad \sigma_{xx} = -\frac{32\, M_{zz}}{\pi\, d^3} = -\frac{32\,(47)}{\pi\,(.04)^3} = -7.5 \text{ MPa}$$

For the critical shaft fillet location:

$$\frac{r}{d} = \frac{5}{40} = 0.125 \qquad \frac{D}{d} = \frac{80}{40} = 2.0$$

From chart:



$$K_{t_b} = 1.65$$

Thus,

$$\sigma_{xx} = -K_{t_b}\,\frac{M_{zz}}{Z} = -K_{t_b}\,\frac{32\,M_{zz}}{\pi\,d^3} = (1.65)(7.5) = -12.4 \text{ MPa}$$

The above stress concentration factor is theoretical based on a theoretical elastic, homogeneous, isotropic material.

End Example □

5.7.4 Stress Concentration for Multiple Notches

In many problems, one stress riser is superimposed upon another. Accurate calculation of the overall stress concentration factor is difficult for such combinations, but reasonable estimates can be made. For such cases one finds the stress concentration factor for each stress riser individually,

$$K_{t_1}, \quad K_{t_2}, \ldots, \quad K_{t_n}$$

The total stress concentration factor can be obtained by multiplying all factors. In order words, the combined theoretical stress concentration factor K_t for the multiple notch can be approximated by the product of the stress concentration factors for the all notches considered individually:

$$K_t = K_{t_1}\, K_{t_2} \ldots K_{t_n}$$

As for an example, suppose a shaft has a radial hole and a groove subject to bending. Then

$$\text{Due to radial hole subject to bending} \quad \rightarrow K_{t_1}$$

$$\text{Due to groove subject to bending} \quad \rightarrow K_{t_2}$$

The actual stress will be given by:

$$\sigma_{xx} = -\, K_t \frac{M_{zz}}{I_{zz}}\, y = -\, K_{t_1}\, K_{t_2} \frac{M_{zz}}{I_{zz}}\, y$$

Recall the minus sign remain because it is consistent with our sign convention.

5.8 References

Collins, J. A., *Mechanical Design of Machine Elements and Machines*, 2003, John Wiley and Sons, New York, NY.

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Juvinall, R. C., and Marsheck, K. A., *Fundamentals of Machine Component Design*, 2000, John Wiley and Sons, New York, NY.

Shigley, J. E., Mischke, C. R., and Budynas, R, G., *Mechanical Engineering Design*, 2004, Seventh Edition, Mc-Graw Hill, New York, NY.

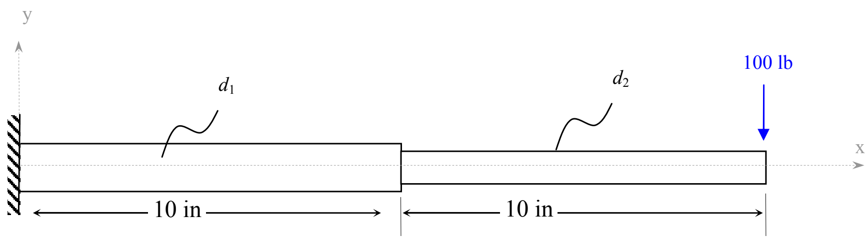
Thomas, G. B., Finney R. L., Weir, M. D., and Giordano F. R., *Thomas Calculus, Early Transcendentals Update*, 2003, Tenth Edition, Addison-Wesley, Massachusetts. Entire book.

5.9 Suggested Problems

Problem 5.1.

A circular-steel solid shaft is loaded by a vertical force (a downward vertical force of 100 lb at $x = 20$ in as shown in Figure. The shaft is composed of two different diameter cross-sections. The shaft has a diameter of d_1 for $0 \leq x \leq 10$ and a diameter of d_2 for $10 \leq x \leq 20$. All loads are applied at the shafts neutral axis. Take:

$$d_2 = 3\,d_1$$



(a) Determine the cross sectional area A_x, the second moment of inertia I_{zz}, and the polar moment of inertia J_{xx} for each cross section

(b) Determine the reaction forces

(c) Draw the shear force diagram: $V_y(x)$ for $0 < x < 20$.

(d) Draw the bending moment diagram: $M_{zz}(x)$ for $0 < x < 20$.

(e) Using the above information determine the location(s) in the shaft's axis (x-axis) where the stresses are critical. Justify your answer with a sentence or two.

□

Problem 5.2.

Figure below shows a steel shaft supported by self-aligning bearings and subjected to a uniformly distributed loads. If $d = 2$ in,

(a) Determine the cross sectional area A_x, the second moment of inertia I_{zz}, and the polar moment of inertia J_{xx} for each cross section

(b) Determine the reaction forces

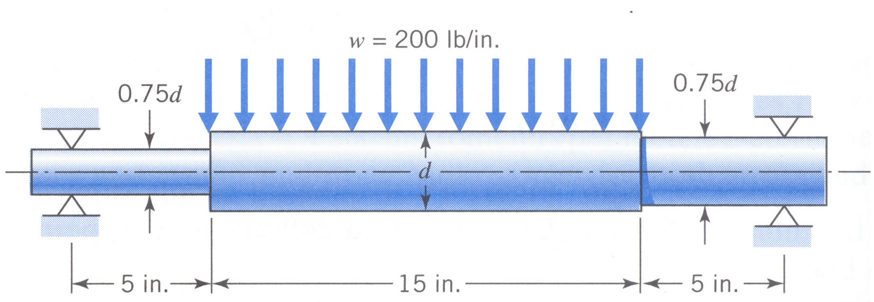
(c) Draw the shear force diagram: $V_y(x)$ for $0 < x < 25$.

(d) Draw the bending moment diagram: $M_{zz}(x)$ for $0 < x < 25$.

(e) Using the above information determine the location(s) in the shaft's axis (x-axis) where the stresses are critical. Justify your answer with a sentence or two.

(f) At the critical location, determine the state of stress at locations where the shear stress due to shear is zero.

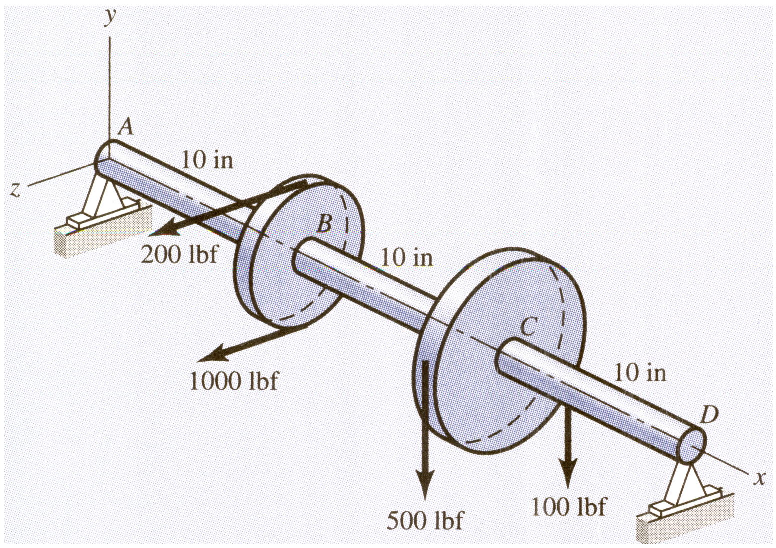
(g) At the critical location, determine the state of stress at locations where the normal stress due to bending is zero.



□

Problem 5.3.

Illustrated in the figure is a 1.25 in diameter steel countershaft that supports two pulley. Two pulleys are keyed to the shaft where pulley B is of diameter 4.0 in and pulley C is of diameter 8.0 in. Pulley C delivers power to a machine causing a tension of 500 lb in the tight side of the belt and 100 lb–in the loose side, as indicated. Pulley C receives power from a motor. The belt tensions on pulley C have a tension of 1000 lb in the tight side of the belt and 200 lb–in the loose side, as indicated. Assume that the bearings constitute simple supports, determine the critical location throughout the shaft and the cross-section.



□

Problem 5.4.

Figure below shows a steel shaft supported by self-aligning bearings and subjected to a point loads.

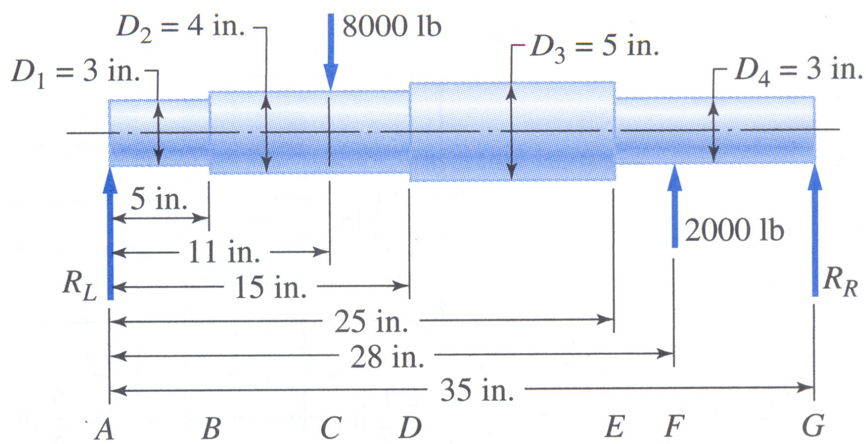
(a) Determine the cross sectional area A_x, the second moment of inertia I_{zz}, and the polar moment of inertia J_{xx} for each cross section

(b) Determine the reaction forces

(c) Using the above information determine the location(s) in the shaft's axis (x-axis) where the stresses are critical. Justify your answer with a sentence or two.

(d) At the critical location, determine the state of stress at locations where the shear stress due to shear is zero.

(e) At the critical location, determine the state of stress at locations where the normal stress due to bending is zero.



□

Problem 5.5.

Determine the torque for a engine supplying a 4.0 hp and with a rotating speed of 1500 rpm.

□

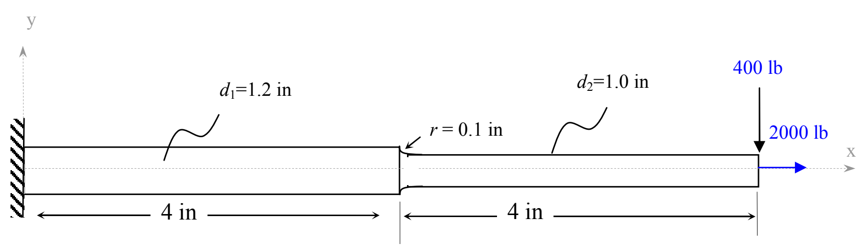
Problem 5.6.

Problem 4.45 of textbook.

□

Problem 5.7.

Find the most critically stressed location.



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Chapter 6
Deflection Analysis

6.1 Slope and deflection profiles

For the Euler Bernoulli assumption, Eq. (5.10) can be written as

$$\left\{\begin{array}{c} \varepsilon^{\circ}_{xx} \\ -\kappa^{\circ}_{zz} \\ \kappa^{\circ}_{yy} \end{array}\right\} = \frac{1}{E_0}\left[\begin{array}{ccc} \frac{1}{A} & 0 & 0 \\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}} \end{array}\right]\left\{\begin{array}{c} N_{xx}+N^t_{xx} \\ M_{yy}+M^t_{yy} \\ M_{zz}-M^t_{zz} \end{array}\right\}$$

$$\left\{\begin{array}{c} u' \\ -v'' \\ -w'' \end{array}\right\} = \frac{1}{E_0}\left[\begin{array}{ccc} \frac{1}{A} & 0 & 0 \\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}} \end{array}\right]\left\{\begin{array}{c} N_{xx}+N^t_{xx} \\ M_{yy}+M^t_{yy} \\ M_{zz}-M^t_{zz} \end{array}\right\}$$

or

$$\left\{\begin{array}{c} u' \\ v'' \\ w'' \end{array}\right\} = \frac{1}{E_0}\left[\begin{array}{ccc} \frac{1}{A} & 0 & 0 \\ 0 & \frac{1}{R_{yz}} & \frac{1}{R_{yy}} \\ 0 & -\frac{1}{R_{zz}} & -\frac{1}{R_{yz}} \end{array}\right]\left\{\begin{array}{c} N_{xx}+N^t_{xx} \\ M_{yy}+M^t_{yy} \\ M_{zz}-M^t_{zz} \end{array}\right\} \tag{6.1}$$

Note that Eq. (6.1) contains lower order derivatives in the dependent variables; and thus may be solved by integrating directly and applying displacement boundary conditions to account for integration constants.

The displacement and slope differential equations are:

$$\frac{du}{dx} = \frac{N_{xx} + N^t_{xx}}{E_0\, A}$$

$$\frac{d^2 v}{dx^2} = \frac{M_{yy} + M^t_{yy}}{E_0\, R_{yz}} + \frac{M_{zz} - M^t_{zz}}{E_0\, R_{yy}}$$

$$\frac{d^2 w}{dx^2} = -\frac{M_{yy} + M^t_{yy}}{E_0\, R_{zz}} - \frac{M_{zz} - M^t_{zz}}{E_0\, R_{yz}}$$

When cross-sectional properties are constant (do not change with x), the displacements and slopes are found by solving the following differential equations:

$$E_0\, A\, \frac{du}{dx} = N_{xx} + N^t_{xx} = P_u(x)$$

$$E_0\, R_{yy}\, \frac{d^2 v}{dx^2} = \frac{R_{yy}}{R_{yz}} \left(M_{yy} + M^t_{yy}\right) + \left(M_{zz} - M^t_{zz}\right) = M_v(x)$$

$$E_0\, R_{zz}\, \frac{d^2 w}{dx^2} = -\left(M_{yy} + M^t_{yy}\right) - \frac{R_{zz}}{R_{yz}} \left(M_{zz} - M^t_{zz}\right) = -M_w(x)$$

where $P_u(x)$, $M_v(x)$ and $M_w(x)$ are introduced to simplify our analysis. Note that

$$u = u(x) \qquad v = v(x) \qquad w = w(x)$$

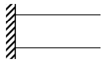
$$P_u = P_u(x) \qquad M_v = M_v(x) \qquad M_w = M_w(x)$$

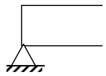
Thus, the axial displacement (u) equation is

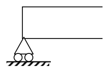
$$E_0\, A\, u' = P_u$$

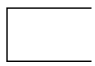
$$E_0\, A\, u(x) = \int P_u\, dx + A_1$$

The constant A_1 is found from boundary and continuity conditions (for symmetric cross-sections):

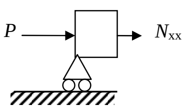

Fixed: $u = 0$


Pinned: $u = 0$


Roller: $N_{xx} = E_0\, A\, u' = 0$


Free: $N_{xx} = E_0\, A\, u' = 0$

If there is was an external load applied then



$$\overset{+}{\rightarrow}\sum F_x = 0 \quad \Rightarrow \quad P + N_{xx} = 0 \quad \Rightarrow \quad N_{xx} = E_0\, A\, u' = -P$$

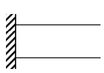
The transverse displacement (v) and slope equations (v') are

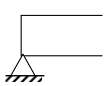
$$E_0\, R_{yy}\, v'' = M_v$$

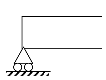
$$E_0\, R_{yy}\, v'(x) = \int M_v\, dx + B_1$$

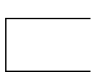
$$E_0\, R_{yy}\, v(x) = \int \left\{ \int M_v\, dx \right\} dx + B_1\, x + B_2$$

The constants B_1 and B_2 are found from boundary and continuity conditions (for symmetric cross-sections):


Fixed: $v = 0$ $\quad v' = 0$


Pinned: $v = 0$ $\quad M_{zz} = E_0\, R_{yy}\, v'' = 0$


Roller: $v = 0$ $\quad M_{zz} = E_0\, R_{yy}\, v'' = 0$


Free: $M_{zz} = E_0\, R_{yy}\, v'' = 0$ $\quad V_y = E_0\, R_{yy}\, v''' = 0$

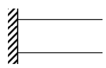
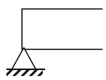
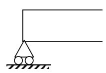
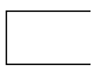
The lateral displacement (w) and slope equations (w') are

$$E_0\, R_{zz}\, w'' = M_w$$

$$E_0\, R_{zz}\, w'(x) = -\int M_w\, dx + C_1$$

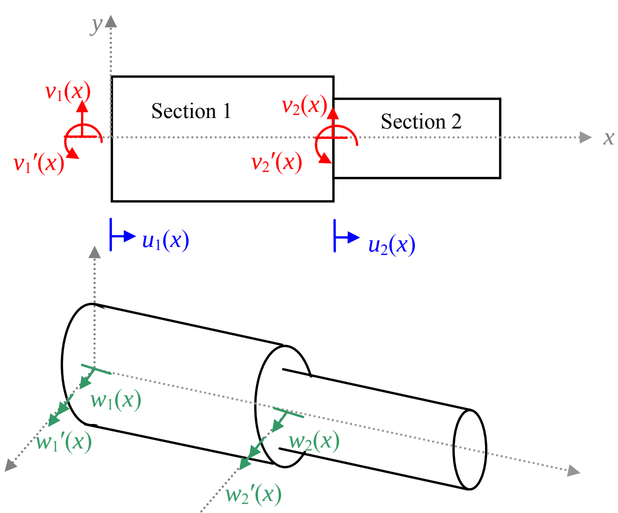
$$E_0\, R_{zz}\, w(x) = -\int \left\{ \int M_w\, dx \right\} dx + C_1\, x + C_2$$

The constants C_1 and C_2 are found from boundary and continuity conditions (for symmetric cross-sections):

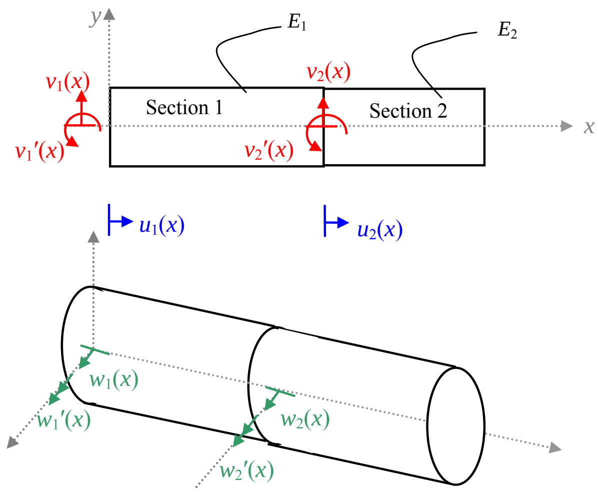
	Fixed:	$w = 0$	$w' = 0$
	Pinned:	$w = 0$	$M_{yy} = E_0\, R_{zz}\, w'' = 0$
	Roller:	$w = 0$	$M_{yy} = E_0\, R_{zz}\, w'' = 0$
	Free:	$M_{yy} = E_0\, R_{zz}\, w'' = 0$	$V_z = E_0\, R_{zz}\, w''' = 0$

In some cases we may encounter, at least, four different types of discontinuities. These discontinuities will require additional compatibility equations, these come from continuity conditions. These discontinuities are

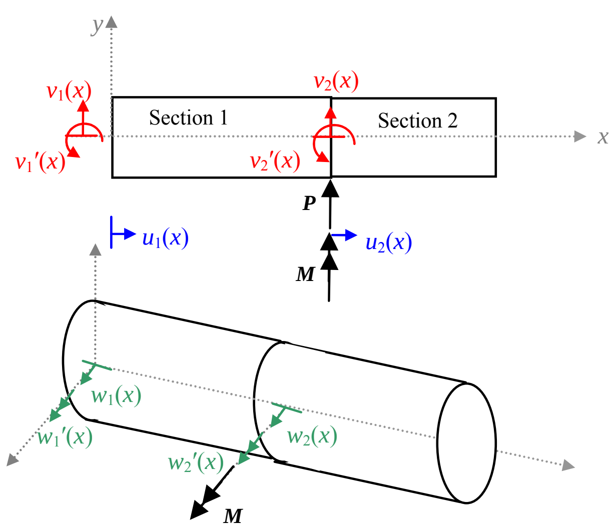
1. Geometric



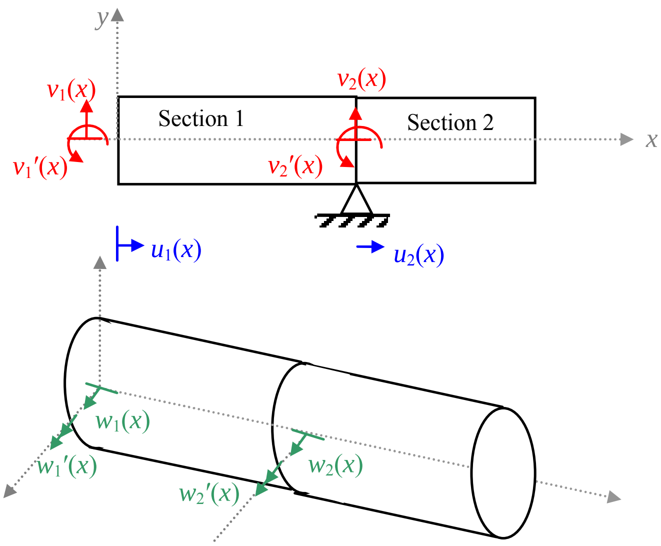
2. Material



3. Load



4. Boundary



Thus continuity conditions for a bar are such that when two different sections (say section 1 and section 2) are considered at x_1, then:

$$u_1(x_1) = u_2(x_1)$$

Continuity conditions for a beam are such that when two different sections (say section 1 and section 2) are considered at x_1, then:

$$v_1(x_1) = v_2(x_1) \qquad v_1'(x_1) = v_2'(x_1)$$

$$w_1(x_1) = w_2(x_1) \qquad w_1'(x_1) = w_2'(x_1)$$

Example 6.1.

For example in Section 4.3 and 5.2: Students have approximated a machine component using a beam model as shown in Fig. 6.1. The cantilever beam's squared cross section is uniform. These engineers need your help to analyze this component and they have a five-day deadline to complete the analysis. Take $a = 25$ mm, $b = 5$ mm. Use the stress convention and show all your steps.

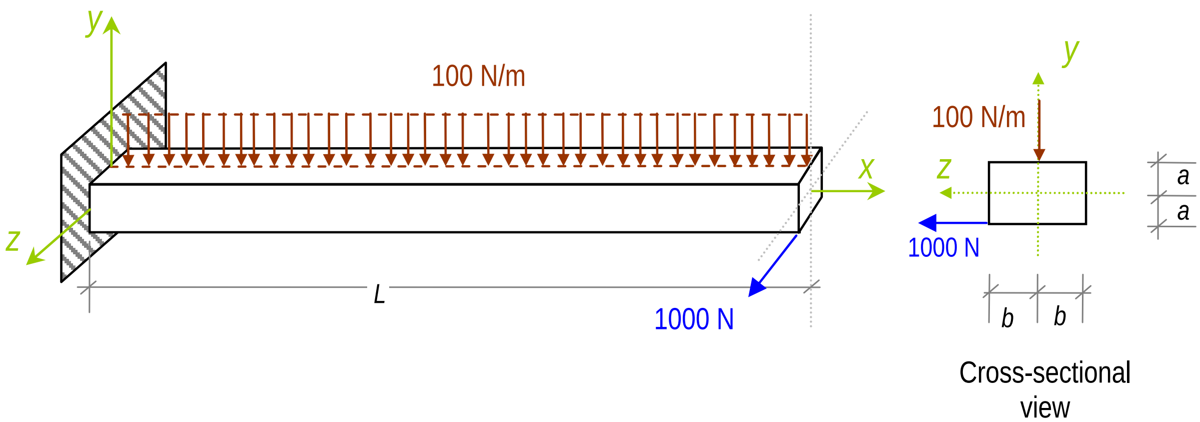


Figure 6.1: Machine component for example below.

a) Plot the deflection and slope profile in the y-direction.

The slope and deflections can be found by using:

$$\begin{Bmatrix} u' \\ -\theta'_z \\ \theta'_y \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & -\frac{1}{R_{yz}} & -\frac{1}{R_{yy}} \\ 0 & \frac{1}{R_{zz}} & \frac{1}{R_{yz}} \end{bmatrix}\begin{Bmatrix} N_{xx} \\ M_{yy} \\ M_{zz} \end{Bmatrix}$$

For a symmetric cross section the displacement and slope in the y–direction are obtained from:

$$E\, I_{zz}\, v''(x) = M_{zz}$$

The equation for deflection and slope in the y-direction is $(0 \le x \le L)$:

$$E\,I_{zz}\,v''(x) = M_{zz}(x) = -50\,L^2 + 100\,L\,x - 50\,x^2$$

$$E\,I_{zz}\,v'(x) = E\,I_{zz} \int v''(x)\,dx + A_1$$

$$= -50\,L^2\,x + 50\,L\,x^2 - \frac{50\,x^3}{3} + A_1$$

$$E\,I_{zz}\,v(x) = E\,I_{zz} \int \left(\int v''(x)\,dx \right) dx + A_1\,x + B_1$$

$$= -25\,L^2\,x^2 + \frac{50\,L\,x^3}{3} - \frac{25\,x^4}{6} + A_1\,x + B_1$$

Now we use boundary conditions to determine the coefficients:

$$v(x)\Big|_{x=0} = 0 \qquad \rightarrow \qquad B_1 = 0 \tag{6.2}$$

$$v'(x)\Big|_{x=0} = 0 \qquad \rightarrow \qquad A_1 = 0 \tag{6.3}$$

Thus the deflection in the y-direction is:

$$E\,I_{zz}\,v(x) = -25\,L^2\,x^2 + \frac{50\,L\,x^3}{3} - \frac{25\,x^4}{6}$$

$$= 25\,L^4 \left\{ -\left(\frac{x}{L}\right)^2 + \frac{2}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{6}\left(\frac{x}{L}\right)^4 \right\}$$

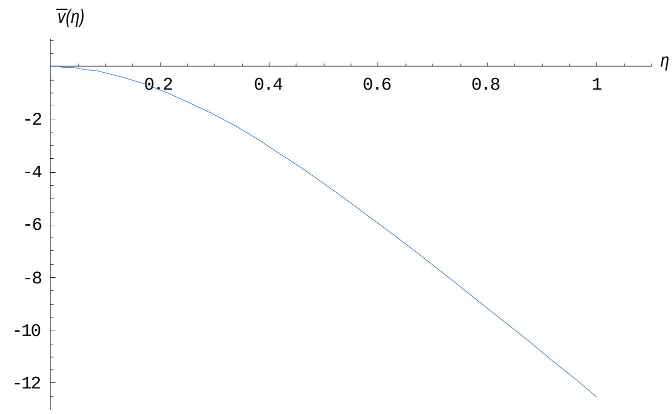
and the slope in the y-direction is:

$$E\,I_{zz}\,v'(x) = -50\,L^2\,x + 50\,L\,x^2 - \frac{50\,x^3}{3}$$

$$= 50\,L^3 \left\{ -\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^3 \right\}$$

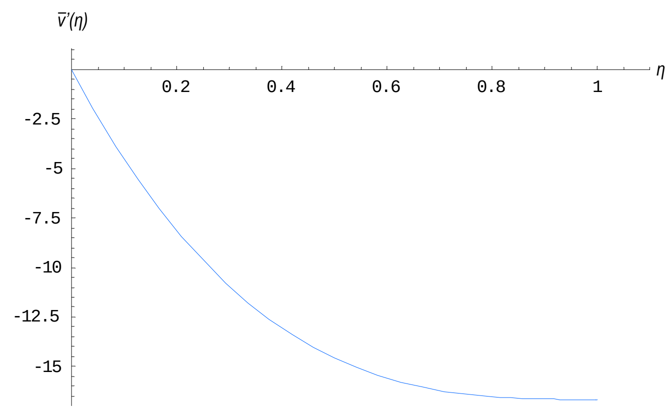
In general, it is convenient to give nondimensional quantities. By normalizing the length to one, the dimensionless deflection profile in the y–direction is

$$\overline{v}(\eta) = \frac{E\,I_{zz}\,v(x)}{L^4} = -25\,\eta^2 + \frac{50}{3}\,\eta^3 - \frac{25}{6}\,\eta^4$$



and the dimensionless slope profile in the y–direction is

$$\overline{v}'(\eta) = \frac{E\,I_{zz}\,v'(x)}{L^3} = -50\,\eta + 50\,\eta^2 - \frac{50}{3}\,\eta^3$$



b) Plot the deflection and slope profile in the z-direction.

For a symmetric cross section the displacement and slope in the z–direction are obtained from:

$$E\,I_{yy}\,w''(x) = -M_{yy}$$

The equation for deflection and slope in the z-direction is $(0 \le x \le L)$:

$$E\,I_{yy}\,w''(x) = -M_{yy}(x) = 1000\,L - 1000\,x$$

$$E\,I_{yy}\,w'(x) = E\,I_{yy}\int w''(x)\,dx + C_1$$

$$= 1000\,L\,x - 500\,x^2 + C_1$$

$$E\,I_{yy}\,w(x) = E\,I_{yy}\int\left(\int w''(x)\,dx\right)dx + C_1\,x + D_1$$

$$= 500\,L\,x^2 - \frac{500\,x^3}{3} + C_1\,x + D_1$$

Now we use boundary conditions to determine the coefficients:

$$w(x)\Big|_{x=0} = 0 \qquad \rightarrow \qquad D_1 = 0 \tag{6.4}$$

$$w'(x)\Big|_{x=0} = 0 \qquad \rightarrow \qquad C_1 = 0 \tag{6.5}$$

Thus the deflection in the z-direction is:

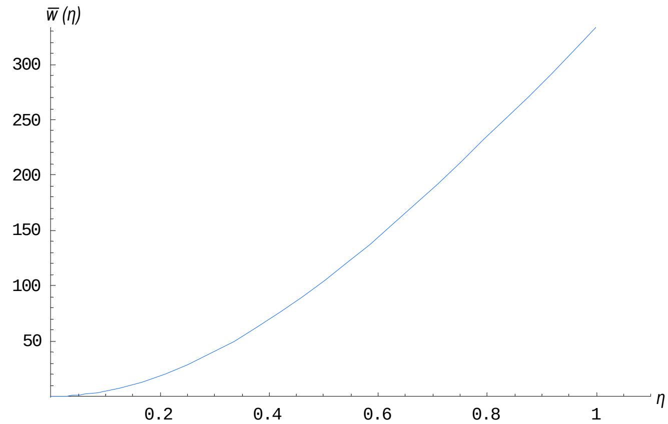
$$E\,I_{yy}\,w(x) = 500\,L\,x^2 - \frac{500\,x^3}{3} = 500\,L^3\left\{\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^3\right\}$$

and the slope in the z-direction is:

$$E\,I_{yy}\,w'(x) = 1000\,L\,x - 500\,x^2 = 500\,L^2\left\{2\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right\}$$

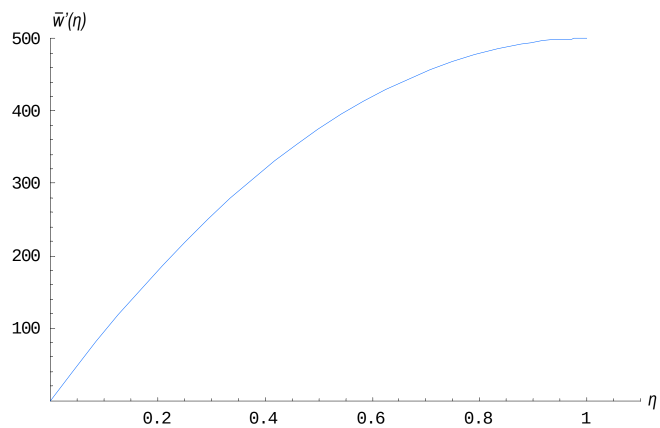
The dimensionless deflection profile is

$$\overline{w}(\eta) = \frac{E\,I_{yy}\,w(x)}{L^3} = 500\,\eta^2 - \frac{500}{3}\,\eta^3$$



and the dimensionless slope profile is

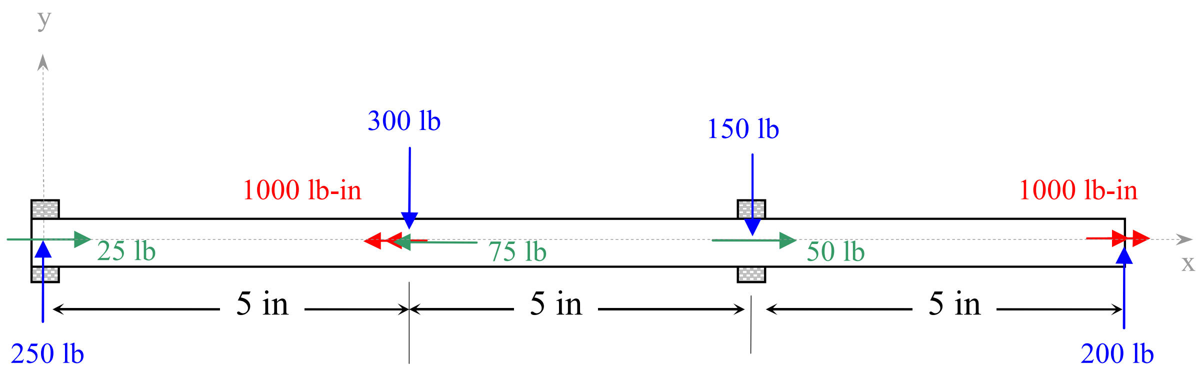
$$\overline{w}'(\eta) = \frac{E\,I_{yy}\,w'(x)}{L^2} = 1000\,\eta - 500\,\eta^2$$



End Example □

Example 6.2.

For example in Section 5.5: A circular-steel solid one-inch diameter shaft is loaded by the



four vertical forces (an upward vertical force of 250 lb at $x = 0$ in, a downward vertical force of 300 lb at $x = 5$ in, a downward vertical force of 150 lb at $x = 10$ in, an upward vertical force of 200 lb at $x = 15$ in), three axial forces (an outward axial force of 25 lb at $x = 0$ in, an inward axial force of 75 lb at $x = 5$ in, an outward axial force of 50 lb at $x = 10$ in), and torques (an inward torque of 1000 lb–in at $x = 5$ in, an outward torque of 1000 lb–in at $x = 15$ in) as shown in Figure, which result from the actions of helical gears and shaft's rolling-element bearing supports. All loads are applied at the shafts neutral axis. Draw the deflection diagram: $v(x)$ for $0 < x < 15$.

Draw the deflection diagram: $v(x)$ for $0 < x < 15$.

Since the shaft has a symmetric cross-section $I_{yz} = 0$, and from the loading conditions, $M_{yy} = 0$. Thus

$$\begin{Bmatrix} -v'' \\ -w'' \end{Bmatrix} = \frac{1}{E\,(I_{yy}\,I_{zz})} \begin{bmatrix} 0 & -I_{yy} \\ I_{zz} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ M_{zz} \end{Bmatrix} = \frac{1}{E\,(I_{yy}\,I_{zz})} \begin{Bmatrix} -M_{zz}\,I_{yy} \\ 0 \end{Bmatrix}$$

Thus

$$v''(x) = \frac{1}{E\,I_{zz}}\,M_{zz}(x) \quad \rightarrow \quad v'(x) = \frac{1}{E\,I_{zz}} \int M_{zz}(x)\,dx + A$$

$$v(x) = \int \left\{ \frac{1}{E\,I_{zz}} \int M_{zz}(x)\,dx \right\} dx + A\,x + B$$

(You may need to use Table 3.9 of your textbook, page 136.) After integrating twice, use boundary conditions to determine the two constants of integration:

$$v(x)\Big|_{x=0} = 0 \qquad\qquad v(x)\Big|_{x=10} = 0$$

$$v''(x) = \frac{1}{E\,I_{zz}}\,M_{zz}(x) \quad\rightarrow\quad v'(x) = \frac{1}{E\,I_{zz}}\int M_{zz}(x)\,dx + A$$

$$v(x) = \int\left\{\frac{1}{E\,I_{zz}}\int M_{zz}(x)\,dx\right\}dx + A\,x + B$$

Note that we will have three different displacement functions. This because the shear and moments are different in three regions.

The equation for deflection is $(0 \le x \le 5)$

$$v_1''(x) = \frac{1}{E\,I_{zz}}\,M_{zz_1}(x) = \frac{250\,x}{E\,I_{zz}}$$

$$v_1'(x) = \frac{125\,x^2}{E\,I_{zz}} + A_1$$

$$v_1(x) = \frac{125\,x^3}{3\,E\,I_{zz}} + A_1\,x + B_1$$

The equation for deflection is $(5 \le x \le 10)$

$$v_2''(x) = \frac{1}{E\,I_{zz}}\,M_{zz_2}(x) = \frac{-50\,x + 1500}{E\,I_{zz}} = -\frac{50\,x}{E\,I_{zz}} + \frac{1500}{E\,I_{zz}}$$

$$v_2'(x) = -\frac{25\,x^2}{E\,I_{zz}} + \frac{1500\,x}{E\,I_{zz}} + A_2$$

$$v_2(x) = -\frac{25\,x^3}{3\,E\,I_{zz}} + \frac{750\,x^2}{E\,I_{zz}} + A_2\,x + B_2$$

The equation for deflection is $(10 \le x \le 15)$

$$v_3''(x) = \frac{1}{E\,I_{zz}}\,M_{zz_3}(x) = \frac{-200\,x + 3000}{E\,I_{zz}} = -\frac{200\,x}{E\,I_{zz}} + \frac{3000}{E\,I_{zz}}$$

$$v_3'(x) = -\frac{100\,x^2}{E\,I_{zz}} + \frac{3000\,x}{E\,I_{zz}} + A_3$$

$$v_3(x) = -\frac{100\,x^3}{3\,E\,I_{zz}} + \frac{1500\,x^2}{E\,I_{zz}} + A_3\,x + B_3$$

Now we use boundary conditions to determine the coefficients:

$$v_1(x)\Big|_{x=0} = 0 \qquad \rightarrow \qquad B_1 = 0 \tag{6.6}$$

$$v_2(x)\Big|_{x=10} = 0 \qquad \rightarrow \qquad 10\,A_2 + B_2 + \frac{200000}{3\,E\,I_{zz}} = 0 \tag{6.7}$$

$$v_3(x)\Big|_{x=10} = 0 \qquad \rightarrow \qquad 10\,A_3 + B_3 + \frac{350000}{3\,E\,I_{zz}} = 0 \tag{6.8}$$

$$v_1(x)\Big|_{x=5} = v_2(x)\Big|_{x=5} \qquad \rightarrow \qquad 5\,A_1 + B_1 + \frac{15625}{3\,E\,I_{zz}} = 5\,A_2 + B_2 + \frac{53125}{3\,E\,I_{zz}} \tag{6.9}$$

$$v_1'(x)\Big|_{x=5} = v_2'(x)\Big|_{x=5} \qquad \rightarrow \qquad A_1 + \frac{3125}{E\,I_{zz}} = A_2 + \frac{6875}{E\,I_{zz}} \tag{6.10}$$

$$v_2'(x)\Big|_{x=10} = v_3'(x)\Big|_{x=10} \qquad \rightarrow \qquad A_2 + \frac{12500}{E\,I_{zz}} = A_3 + \frac{20000}{E\,I_{zz}} \tag{6.11}$$

From Eq. (6.6): $B_1 = 0$.

From Eq. (6.10):

$$A_1 = A_2 + \frac{3125}{E\,I_{zz}}$$

From Eqs. (6.7) and (6.9):

$$A_2 = -\frac{21875}{E\,I_{zz}} \qquad B_2 = \frac{6250}{E\,I_{zz}}$$

From Eq. (6.11):

$$A_3 = -\frac{44375}{3\,E\,I_{zz}}$$

From Eq. (6.8):

$$B_3 = \frac{31250}{E\,I_{zz}}$$

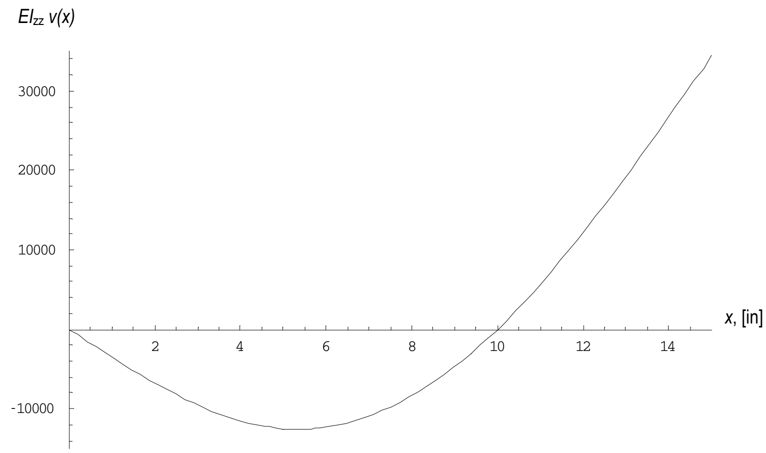
The equation for deflection is

$$0 \le x \le 5 \quad : \quad v_1(x) = -\frac{10625\,x}{3\,E\,I_{zz}} + \frac{125\,x^3}{3\,E\,I_{zz}}$$

$$5 \le x \le 10 \quad : \quad v_2(x) = \frac{6250}{E\,I_{zz}} - \frac{21875\,x}{3\,E\,I_{zz}} + \frac{750\,x^2}{E\,I_{zz}} - \frac{25\,x^3}{3\,E\,I_{zz}}$$

$$10 \le x \le 15 \quad : \quad v_3(x) = \frac{31250}{E\,I_{zz}} - \frac{44375\,x}{3\,E\,I_{zz}} + \frac{1500\,x^2}{E\,I_{zz}} - \frac{100\,x^3}{3\,E\,I_{zz}}$$

Deflection profile in the y–direction is



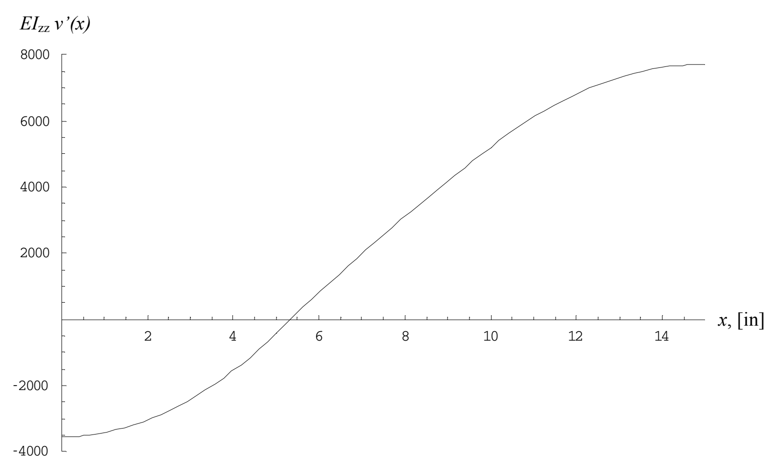
The equation for slope is

$$0 \leq x \leq 5 \quad : \quad v_1'(x) = -\frac{10625}{3\,E\,I_{zz}} + \frac{125\,x^2}{E\,I_{zz}}$$

$$5 \leq x \leq 10 \quad : \quad v_2'(x) = -\frac{21875}{3\,E\,I_{zz}} + \frac{1500\,x}{E\,I_{zz}} - \frac{25\,x^2}{E\,I_{zz}}$$

$$10 \leq x \leq 15 \quad : \quad v_3'(x) = -\frac{44375}{3\,E\,I_{zz}} + \frac{3000\,x}{E\,I_{zz}} - \frac{100\,x^2}{E\,I_{zz}}$$

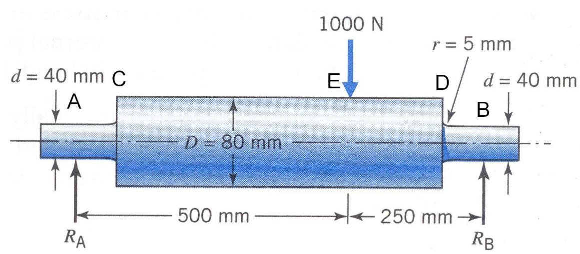
Slope profile in the y–direction is



End Example □

Example 6.3.

For Example 5.6: A shaft is supported by bearings at locations A and B and is loaded with a downward 1000 N force at E, as shown in Figure. Fillets at C and D are identical. The distance between A and C is 70 mm and D and B is 70 mm. Identify the critical location and find the maximum stress at the most critical shaft fillet.



Determine the displacement and slope diagrams.

Note geometric discontinuity. Let

$$EI_{zz} = EI_{zz_1}$$

and note that:

$$I_{zz_1} = \frac{\pi\, d^4}{64} = 1.25664 \times 10^{-7}\ \text{m}^4 \qquad I_{zz_2} = \frac{\pi\, D^4}{64} = 2.01062 \times 10^{-6}\ \text{m}^4$$

and the ratio of these two are

$$\frac{I_{zz_2}}{I_{zz_1}} = 16 \qquad \rightarrow \qquad I_{zz_2} = 16\, I_{zz_1}$$

Thus

$$\text{section } AC: \qquad EI_{zz_1} = EI_{zz}$$

$$\text{section } CE: \qquad EI_{zz_2} = 16\, EI_{zz}$$

$$\text{section } ED: \qquad EI_{zz_3} = 16\, EI_{zz}$$

$$\text{section } DB: \qquad EI_{zz_4} = EI_{zz}$$

The equations for slope and deflection are $(0 \leq x \leq 0.180)$

$$v_1''(x) = \frac{1}{E\,I_{zz_1}}\,M_{zz_1}(x) = \frac{333}{E\,I_{zz_1}}\,x$$

$$v_1'(x) = \frac{333}{2\,E\,I_{zz_1}}\,x^2 + A_1 = \frac{333}{2\,E\,I_{zz}}\,x^2 + A_1$$

$$v_1(x) = \frac{111}{2\,E\,I_{zz}}\,x^3 + A_1\,x + B_1$$

The equations for slope and deflection are $(0.180 \leq x \leq 0.500)$

$$v_2''(x) = \frac{1}{E\,I_{zz_2}}\,M_{zz_2}(x) = \frac{333}{E\,I_{zz_2}}\,x$$

$$v_2'(x) = \frac{333}{2\,E\,I_{zz_2}}\,x^2 + A_2 = \frac{333}{32\,E\,I_{zz}}\,x^2 + A_2$$

$$v_2(x) = \frac{111}{32\,E\,I_{zz}}\,x^3 + A_2\,x + B_2$$

The equations for slope and deflection are $(0.500 \leq x \leq 0.570)$

$$v_3''(x) = \frac{1}{E\,I_{zz_3}}\,M_{zz_3}(x) = \frac{500 - 667\,x}{E\,I_{zz_3}}$$

$$v_3'(x) = \frac{500}{E\,I_{zz_3}}\,x - \frac{667}{2\,E\,I_{zz_3}}\,x^2 + A_3 = -\frac{667}{32\,E\,I_{zz}}\,x^2 + \frac{125}{4\,E\,I_{zz}}\,x + A_3$$

$$v_3(x) = -\frac{667}{96\,E\,I_{zz}}\,x^3 + \frac{125}{8\,E\,I_{zz}}\,x^2 + A_3\,x + B_3$$

The equations for slope and deflection are $(0.570 \leq x \leq 0.750)$

$$v_4''(x) = \frac{1}{E\,I_{zz_4}}\,M_{zz_4}(x) = \frac{500 - 667\,x}{E\,I_{zz_4}}$$

$$v_4'(x) = \frac{500}{E\,I_{zz_4}}\,x - \frac{667}{2\,E\,I_{zz_4}}\,x^2 + A_4 = -\frac{667}{2\,E\,I_{zz}}\,x^2 + \frac{500}{E\,I_{zz}}\,x + A_4$$

$$v_4(x) = -\frac{667}{6\,E\,I_{zz}}\,x^3 + \frac{250}{E\,I_{zz}}\,x^2 + A_4\,x + B_4$$

Now we use boundary conditions to determine the coefficients:

$$v_1(x)\Big|_{x=0} = 0 \qquad v_4(x)\Big|_{x=0.750} = 0$$

$$v_1(x)\Big|_{x=0.070} = v_2(x)\Big|_{x=0.070} \qquad v_2(x)\Big|_{x=0.500} = v_3(x)\Big|_{x=0.500}$$

$$v_3(x)\Big|_{x=0.680} = v_4(x)\Big|_{x=0.680} \qquad v_1'(x)\Big|_{x=0.070} = v_2'(x)\Big|_{x=0.070}$$

$$v_2'(x)\Big|_{x=0.500} = v_3'(x)\Big|_{x=0.500} \qquad v_3'(x)\Big|_{x=0.680} = v_4'(x)\Big|_{x=0.680}$$

Solving the above the system of equations, the constants are:

$$A_1 = \frac{-2.54599}{EI_{zz}} \quad B_1 = 0 \qquad A_2 = \frac{-1.78113}{EI_{zz}} \quad B_2 = \frac{-0.0356934}{EI_{zz}}$$

$$A_3 = \frac{-9.59363}{EI_{zz}} \quad B_3 = \frac{1.26639}{EI_{zz}} \quad A_4 = \frac{-183.771}{EI_{zz}} \quad B_4 = \frac{44.102}{EI_{zz}}$$

Thus the equations for deflection and slope are:

$$0.000 \leq x \leq 0.070 \quad EI_{zz}\, v_1(x) = -2.54599\, x + 55.5\, x^3$$

$$EI_{zz}\, v_1'(x) = -2.54599 + 166.5\, x^2$$

$$0.070 \leq x \leq 0.500 \quad EI_{zz}\, v_2(x) = -0.0356934 - 1.78113\, x + 3.46875\, x^3$$

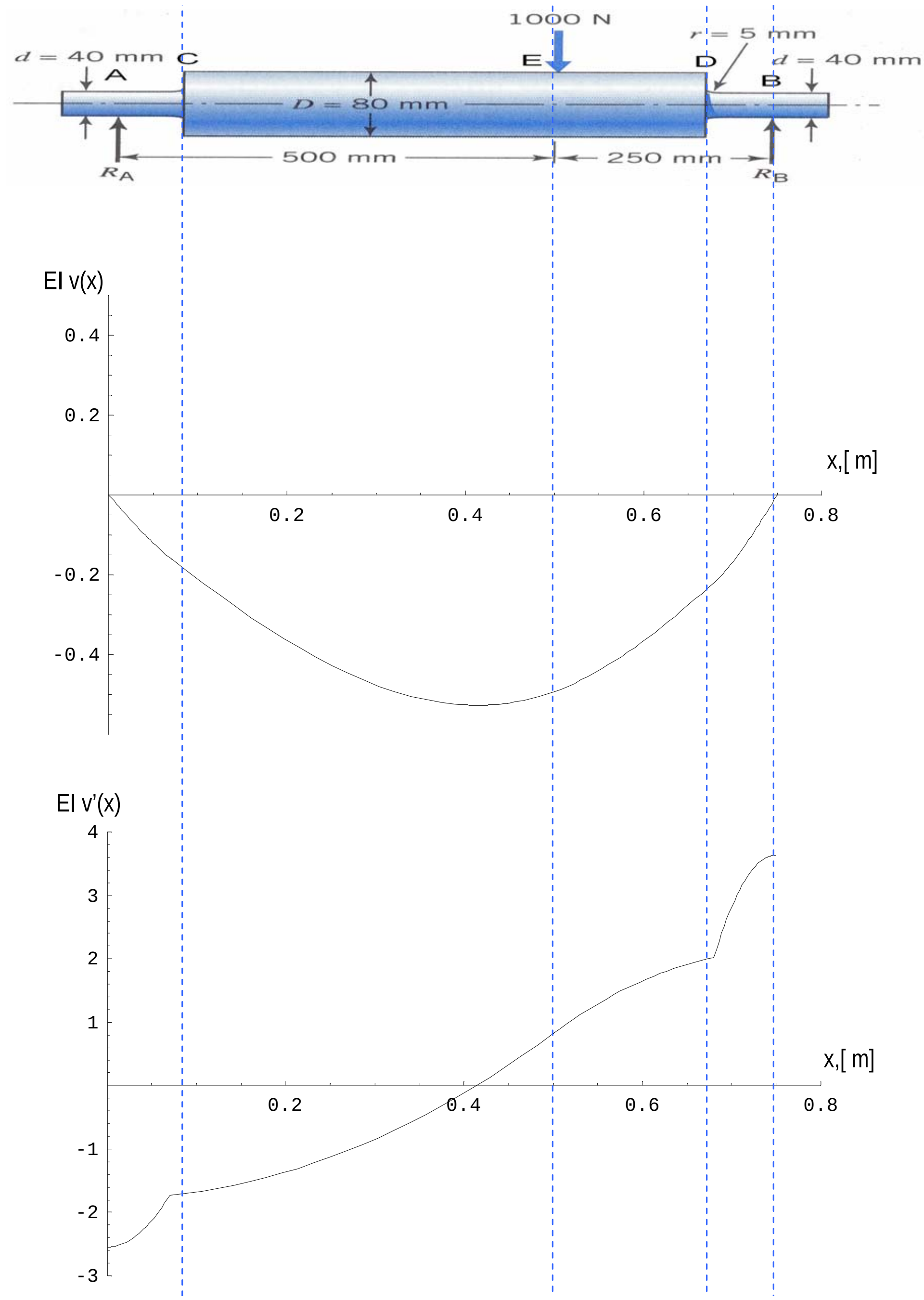
$$EI_{zz}\, v_2'(x) = -1.78113 + 10.4063\, x^2$$

$$0.500 \leq x \leq 0.680 \quad EI_{zz}\, v_3(x) = 1.26639 - 9.59363\, x + 15.625\, x^2 - 6.94792\, x^3$$

$$EI_{zz}\, v_3'(x) = -9.59363 + 31.25\, x - 20.8438\, x^2$$

$$0.680 \leq x \leq 0.750 \quad EI_{zz}\, v_4(x) = 44.102 - 183.771\, x + 250.\, x^2 - 111.167\, x^3$$

$$EI_{zz}\, v_4'(x) = -183.771 + 500.\, x - 333.5\, x^2$$


1000 N
r = 5 mm
d = 40 mm
C
E
D
d = 40 mm
A
B
D = 80 mm
500 mm
250 mm
RA
RB
EI v(x)
x,[m]
EI v'(x)
x,[m]

End Example □

6.2 Castigliano's Theorem

In basic Strength of Material courses, students learn how to find deformations and determine the values of indeterminate reactions. In general, these techniques were based upon geometric considerations. There are, however, many types of problems that can be solved more efficiently through techniques based upon relations between the work done by the external forces and the internal strain energy stored within the body during the deformation process.

Statically indeterminate beams and beams of varying material properties or cross-sections cannot be successfully analyzed by using the methods discussed in previously. Also, when a loading is energy-related, such as an object striking a beam with a given initial velocity, the exact forces in the loadings are not known. For this reason energy methods are often extremely useful.

6.2.1 Internal Strain Energy

When loads are applied to a machine element, the material of the machine element will deform. In the process the external work done by the loads will be converted by the action of either normal or shear stress into internal work called strain energy, provided that no energy is lost in the form of heat. This strain energy is stored in the body. The unit of strain energy is N–m in SI units and lb–in in English units. Strain energy is always a positive scalar quantity even if the stress is compressive because stress and strain are always in the same direction. The symbol U is used to designate strain energy. The strain energy density is expressed with u and is shown in Fig. 6.2.

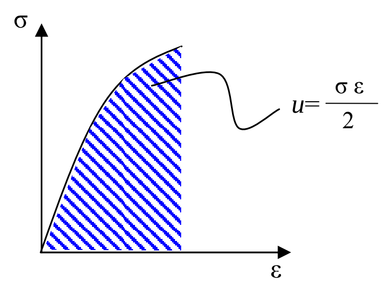


Figure 6.2: Strain energy density.

When an external force acts upon an elastic body and deforms it, the work done by the force is stored within the body in the form of strain energy. In the case of elastic deformation, the total strain energy density due to a general state of stress is

$$u = \frac{1}{2}\underline{\boldsymbol{\sigma}}^{\mathrm{T}}\underline{\boldsymbol{\varepsilon}} = \frac{1}{2}\left\{\sigma_{\mathrm{xx}}\,\varepsilon_{\mathrm{xx}} + \sigma_{\mathrm{yy}}\,\varepsilon_{\mathrm{yy}} + \sigma_{\mathrm{zz}}\,\varepsilon_{\mathrm{zz}} + \tau_{\mathrm{xy}}\,\varepsilon_{\mathrm{xy}} + \tau_{\mathrm{xz}}\,\varepsilon_{\mathrm{xz}} + \tau_{\mathrm{yz}}\,\varepsilon_{\mathrm{yz}}\right\} \tag{6.12}$$

Then total strain energy due to a general state of stress is

$$U = \iiint_{\text{Vol}} \frac{1}{2} \left\{ \sigma_{\text{xx}}\, \varepsilon_{\text{xx}} + \sigma_{\text{yy}}\, \varepsilon_{\text{yy}} + \sigma_{\text{zz}}\, \varepsilon_{\text{zz}} + \tau_{\text{xy}}\, \varepsilon_{\text{xy}} + \tau_{\text{xz}}\, \varepsilon_{\text{xz}} + \tau_{\text{yz}}\, \varepsilon_{\text{yz}} \right\} d\text{Vol} \tag{6.13}$$

For isotropic material, it can be further expressed as:

$$\begin{aligned} U = \iiint_{\text{Vol}} & \left\{ \frac{1}{2\,E} \left(\sigma_{\text{xx}}^2 + \sigma_{\text{yy}}^2 + \sigma_{\text{zz}}^2 \right) - \frac{\nu}{E} \left(\sigma_{\text{xx}}\, \sigma_{\text{yy}} + \sigma_{\text{yy}}\, \sigma_{\text{zz}} + \sigma_{\text{xx}}\, \sigma_{\text{zz}} \right) \right. \\ & \left. + \frac{1}{2\,G} \left(\tau_{\text{xy}}^2 + \tau_{\text{xz}}^2 + \tau_{\text{yz}}^2 \right) \right\} d\text{Vol} \end{aligned} \tag{6.14}$$

or in terms of the principal stresses

$$U = \iiint_{\text{Vol}} \left\{ \frac{1}{2\,E} \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{\nu}{E} \left(\sigma_1\, \sigma_2 + \sigma_2\, \sigma_3 + \sigma_1\, \sigma_3 \right) \right\} d\text{Vol} \tag{6.15}$$

Internal strain energy for an axial load

The only stresses involved in axial loading is: σ_{xx}. Thus

$$U_{\text{axial}} = \iiint_{\text{Vol}} \left\{ \frac{1}{2\,E}\, \sigma_{\text{xx}}^2 \right\} d\text{Vol}$$

For axial loading the stresses are

$$\sigma_{\text{xx}} = \frac{N_{\text{xx}}(x)}{A}$$

and are only in the x-direction. Thus the internal energy becomes

$$\begin{aligned} U_{\text{axial}} &= \int_{\text{x}} \iint_{\text{A}} \left\{ \frac{1}{2\,E} \left(\frac{N_{\text{xx}}(x)}{A} \right)^2 \right\} d\text{A}\, dx = \int_{\text{x}} \left\{ \frac{1}{2\,E} \left(\frac{N_{\text{xx}}^2}{A^2} \right) \right\} \iint_{\text{A}} d\text{A}\, dx \\ &= \int_0^L \left\{ \frac{1}{2\,E} \left(\frac{N_{\text{xx}}^2}{A^2} \right) \right\} A\, dx \end{aligned}$$

Thus

$$U_{\text{axial}} = \int_0^L \left\{ \frac{N_{\text{xx}}^2}{2\,E\,A} \right\} dx \tag{6.16}$$

Internal strain energy for a bending moment

The only stresses involved in bending moment is: σ_{xx}. Thus

$$U_{\text{bending}} = \iiint\limits_{\text{Vol}} \left\{ \frac{1}{2\,E}\,\sigma_{\text{xx}}^2 \right\} d\text{Vol}$$

For axial loading the stresses are

$$\sigma_{\text{xx}} = -\frac{M_{\text{zz}}(x)\,y}{I_{\text{zz}}}$$

and are only in the x-direction. Thus the internal energy becomes

$$U_{\text{bending}} = \int\limits_{\text{x}} \iint\limits_{\text{A}} \left\{ \frac{1}{2\,E} \left(-\frac{M_{\text{zz}}(x)\,y}{I_{\text{zz}}} \right)^2 \right\} d\text{A}\,dx = \int\limits_{\text{x}} \iint\limits_{\text{A}} \left\{ \frac{1}{2\,E} \left(\frac{M_{\text{zz}}^2\,y^2}{I_{\text{zz}}^2} \right) \right\} d\text{A}\,dx$$

$$= \int_0^L \left\{ \frac{1}{2\,E} \left(\frac{M_{\text{zz}}^2}{I_{\text{zz}}^2} \right) \right\} \iint\limits_{\text{A}} y^2 d\text{A}\,dx = \int_0^L \left\{ \frac{1}{2\,E} \left(\frac{M_{\text{zz}}^2}{I_{\text{zz}}^2} \right) \right\} I_{\text{zz}}\,dx$$

Thus

$$U_{\text{bending}} = \int_0^L \left\{ \frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}} \right\} dx \tag{6.17}$$

Internal strain energy for a shear

It can be shown that the strain energy due to transverse shear is

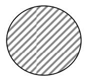
$$U_{\text{shear}} = \int_0^L \left\{ \frac{V_{\text{y}}^2}{2\,k_s\,G\,A} \right\} dx \tag{6.18}$$

where k_s is the shear correction factor and is defined as follows

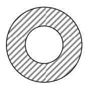
$$\frac{1}{k_s} = \frac{A}{I_{\text{zz}}} \iint\limits_{\text{A}} \frac{Q^2}{b^2}\,d\text{A}$$

This represents a dimensionless quantity specific to a given cross-section geometry:

Solid rectangular: $k_s = \dfrac{5}{6}$

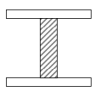


Solid circular: $k_s = \dfrac{9}{10}$



Tubular tube: $k_s = 1$

Thin-walled sections: $k_s = \dfrac{1}{2}$



I-web: $k_s = \dfrac{A_{\text{web}}}{A_{\text{entire cross section}}}$

However, it should be noted that the energy associated with shear is far smaller than the one associated with that of bending moment. Thus it is negligible.

Internal strain energy for a torsional moment

The only stresses involved in torsional moment is: τ_{xy} or τ_{xz}. Thus

$$U_{\text{torsion}} = \iiint\limits_{\text{Vol}} \left\{ \frac{1}{2\,G}\, \tau_{\text{xy}}^2 \right\} d\text{Vol}$$

For axial loading the stresses are

$$\tau_{\text{xy}} = \frac{M_{\text{xx}}(x)\, r}{J_{\text{xx}}}$$

and are only in the x-direction. Thus the internal energy becomes

$$U_{\text{torsion}} = \int\limits_{\text{x}} \iint\limits_{\text{A}} \left\{ \frac{1}{2\,G} \left(\frac{M_{\text{xx}}(x)\, r}{J_{\text{xx}}} \right)^2 \right\} d\text{A}\, dx = \int\limits_{\text{x}} \iint\limits_{\text{A}} \left\{ \frac{1}{2\,G} \left(\frac{M_{\text{xx}}^2\, r^2}{J_{\text{xx}}^2} \right) \right\} d\text{A}\, dx$$

$$= \int_0^L \left\{ \frac{1}{2\,G} \left(M_{\text{xx}}^2 \right) \right\} \iint\limits_{\text{A}} \frac{r^2}{J_{\text{xx}}^2} d\text{A}\, dx = \int_0^L \left\{ \frac{M_{\text{xx}}^2}{2\,G} \right\} \frac{1}{J_{\text{xx}}}\, dx$$

Thus

$$U_{\text{torsion}} = \int_0^L \left\{ \frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}} \right\} dx \tag{6.19}$$

For circular cross-sections $J_{\text{xx}} = J_{\text{xx}}$ and expressions for J_{xx} are found in Tables.

Total internal strain energy for a beam

The total internal strain energy for a beam is the sum of all internal energy contribution due to each loading condition:

$$U = U_{\text{shear}} + U_{\text{torsion}} + U_{\text{axial}} + U_{\text{bending}}$$

$$U = \int_0^L \left\{ \frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}} \right\} dx + \int_0^L \left\{ \frac{V_{\text{y}}^2}{2\,k_s\,G\,A} \right\} dx + \int_0^L \left\{ \frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}} \right\} dx + \int_0^L \left\{ \frac{N_{\text{xx}}^2}{2\,E\,A} \right\} dx \tag{6.20}$$

However, it has been shown that in presence of bending the internal strain energy for axial and shear is insignificant. Thus the total internal strain energy can be written as

$$U = U_{\text{torsion}} + U_{\text{bending}} = \int_0^L \left\{ \frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}} \right\} dx + \int_0^L \left\{ \frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}} \right\} dx \tag{6.21}$$

More generally,

$$U = \int_0^L \left\{ \frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}} \right\} dx + \int_0^L \left\{ \frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}} \right\} dx + \int_0^L \left\{ \frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}} \right\} dx \tag{6.22}$$

6.2.2 Second Castigliano's Theorem

Sign Conventions

Strain energy methods are particularly well suited to problems involving several structural members at various angles to one another. The fact that the members may be curved in their planes presents no additional difficulties. One of the great advantages of strain energy methods is that independent coordinate systems may be established for each member without regard for consistency of positive directions of the various coordinate systems. Also, deflections and the related loads are always taken in the same direction. This advantage is essentially due to the fact that the strain energy is always a positive scalar quantity, and hence algebraic signs of external forces need be consistent only within each structural member.

Definition

This theorem is extremely useful for finding displacements of elastic bodies subject to axial loads, torsion, bending, or any combination of these loadings. The theorem states that the partial derivative of the total internal strain energy with respect to any external applied force yields the displacement under the point of application of that force in the direction of that force. Here, the terms force and displacement are used in their generalized sense and could either indicate a usual force and its linear displacement, or a couple and the corresponding angular displacement. In equation form the displacement under the point of application of the force Q_n is given according to this theorem by

$$\Delta_n = \frac{\partial U}{\partial Q_n}$$

Application to Statically Determinate Problems

In statically determinate problems all external reactions can be found by application of the equations of statics. After this has been done, the deflection under the point of application of any external applied force can be found directly by use of Castigliano's theorem:

$$\begin{aligned}\Delta_n = \frac{\partial U}{\partial Q_n} = &\int_0^L \frac{M_{\text{xx}}}{G\,J_{\text{xx}}}\left(\frac{\partial M_{\text{xx}}}{\partial Q_n}\right)dx + \int_0^L \frac{M_{\text{yy}}}{E\,I_{\text{yy}}}\left(\frac{\partial M_{\text{yy}}}{\partial Q_n}\right)dx \\ &+ \int_0^L \frac{M_{\text{zz}}}{E\,I_{\text{zz}}}\left(\frac{\partial M_{\text{zz}}}{\partial Q_n}\right)dx + \int_0^L \frac{N_{\text{xx}}}{E\,A}\left(\frac{\partial N_{\text{xx}}}{\partial Q_n}\right)dx \\ &+ \int_0^L \frac{V_{\text{y}}}{k_s\,G\,A}\left(\frac{\partial V_{\text{y}}}{\partial Q_n}\right)dx + \int_0^L \frac{V_{\text{z}}}{k_s\,G\,A}\left(\frac{\partial V_{\text{z}}}{\partial Q_n}\right)dx\end{aligned} \tag{6.23}$$

If the deflection is desired at some point where there is no applied force, then it is necessary to introduce an auxiliary (i.e., fictitious) force at that point and, treating that force just as one of the real ones, use Castigliano's theorem to determine the deflection at that point. At the end of the problem the auxiliary force is set equal to zero.

The procedure can be summarized as follows:

1. Apply a fictitious load Q_n at the point and in the direction of the desired deflection
2. Obtain all internal loads acting on the member:

$$N_{\text{xx}}(x, Q_n) \qquad V_{\text{y}}(x, Q_n) \qquad M_{\text{xx}}(x, Q_n) \qquad M_{\text{zz}}(x, Q_n)$$

3. Obtain an expression for the total internal strain energy.
4. Obtain the deflection using Eq. (6.23).
5. Set $Q_n = 0$ and solve the resulting equation:

$$\Delta_n = \left.\frac{\partial U}{\partial Q_n}\right|_{Q_n=0} \tag{6.24}$$

Note that the fictitious load could be a force or a moment and the deflection displacement or rotation, respectively.

Application to Statically Indeterminate Problems

Castigliano's theorem is extremely useful for determining the indeterminate reactions in such problems. This is because the theorem can be applied to each reaction, and the displacement corresponding to each reaction is known beforehand and is usually zero. In this manner it is possible to establish as many equations as there are redundant reactions, and these equations together with those found from statics yield the solution for all reactions. After the values of all reactions have been found, the deflection at any desired point can be found by direct use of Castigliano's theorem.

The procedure can be summarized as follows:

1. Choose the redundant reaction(s).
2. Remove reaction(s) and place assumed load(s) Q_n(s) at the point.
3. Obtain all internal loads acting on the member:

$$N_{\mathrm{xx}}(x, Q_n) \qquad V_{\mathrm{y}}(x, Q_n) \qquad M_{\mathrm{xx}}(x, Q_n) \qquad M_{\mathrm{zz}}(x, Q_n)$$

4. Obtain an expression for the total internal strain energy.
5. Obtain the deflection using Eq. (6.23).
6. Usually the deflection(s) is(are) zero or known, thus solve for the Q_n's.

Note that the fictitious load could be a force or a moment and the deflection displacement or rotation, respectively.

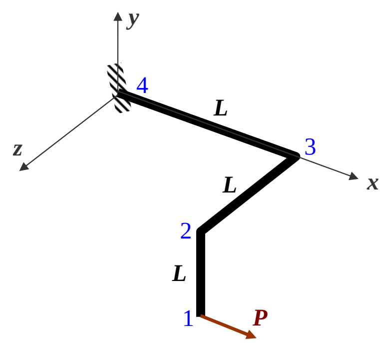
Assumptions and Limitations

Throughout this chapter it is assumed that the material is a linear elastic one obeying Hooke's law. Further, it is necessary that the entire system obey the law of superposition. This implies that certain unusual systems n cannot be treated by the techniques discussed here. Note that Castigliano's Theorem is based on Energy Methods and these methods can only be used for conservative systems. However, the Principle of Virtual Work and/or the Principle of Complementary Virtual Work applies to all types of problems.

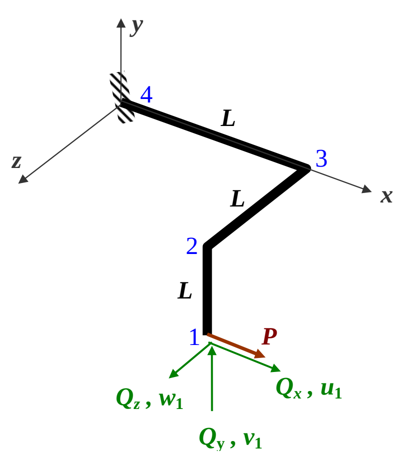
Example 6.4.

Statically Determinate Structures

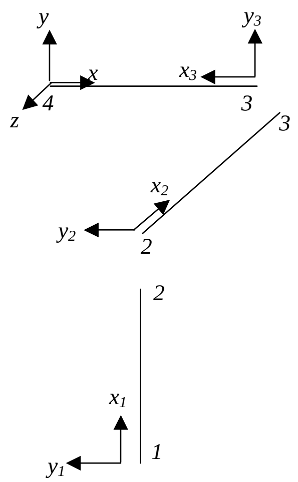
The structure is made of a solid circular steel rod with a uniform cross section of diameter d. The three dimensional frame consists of two right-angle bends, with point 4 built in. A load P in the x-direction acts at point 1. Using the Second Castigliano's Theorem to find the three components of translational displacement (u_1, v_1, w_1) of point 1 in terms of P, L, d, and E. Take a Poisson's ratio of $\nu = 0.25$.

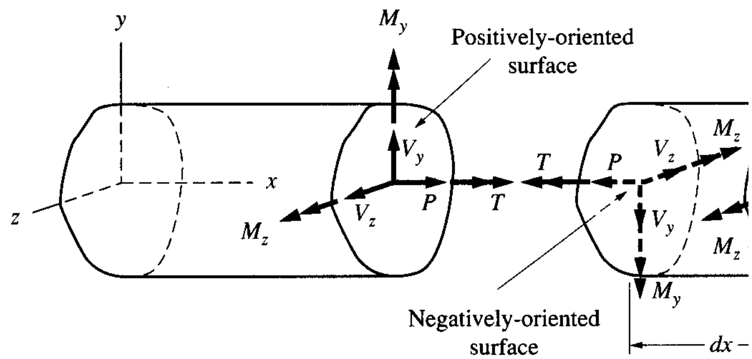


Dummy Loads Place dummy loads at the point where we are interested in calculating displacements:



Internal Loads The idea is to change the coordinate system such that we are consistent with our sign convention. You can use any coordinate system as far as you are consistent throughout the entire problem.





BAR 1-2: $0 \leq x_1 \leq L$

$$V_{\text{y}}(x_1) = P + Q_x \qquad V_{\text{z}}(x_1) = -Q_z \qquad N_{\text{xx}}(x_1) = -Q_y$$

$$M_{\text{yy}}(x_1) = -Q_z\, x_1 \qquad M_{\text{zz}}(x_1) = -Q_x\, x_1 - P\, x_1 \qquad M_{\text{xx}}(x_1) = 0$$

BAR 2-3: $0 \leq x_2 \leq L$

$$V_{\text{y}}(x_2) = P + Q_x \qquad V_{\text{z}}(x_2) = -Q_y \qquad N_{\text{xx}}(x_2) = Q_z$$

$$M_{\text{yy}}(x_2) = -Q_z\, L - Q_y\, x_2 \qquad M_{\text{zz}}(x_2) = -P\, x_2 - Q_x\, x_2 \qquad M_{\text{xx}}(x_2) = P\, L + Q_x\, L$$

BAR 3-4: $0 \leq x_3 \leq L$

$$V_{\text{y}}(x_3) = -Q_y \qquad V_{\text{z}}(x_3) = Q_z$$

$$N_{\text{xx}}(x_3) = P + Q_x \qquad M_{\text{yy}}(x_3) = -P\, L + Q_z\, x_3 - Q_x\, L$$

$$M_{\text{zz}}(x_3) = P\, L + Q_y\, x_3 + Q_x\, L \qquad M_{\text{xx}}(x_3) = -Q_y\, L - Q_z\, L$$

Derivative With Respect To Dummy Loads

BAR 1-2: $0 \leq x_1 \leq L$

$$\frac{\partial V_{\text{y}}(x_1)}{\partial Q_x} = 1 \qquad \frac{\partial V_{\text{y}}(x_1)}{\partial Q_y} = 0 \qquad \frac{\partial V_{\text{y}}(x_1)}{\partial Q_z} = 0$$

$$\frac{\partial V_{\text{z}}(x_1)}{\partial Q_x} = 0 \qquad \frac{\partial V_{\text{z}}(x_1)}{\partial Q_y} = 0 \qquad \frac{\partial V_{\text{z}}(x_1)}{\partial Q_z} = -1$$

$$\frac{\partial V_x(x_1)}{\partial Q_x} = 0 \qquad \frac{\partial V_x(x_1)}{\partial Q_y} = -1 \qquad \frac{\partial V_x(x_1)}{\partial Q_z} = 0$$

$$\frac{\partial M_{\text{yy}}(x_1)}{\partial Q_x} = 0 \qquad \frac{\partial M_{\text{yy}}(x_1)}{\partial Q_y} = 0 \qquad \frac{\partial M_{\text{yy}}(x_1)}{\partial Q_z} = -x_1$$

$$\frac{\partial M_{\text{zz}}(x_1)}{\partial Q_x} = -x_1 \qquad \frac{\partial M_{\text{zz}}(x_1)}{\partial Q_y} = 0 \qquad \frac{\partial M_{\text{zz}}(x_1)}{\partial Q_z} = 0$$

$$\frac{\partial M_{\text{xx}}(x_1)}{\partial Q_x} = 0 \qquad \frac{\partial M_{\text{xx}}(x_1)}{\partial Q_y} = 0 \qquad \frac{\partial M_{\text{xx}}(x_1)}{\partial Q_z} = 0$$

BAR 2-3: $0 \leq x_2 \leq L$

$$\frac{\partial V_{\text{y}}(x_2)}{\partial Q_x} = 1 \qquad \frac{\partial V_{\text{y}}(x_2)}{\partial Q_y} = 0 \qquad \frac{\partial V_{\text{y}}(x_2)}{\partial Q_z} = 0$$

$$\frac{\partial V_{\text{z}}(x_2)}{\partial Q_x} = 0 \qquad \frac{\partial V_{\text{z}}(x_2)}{\partial Q_y} = -1 \qquad \frac{\partial V_{\text{z}}(x_2)}{\partial Q_z} = 0$$

$$\frac{\partial V_x(x_2)}{\partial Q_x} = 0 \qquad \frac{\partial V_x(x_2)}{\partial Q_y} = 0 \qquad \frac{\partial V_x(x_2)}{\partial Q_z} = 1$$

$$\frac{\partial M_{\rm yy}(x_2)}{\partial Q_x}=0 \qquad \frac{\partial M_{\rm yy}(x_2)}{\partial Q_y}=-x_2 \qquad \frac{\partial M_{\rm yy}(x_2)}{\partial Q_z}=-L$$

$$\frac{\partial M_{\rm zz}(x_2)}{\partial Q_x}=-x_2 \qquad \frac{\partial M_{\rm zz}(x_2)}{\partial Q_y}=0 \qquad \frac{\partial M_{\rm zz}(x_2)}{\partial Q_z}=0$$

$$\frac{\partial M_{\rm xx}(x_2)}{\partial Q_x}=L \qquad \frac{\partial M_{\rm xx}(x_2)}{\partial Q_y}=0 \qquad \frac{\partial M_{\rm xx}(x_2)}{\partial Q_z}=0$$

<u>BAR 3-4</u>: $0 \le x_3 \le L$

$$\frac{\partial V_{\rm y}(x_3)}{\partial Q_x}=0 \qquad \frac{\partial V_{\rm y}(x_3)}{\partial Q_y}=-1 \qquad \frac{\partial V_{\rm y}(x_3)}{\partial Q_z}=0$$

$$\frac{\partial V_{\rm z}(x_3)}{\partial Q_x}=0 \qquad \frac{\partial V_{\rm z}(x_3)}{\partial Q_y}=0 \qquad \frac{\partial V_{\rm z}(x_3)}{\partial Q_z}=1$$

$$\frac{\partial V_x(x_3)}{\partial Q_x}=1 \qquad \frac{\partial V_x(x_3)}{\partial Q_y}=0 \qquad \frac{\partial V_x(x_3)}{\partial Q_z}=0$$

$$\frac{\partial M_{\rm yy}(x_3)}{\partial Q_x}=-L \qquad \frac{\partial M_{\rm yy}(x_3)}{\partial Q_y}=0 \qquad \frac{\partial M_{\rm yy}(x_3)}{\partial Q_z}=x_3$$

$$\frac{\partial M_{\rm zz}(x_3)}{\partial Q_x}=L \qquad \frac{\partial M_{\rm zz}(x_3)}{\partial Q_y}=x_3 \qquad \frac{\partial M_{\rm zz}(x_3)}{\partial Q_z}=0$$

$$\frac{\partial M_{\rm xx}(x_3)}{\partial Q_x}=0 \qquad \frac{\partial M_{\rm xx}(x_3)}{\partial Q_y}=-L \qquad \frac{\partial M_{\rm xx}(x_3)}{\partial Q_z}=-L$$

Internal Strain Energy

$$U = \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{1-2}} dx_1$$

$$+ \int_0^L \left\{\frac{N_{\text{xx}}^2}{2\,E\,A}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{V_{\text{z}}^2}{2\,k_s\,G\,A}\right\}_{\text{1-2}} dx_1$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{2-3}} dx_2$$

$$+ \int_0^L \left\{\frac{N_{\text{xx}}^2}{2\,E\,A}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{V_{\text{z}}^2}{2\,k_s\,G\,A}\right\}_{\text{2-3}} dx_2$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{3-4}} dx_3$$

$$+ \int_0^L \left\{\frac{N_{\text{xx}}^2}{2\,E\,A}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{V_{\text{z}}^2}{2\,k_s\,G\,A}\right\}_{\text{3-4}} dx_3$$

Usually the contribution to shear and stretching is small when compared to bending effects. Thus ignoring shear and stretching effects:

$$U \approx \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{1-2}} dx_1$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{2-3}} dx_2$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{3-4}} dx_3$$

Further using the actual loads:

$$U \approx \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{1-2}} dx_1 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{1-2}} dx_1$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{2-3}} dx_2 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{2-3}} dx_2$$

$$+ \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{yy}}^2}{2\,E\,I_{\text{yy}}}\right\}_{\text{3-4}} dx_3 + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\}_{\text{3-4}} dx_3$$

Second Castigliano's Theorem The first equation is obtained for the displacement in the x-direction:

$$u_1 = \frac{\partial U}{\partial Q_x} = \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_x} \right) \right\}_{1\text{-}2} dx_1 + \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_x} \right) \right\}_{1\text{-}2} dx_1$$
$$+ \int_0^L \left\{ \frac{M_{xx}}{G\,J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_x} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_x} \right) \right\}_{2\text{-}3} dx_2$$
$$+ \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_x} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{xx}}{G\,J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_x} \right) \right\}_{3\text{-}4} dx_3$$
$$+ \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_x} \right) \right\}_{3\text{-}4} dx_3 + \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_x} \right) \right\}_{3\text{-}4} dx_3$$

Thus

$$u_1 = \int_0^L \left[\frac{(-P\,x_1 - Q_x\,x_1)(-x_1)}{EI_{zz}} \right]_{1\text{-}2} dx_1 + \int_0^L \left[\frac{(P\,L + Q_x\,L)(L)}{G\,J_{xx}} \right]_{2\text{-}3} dx_2$$
$$+ \int_0^L \left[\frac{(-P\,x_2 - Q_x\,L)(-x_2)}{EI_{zz}} \right]_{2\text{-}3} dx_2 + \int_0^L \left[\frac{(P\,L + Q_y\,x_3 + Q_x\,L)(L)}{EI_{zz}} \right]_{3\text{-}4} dx_3$$
$$+ \int_0^L \left[\frac{(-P\,L + Q_z\,x_3 - Q_x\,L)(-L)}{EI_{yy}} \right]_{3\text{-}4} dx_3$$

The second equation is obtained for the displacement in the y-direction:

$$v_1 = \frac{\partial U}{\partial Q_y} = \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_y} \right) \right\}_{1\text{-}2} dx_1 + \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_y} \right) \right\}_{1\text{-}2} dx_1$$
$$+ \int_0^L \left\{ \frac{M_{xx}}{G\,J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_y} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_y} \right) \right\}_{2\text{-}3} dx_2$$
$$+ \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_y} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{xx}}{G\,J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_y} \right) \right\}_{3\text{-}4} dx_3$$
$$+ \int_0^L \left\{ \frac{M_{yy}}{E\,I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_y} \right) \right\}_{3\text{-}4} dx_3 + \int_0^L \left\{ \frac{M_{zz}}{E\,I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_y} \right) \right\}_{3\text{-}4} dx_3$$

Thus

$$v_1 = \int_0^L \left[\frac{(P\,L + Q_y\,x_3 + Q_x\,L)(x_3)}{EI_{zz}} \right]_{3\text{-}4} dx_3$$

The third equation is obtained for the displacement in the z-direction:

$$w_1 = \frac{\partial U}{\partial Q_z} = \int_0^L \left\{ \frac{M_{zz}}{E\, I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_z} \right) \right\}_{1\text{-}2} dx_1 + \int_0^L \left\{ \frac{M_{yy}}{E\, I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_z} \right) \right\}_{1\text{-}2} dx_1$$

$$+ \int_0^L \left\{ \frac{M_{xx}}{G\, J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_z} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{yy}}{E\, I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_z} \right) \right\}_{2\text{-}3} dx_2$$

$$+ \int_0^L \left\{ \frac{M_{zz}}{E\, I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_z} \right) \right\}_{2\text{-}3} dx_2 + \int_0^L \left\{ \frac{M_{xx}}{G\, J_{xx}} \left(\frac{\partial M_{xx}}{\partial Q_z} \right) \right\}_{3\text{-}4} dx_3$$

$$+ \int_0^L \left\{ \frac{M_{yy}}{E\, I_{yy}} \left(\frac{\partial M_{yy}}{\partial Q_z} \right) \right\}_{3\text{-}4} dx_3 + \int_0^L \left\{ \frac{M_{zz}}{E\, I_{zz}} \left(\frac{\partial M_{zz}}{\partial Q_z} \right) \right\}_{3\text{-}4} dx_3$$

Thus

$$w_1 = \int_0^L \left[\frac{(-P\, L + Q_z\, x_3 - Q_x\, L)(x_3)}{EI_{yy}} \right]_{3\text{-}4} dx_3$$

For a circular cross section:

$$[EI_{zz}]_1 = [EI_{zz}]_2 = [EI_{zz}]_3 = [EI_{zz}]_3 \qquad I_{zz} = I_{zz} = \frac{\pi d^4}{64} \qquad J_{xx} = \frac{\pi d^4}{32}$$

For isotropic material:

$$G = \frac{E}{2(1+\nu)}$$

Therefore,

$$[EI_{zz}]_1 = [EI_{zz}]_2 = [EI_{zz}]_3 = [EI_{zz}]_3 = E\, d^4 \frac{\pi}{64} \qquad GK = Ed^4 \frac{\pi}{64(1+\nu)}$$

Solution For $\nu = 1/4$ the three equations became:

$$u_1 = \frac{\partial U}{\partial Q_x}\Big|_{Q_x=0, Q_y=0, Q_z=0} = \frac{752\, L^3\, P}{3\, d^4\, \pi\, E}$$

$$v_1 = \frac{\partial U}{\partial Q_y}\Big|_{Q_x=0, Q_y=0, Q_z=0} = \frac{32\, L^3\, P}{d^4\, \pi\, E}$$

$$w_1 = \frac{\partial U}{\partial Q_z}\Big|_{Q_x=0, Q_y=0, Q_z=0} = -\frac{32\, L^3\, P}{d^4\, \pi\, E} \qquad \text{(in the positive z-direction)}$$

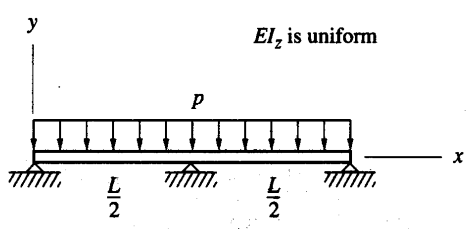
End Example □

Example 6.5.

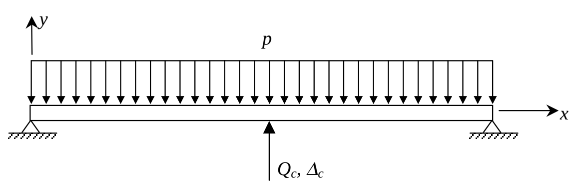
Statically Indeterminate Structures

Suppose the center support moves downward by the amount Δ_c and remains attached to the beam. Using the Second Castigliano's Theorem to determine the reactions at the left and right supports.

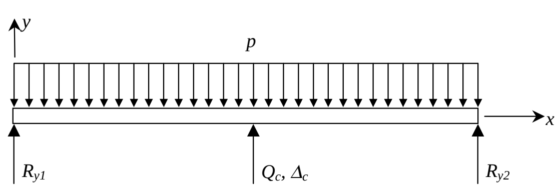
$$\Delta_c = \frac{p\,L^4}{100\,EI_{zz}} \downarrow$$



Redundant Loads Note that it is an indeterminate problem. Therefore, assume that Q_c is known



Internal Loads Calculate the reaction forces:

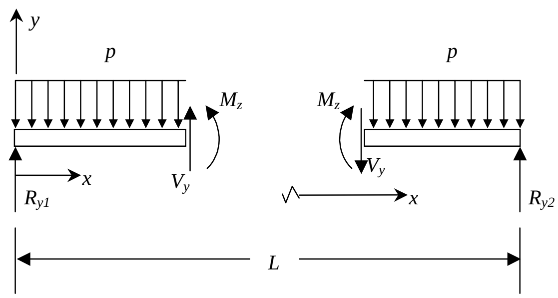


$$\uparrow \sum F_y = 0 = R_{y1} + R_{y2} + Q_c - p\,L \qquad\qquad R_{y1} = \frac{p\,L}{2} - \frac{Q_c}{2}$$

$$\Rightarrow$$

$$\circlearrowleft \sum M_1 = 0 = L\,R_{y2} - \frac{p\,L}{2} + \frac{L}{2}\,Q_c \qquad\qquad R_{y2} = \frac{p\,L}{2} - \frac{Q_c}{2}$$

Internal loads are different for the following two sections of the beam: $0 \le x < \frac{L}{2}$ and $\frac{L}{2} < x < L$



For $0 \le x < \frac{L}{2}$

$$\frac{d\,V_y(x)}{d\,x} - p(x) = 0 \qquad V_y(x)\Big|_{x=0} = -R_{y1} \qquad V_y(x) = p\,x - \frac{1}{2}(p\,L - Q_c)$$

$$\frac{d\,M_{zz}(x)}{d\,x} + V_y(x) = 0 \qquad M_{zz}(x)\Big|_{x=0} = 0 \qquad M_{zz}(x) = -\frac{1}{2}\,p\,x^2 + \frac{1}{2}(p\,L - Q_c)\,x$$

For $\frac{L}{2} < x < L$

$$\frac{d\,V_y(x)}{d\,x} - p(x) = 0 \qquad V_y(x)\Big|_{x=L} = R_{y2} \qquad V_y(x) = p\,x - \frac{1}{2}(p\,L + Q_c)$$

$$\frac{d\,M_{zz}(x)}{d\,x} + V_y(x) = 0 \quad M_{zz}(x)\Big|_{x=L} = 0 \qquad M_{zz}(x) = -\frac{1}{2}\,p\,x^2 + \frac{1}{2}(p\,L + Q_c)\,x - \frac{L}{2}\,Q_c$$

Derivative With Respect To Redundant Loads

Virtual Internal loads are different for the following two sections of the beam: $0 \le x < \frac{L}{2}$ and $\frac{L}{2} < x < L$

For $0 \le x < \dfrac{L}{2}$

$$\frac{\partial V_{\text{y}}(x)}{\partial Q_c} = \frac{1}{2} \qquad \frac{\partial M_{\text{zz}}(x)}{\partial Q_c} = -\frac{1}{2}\,x$$

For $\dfrac{L}{2} < x < L$

$$\frac{\partial V_{\text{y}}(x)}{\partial Q_c} = -\frac{1}{2} \qquad \frac{\partial M_{\text{zz}}(x)}{\partial Q_c} = -\frac{1}{2}\,(L-x)$$

Internal Strain Energy

$$U = \int_0^L \left\{\frac{M_{\text{xx}}^2}{2\,G\,J_{\text{xx}}}\right\} dx + \int_0^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\} dx + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx + \int_0^L \left\{\frac{N_{\text{xx}}^2}{2\,E\,A}\right\} dx$$

$$= \int_0^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\} dx + \int_0^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx$$

$$= \int_0^{L/2} \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\} dx + \int_{L/2}^L \left\{\frac{V_{\text{y}}^2}{2\,k_s\,G\,A}\right\} dx + \int_0^{L/2} \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx + \int_{L/2}^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx$$

$$\approx \int_0^{L/2} \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx + \int_{L/2}^L \left\{\frac{M_{\text{zz}}^2}{2\,E\,I_{\text{zz}}}\right\} dx$$

Second Castigliano's Theorem

$$\Delta_c = \frac{\partial U}{\partial Q_c} = \int_0^{L/2} \frac{M_{\text{zz}}}{E\,I_{\text{zz}}}\left(\frac{\partial M_{\text{zz}}}{\partial Q_c}\right) dx + \int_{L/2}^L \frac{M_{\text{zz}}}{E\,I_{\text{zz}}}\left(\frac{\partial M_{\text{zz}}}{\partial Q_c}\right) dx$$

$$\Delta_c = \int_0^{L/2} \frac{1}{EI_{\text{zz}}}\left[-\frac{1}{2}\,p\,x^2 + \frac{1}{2}(p\,L - Q_c)\,x\right]\left(-\frac{1}{2}\,x\right) dx$$
$$+ \int_{L/2}^L \frac{1}{EI_{\text{zz}}}\left[-\frac{1}{2}\,p\,x^2 + \frac{1}{2}(p\,L + Q_c)\,x - \frac{L}{2}\,Q_c\right]\left(-\frac{1}{2}\,(L-x)\right) dx$$

$$= \frac{1}{EI_{\text{zz}}}\int_0^{L/2}\left[\frac{1}{4}\,p\,x^3 - \frac{1}{4}(p\,L - Q_c)\,x^2\right] dx$$

$$+ \frac{1}{EI_{\text{zz}}}\int_{L/2}^L\left[\frac{1}{4}\,p\,(L-x)^3 - \frac{1}{4}(p\,L - Q_c)\,(L-x)^2\right] dx$$

$$\Delta_c = \frac{-5\,L^4\,p}{384\,EI_{\text{zz}}} + \frac{L^3\,Q_c}{48\,EI_{\text{zz}}}$$

Solution

Reaction at the middle support

$$Q_c = \frac{5\,L\,p}{8} + \frac{48\,\Delta_c\,EI_{zz}}{L^3} = \frac{5\,L\,p}{8} + \frac{48\,EI_{zz}}{L^3}\left(-\frac{p\,L^4}{100\,EI_{zz}}\right) = \frac{29\,p\,L}{200} \uparrow$$

Reaction at the left and right supports:

$$R_{y1} = \frac{1}{2}(p\,L - Q_c) = \frac{1}{2}\left(p\,L - \frac{29\,p\,L}{200}\right) = \frac{171\,p\,L}{400} \uparrow$$

$$R_{y2} = \frac{1}{2}(p\,L - Q_c) = \frac{1}{2}\left(p\,L - \frac{29\,p\,L}{200}\right) = \frac{171\,p\,L}{400} \uparrow$$

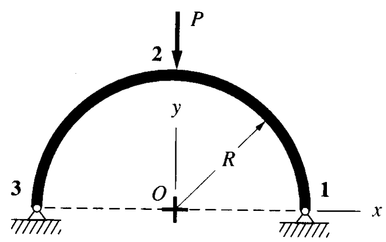
End Example □

Example 6.6.

Indeterminate Arches Structures

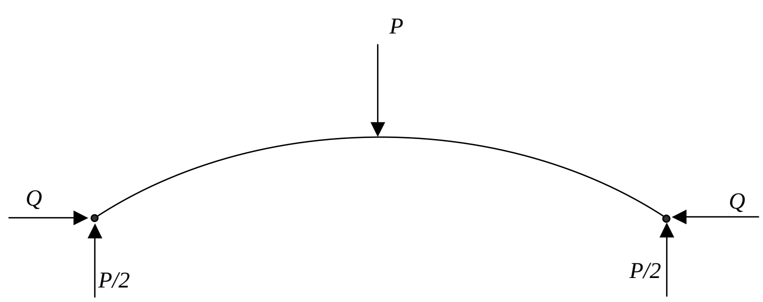
The semicircular frame of radius R is supported by smooth pins at both ends (points 1 and 3). A downward load P is applied to point 2 at the top. Using Castigliano's Second Theorem find the value and location of the maximum bending moment in the frame, in terms of P and R. Assume EI_{zz} is constant:

$$EI_{zz} = EI$$

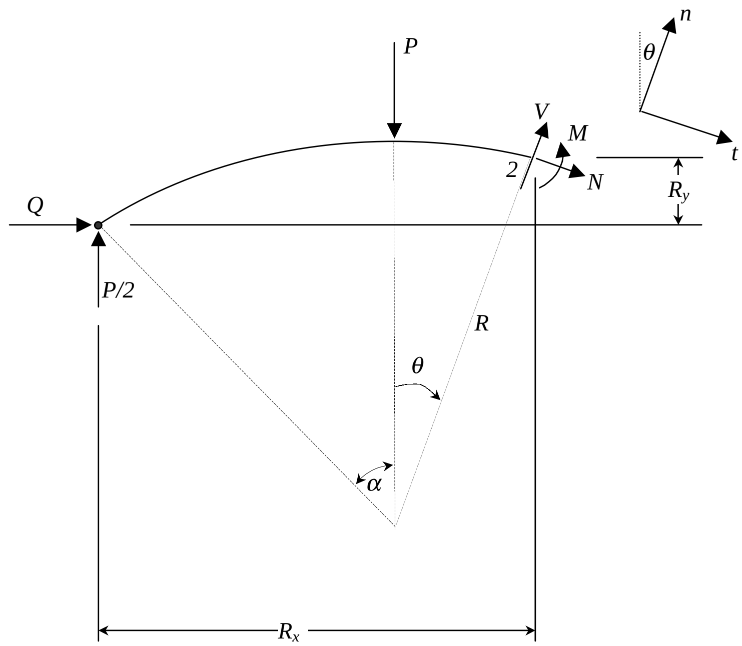


Same as solve in class but $\alpha = 90°$ for this problem.

Redundant Loads



Internal Loads



Now we calculate the internal loads

$$N(\theta) = \begin{cases} \dfrac{P}{2}\sin\theta - Q\cos\theta & : \quad -\alpha \le \theta < 0 \\ -\dfrac{P}{2}\sin\theta - Q\cos\theta & : \quad 0 < \theta \le \alpha \end{cases}$$

$$M(\theta) = \begin{cases} \dfrac{P\,R}{2}(\sin\alpha + \sin\theta) - Q\,R\,(\cos\theta - \cos\alpha) & : \quad -\alpha \le \theta < 0 \\ \dfrac{P\,R}{2}(\sin\alpha - \sin\theta) - Q\,R\,(\cos\theta - \cos\alpha) & : \quad 0 < \theta \le \alpha \end{cases}$$

Derivative With Respect To Redundant Loads

Virtual Internal loads are different for the following two sections of the beam: $-\alpha \le \theta < 0$ and $0 < \theta \le \alpha$

For $-\alpha \le \theta < 0$

$$\frac{\partial N(\theta)}{\partial Q} = -\cos\theta \qquad \frac{\partial M(\theta)}{\partial Q} = -R\,(\cos\theta - \cos\alpha)$$

For $0 < \theta \le \alpha$

$$\frac{\partial N(\theta)}{\partial Q} = -\cos\theta \qquad \frac{\partial M(\theta)}{\partial Q} = -R\,(\cos\theta - \cos\alpha)$$

Internal Strain Energy

$$U = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{M(\theta)^2}{EI}\, R\, d\theta + \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{N(\theta)^2}{EA}\, R\, d\theta$$

Ignoring stretching:

$$U = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{M(\theta)^2}{EI}\, R\, d\theta = \int_{-\alpha}^{0} \frac{M(\theta)^2}{EI}\, R\, d\theta + \int_{0}^{\alpha} \frac{M(\theta)^2}{EI}\, R\, d\theta$$

Second Castigliano's Theorem Ignoring stretching:

$$q = \frac{\partial U}{\partial Q}$$

$$= \int_{-\alpha}^{\alpha} \frac{M(\theta)}{EI}\left(\frac{\partial M}{\partial Q}\right) R\, d\theta = \int_{-\alpha}^{0} \frac{M(\theta)}{EI}\left(\frac{\partial M}{\partial Q}\right) R\, d\theta + \int_{0}^{\alpha} \frac{M(\theta)}{EI}\left(\frac{\partial M}{\partial Q}\right) R\, d\theta$$

$$q = \frac{1}{EI}\int_{-\alpha}^{0}\left[\left(\frac{P\,R}{2}(\sin\alpha + \sin\theta)\right)(-\,R\,(\cos\theta - \cos\alpha))\right] R\, d\theta$$

$$+ \frac{1}{EI}\int_{-\alpha}^{0}\left[(-Q\,R\,(\cos\theta - \cos\alpha))\,(-R\,(\cos\theta - \cos\alpha))\right] R\, d\theta$$

$$+ \frac{1}{EI}\int_{0}^{\alpha}\left[\left(\frac{P\,R}{2}(\sin\alpha - \sin\theta)\right)(-R\,(\cos\theta - \cos\alpha))\right] R\, d\theta$$

$$+ \frac{1}{EI}\int_{0}^{\alpha}\left[(-Q\,R\,(\cos\theta - \cos\alpha))\,(-\,R\,(\cos\theta - \cos\alpha))\right] R\, d\theta$$

$$q = \frac{-\,(P - \pi\, Q)\; R^3}{2\,EI}$$

Solution Since the support is fixed, there will be no displacement:

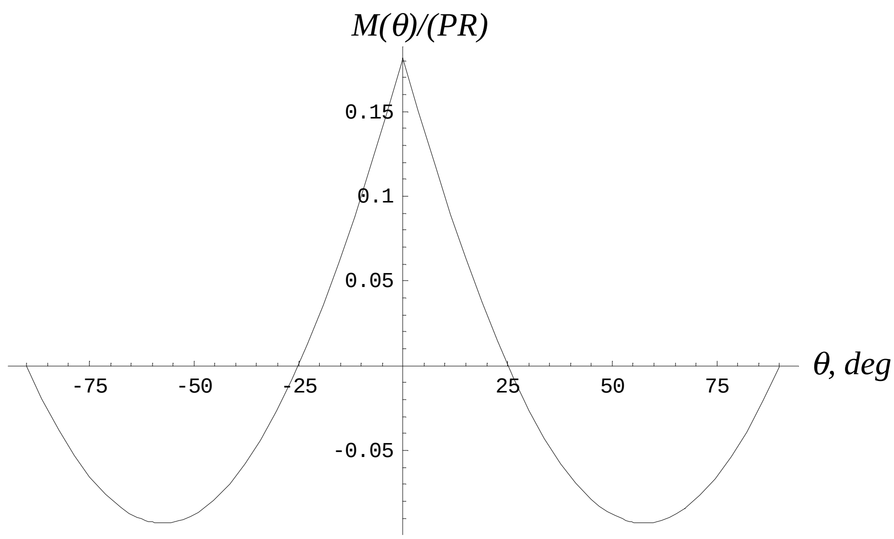
$$q = \frac{\partial U}{\partial Q} = \frac{-\,(P - \pi\, Q)\; R^3}{2\,EI} = 0$$

Thus

$$Q = \frac{P}{\pi}$$

Now we substitute Q into the moment and divide by $P\,R$ to nondimensionalize the moment equation we get

$$\frac{M(\theta)}{P\,R} = \begin{cases} \frac{1}{2} - \frac{\cos\theta}{\pi} + \frac{\sin\theta}{2} & : \quad -\frac{\pi}{2} \le \theta < 0 \\ \frac{1}{2} - \frac{\cos\theta}{\pi} - \frac{\sin\theta}{2} & : \quad 0 < \theta \le \frac{\pi}{2} \end{cases}$$



Therefore, the maximum bending moment occurs at $\theta = 0°$ and has a value of

$$M_{\text{max}} = M(\theta = 0°) = \frac{P\,R}{2} - \frac{P\,R}{\pi} = 0.18169\,P\,R$$

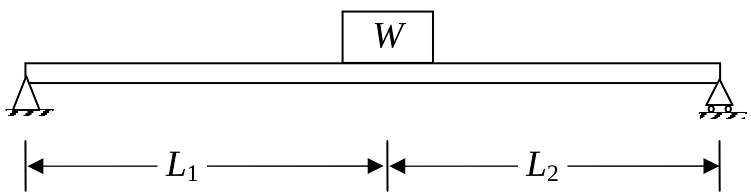
End Example □

6.3 Static deflection

An important concept used in calculating the structural behavior of a system is the static deflection, δ_{st}. This is the deflection of a mechanical system due to gravitational force alone. (The disturbing forces are not considered.) In calculating the static deflection, it is extremely important to distinguish between mass and weight. The static deflection can usually found by using tables, statics, energy methods such Castigliano's Theorem. In general, the static deflection is found by:

$$\delta_{st} = \frac{F}{k}$$

As for an example, consider a beam with a static load placed at the middle ($L_1 = L_2 = L/2$):



The static deflection from tables for a concentrated load at the middle of a beam (neglecting the beam's weight) is

$$\delta_{st} = \frac{W\,L^3}{48\,EI}$$

6.3.1 Effective Stiffness

We can express most structural problems' stiffness in terms of an effective stiffness. The spring rate for members in bending:

$$k = \frac{P}{y_{\text{load}}}$$

For instance the overall spring rate for the beam with a static load placed at the middle ($L_1 = L_2 = L/2$) is

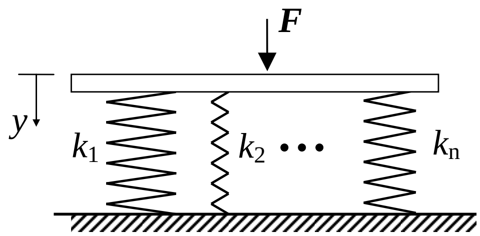
$$k = \frac{W}{\delta_{st}} = \frac{48\,EI}{L^3}$$

The spring rate for torsionally loaded members is:

$$k = \frac{T}{\theta} = \frac{J_{xx}\,G}{L}$$

where J_{xx} depends on the cross-section and is given in tables. The springs may be combined in either series or parallel arrangements.

Springs in Parallel



When springs have a parallel arrangement, the displacements are equal but the total force is split between the springs. Let F be the total force and y the total displacement. Then for a parallel arrangement:

$$F = F_1 + F_2 + \cdots + F_n$$

$$y = y_1 = y_2 = \cdots = y_n$$

Since $F = k\,y$:

$$F = F_1 + F_2 + \cdots + F_n$$

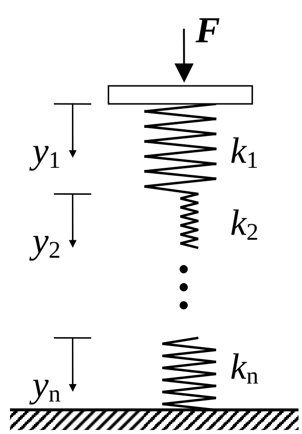
$$k\,y = k_1\,y_1 + k_2\,y_2 + \cdots + k_n\,y_n$$

$$k = k_1 + k_2 + \cdots + k_n$$

Thus when springs are combined in parallel the combined spring rate, the effective stiffness, is the sum of each individual spring rate:

$$k_{\text{eff}} = \sum_{i=1}^{n} k_i$$

Springs in Series



When springs have an arrangement in series, the force is the same on all springs but the displacement is split between the springs. Let F be the total force and y the total displacement. Then for springs in series:

$$F = F_1 = F_2 = \cdots = F_n$$

$$y = y_1 + y_2 + \cdots + y_n$$

Since $F = k\,y$:

$$y = y_1 + y_2 + \cdots + y_n$$

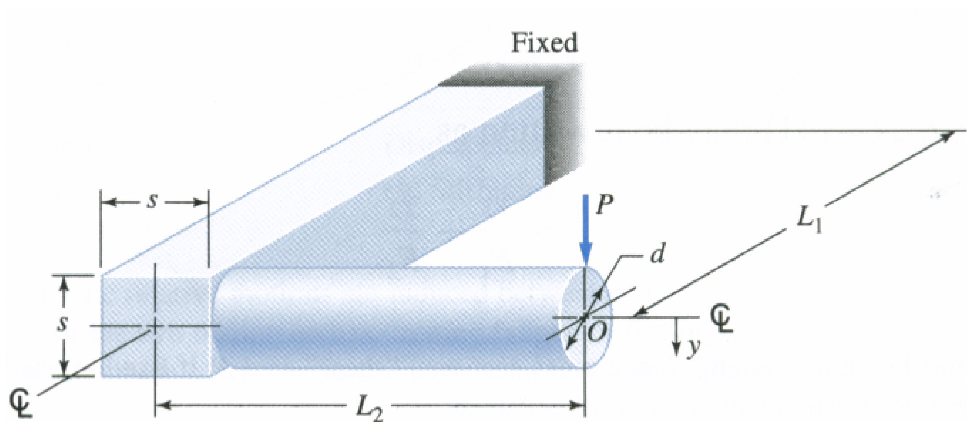
$$\frac{F}{k} = \frac{F_1}{k_1} + \frac{F_2}{k_2} + \cdots + \frac{F_n}{k_n}$$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

Thus when springs are combined in series the combined spring rate, the effective stiffness, is the inverse of the sum of the inverse of each individual spring rate:

$$\frac{1}{k_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{k_i} \quad \rightarrow \quad k_{\text{eff}} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{k_i}}$$

Example 6.7.



The steel right-angle support bracket with bar lengths $L_1 = 10$ inches and $L_2 = 5$ inches, as shown in Figure, is to be used to support the static load $P = 1000$ lb. The load is to be applied vertically at the free end of the cylindrical bar, as shown. Both bracket bar centerlines lie in the same horizontal plane. If the square bar has side $s = 1.25$ inches, and the cylindrical leg has diameter $d = 1.25$ inches, determine the total static deflection.

The total static deflection is defined as:

$$\delta_{\text{st}} = \frac{P}{k_{\text{eff}}}$$

The load P is known and the problem reduces to find the overall spring rate of the system. Note the square bar will be subject to both torsional and bending deflections, while the cylindrical bar is subject to bending only. This can be modeled as spring in series. Thus

$$k_{\text{eff}} = \frac{1}{\displaystyle\sum_{i=1}^{3} \frac{1}{k_i}} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2} + \dfrac{1}{k_3}}$$

where k_1 is the spring rate caused by bending of the square bar, k_2 the spring rate caused by torsion through of the square bar reflected to point **O** through rigid body rotation of cylinder bar length L_2, and k_3 is the spring rate caused by bending of the cylindrical.

For the bending of the squared cross-section,

$$k_1 = \frac{P}{y_1}$$

Using tables,

$$y_1 = \frac{P\,L_1^3}{3\,EI} \to k_1 = \frac{P}{y_1} = \frac{3\,EI}{L_1^3}$$

For a squared cross-section:

$$I = \frac{s^4}{12}$$

Thus

$$k_1 = \frac{E\,s^4}{4\,L_1^3}$$

Next, for the torsion of the square cross-section,

$$k_2 = \frac{P}{y_2}$$

where $y_2 = L_2\,\theta$. The total rotation angle is calculated as

$$\theta = \frac{P\,L_2\,L_1}{J_{xx}\,G}$$

Using this information:

$$k_2 = \frac{P}{y_2} = \frac{P}{L_2\,\theta} = \frac{P}{L_2\left(\dfrac{P\,L_2\,L_1}{J_{xx}\,G}\right)} = \frac{J_{xx}\,G}{L_1\,L_2^2}$$

Using tables for a squared cross-section:

$$J_{xx} = 2.25\left(\frac{s}{2}\right)^4 = 0.14\,s^4$$

Thus

$$k_2 = \frac{0.14\,s^4\,G}{L_1\,L_2^2}$$

For the bending of the circular cross-section,

$$k_3 = \frac{P}{y_3}$$

Using tables,

$$y_3 = \frac{P\,L_2^3}{3\,EI} \to k_3 = \frac{P}{y_3} = \frac{3\,EI}{L_3^3}$$

For a circular cross-section:

$$I = \frac{\pi d^4}{64}$$

Thus

$$k_3 = \frac{3\,\pi\,E\,d^4}{64\,L_2^3}$$

Thus, the overall spring rate is

$$k_{\text{eff}} = \cfrac{1}{\cfrac{1}{\cfrac{E\, s^4}{4\, L_1^3}} + \cfrac{1}{\cfrac{0.14\, s^4\, G}{L_1\, L_2^2}} + \cfrac{1}{\cfrac{3\,\pi\, E\, d^4}{64\, L_2^3}}}$$

$$= \frac{E\, s^4}{L_1^3} \left(\frac{1}{4 + 0.14 \left(\frac{E}{G}\right) \left(\frac{L_2}{L_1}\right)^2 + \frac{64}{3\,\pi} \left(\frac{L_2}{L_1}\right)^3 \left(\frac{s}{d}\right)^4} \right)$$

Using tables,

$$E = 30 \times 10^6 \quad \text{psi} \qquad G = 11.5 \times 10^6 \quad \text{psi}$$

$$k_{\text{eff}} = 7.70 \times 10^3 \quad \frac{\text{lb}}{\text{in}}$$

The total static deflection for the given structure is

$$\delta_{\text{st}} = \frac{P}{k_{\text{eff}}} = 0.13 \quad \text{in} = 0.010833 \quad \text{ft}$$

End Example □

6.4 Elastic Stability and Instability

When a structure is subjected to loading it can fail because local stresses exceed the maximum allowable stress for the material. There exist, however, another type of failure mode where the entire structure suddenly collapses. The critical value of the applied load that triggers this failure mode primarily depends on the geometry of the structure and the stiffness of the material, not its strength. The study of this catastrophic failure mode is known as the theory of elastic stability.

The main objective in stability analysis is to determine whether a system that is perturbed from an equilibrium state will return to that equilibrium steady state. If this is true for small perturbations from equilibrium, then we say that this equilibrium is stable. If a system always returns to that equilibrium, then we say it is globally stable.

In order to understand why columns buckle it is necessary to understand the concept of stability. We are interested to study the stability of the equilibrium state.

6.4.1 Buckling

When a structure (subjected usually to compression) undergoes visibly large displacements transverse to the load then it is said to buckle. Buckling may be demonstrated by pressing the opposite edges of a flat sheet of cardboard towards one another. For small loads the process is elastic since buckling displacements disappear when the load is removed.

Now consider increasing the load slowly. We are interested in the value of the load, called the critical load, at which buckling occurs. That is, we are interested in when a a sequence of equilibrium stable states as a function of the load, one state for each value of the load, ceases to be stable.

Buckling of a structure means

1. Failure due to excessive displacements (loss of structural stiffness), and/or

2. Loss of stability of an equilibrium configuration of the structure.

6.4.2 Definition of Buckling Load

The buckling load is the load at which the current equilibrium state of a structural element or structure suddenly changes from stable to unstable, and is, simultaneously, the load at which the equilibrium state suddenly changes from that previously stable configuration to another stable configuration with or without an accompanying large response. Thus, the buckling load is the largest load for which stability of equilibrium of a structural element or structure exists in its original (or previous) equilibrium configuration.

6.4.3 Stability of equilibrium

Stability of equilibrium means that the response of the structure due to a small disturbance from its equilibrium configuration remains small; the smaller the disturbance the smaller the resulting magnitude of the displacement in the response. If a small disturbance causes large displacement, perhaps even theoretically infinite, then the equilibrium state is unstable. Practical structures are stable at no load. Now consider increasing the load slowly. We are interested in the value of the load, called the critical load, at which buckling occurs. That is, we are interested in when a a sequence of equilibrium stable states as a function of the load, one state for each value of the load, ceases to be stable.

In general, modern aircraft structures consist of thin sheets attached to slender stiffeners. Thus, buckling of these lightweight members can occur at stresses well below the elastic limit. If buckling occurs before the elastic limit of the material, which is roughly the yield stress of the material, then it is called elastic buckling. If buckling occurs beyond the elastic limit, it is called inelastic buckling, or plastic buckling if the material exhibits plasticity during buckling (mainly metals). Most thin-walled structural components buckle in compression below the elastic limit. Therefore, buckling determines the limit state in compression rather than material yielding. In fact, about 50% of an airplane structure is designed based on buckling constraints.

In short, prediction of resistance of compression and shear members to elastic buckling is very important. Note that buckling is a type of failure as is yielding and fracture.

6.4.4 Various Equilibrium Configurations

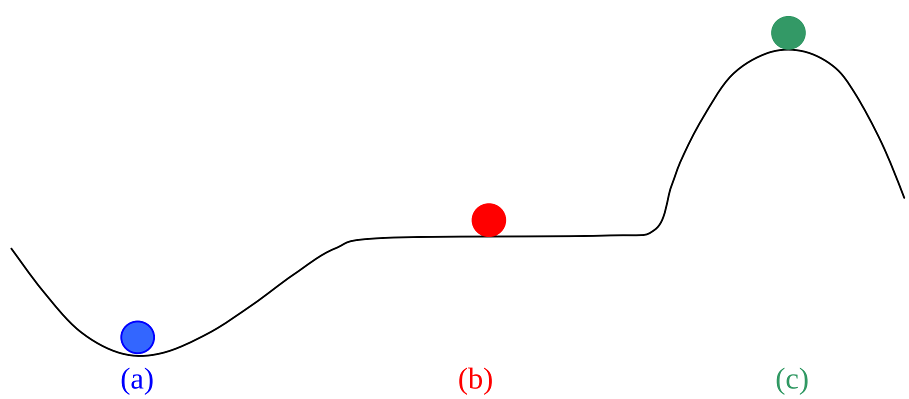


Figure 6.3: Equilibrium states.

To illustrate the concept of stability of the equilibrium configuration, let us consider three cases as shown in Fig. 6.3.

(a) Stable equilibrium: An equilibrium state or configuration of a structural element, structure, or mechanical system is stable if every small disturbance of the system results only in a small response after which the structure always returns to its original equilibrium state. The simplest example of a stable mechanical system is a rigid blue ball in a valley as in Figure. The ball can be disturbed slightly such as by tapping on it, but the ball always returns to the bottom of the valley. Thus, the ball is in a state of stable equilibrium at the bottom of the valley. *The stable equilibrium state is stable with respect to both displacement and/or velocity disturbance.*

(b) Neutral equilibrium: After the rigid red ball has been slightly disturbed from the equilibrium position, it is still in equilibrium at the displaced position, and there is no tendency either to return to the previous position or to move to some other position. Equilibrium is always satisfied. *The neutral equilibrium state is stable with respect to displacement but unstable with respect to velocity disturbance.*

(c) Unstable equilibrium: An equilibrium state or configuration of a structural element, structure, or mechanical system is unstable if any small disturbance of the system results in a sudden change in deformation mode after which the system does not return to its original equilibrium state. The simplest example of an unstable mechanical system is a rigid green ball precariously perched on the top of a hill as in Figure. If the ball is disturbed slightly (an infinitesimal disturbance suffices), the ball will immediately roll down the hill and will never return to the top of the hill. Thus, the ball is in a state of unstable equilibrium at the top of the hill. *The unstable equilibrium state is unstable with respect to both displacement and/or velocity disturbance.*

All the above represent equilibrium paths (states). Two important concepts exist to study the stability of an equilibrium state:

1. System must be in equilibrium.
2. We study the stability of that equilibrium state by given the system a small disturbance.

6.4.5 Methods of stability analysis

Three methods of stability analysis for an equilibrium state exist:

1. ***Dynamic method.*** *What is the value of the load for which the most general free motion of the perfect system in the vicinity of the equilibrium position ceases to be bounded?*

2. ***Adjacent equilibrium method.*** *What is the value of the load for which the perfect system admits nontrivial equilibrium configurations?*

3. ***Energy method.*** *What is the value of the load for which the potential energy ceases to be positive definite?*

Only the dynamic method can be used to study the stability of the perfect system in the vicinity of the equilibrium position of both conservative and non-conservative systems.

6.4.6 Stability of Perfect Beam-Columns (Adjacent Equilibrium Method)

Consider the equilibrium of the perfect column: The column is straight and subject to centric static compressive force P. The beam column obeys the Euler-Bernoulli Theory, is homogeneous, isotropic, and of uniform cross-section. Hence, at equilibrium:

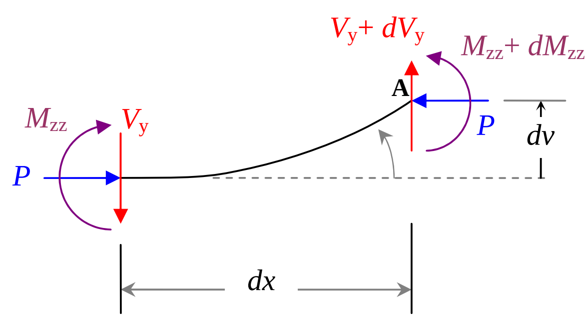
$$u_0(x) = -\frac{P}{EA}\,x, \qquad v_0(x) = 0, \qquad x \in (0, L)$$

Now let us study the stability of the equilibrium state by the method of adjacent equilibrium. Consider infinitesimal variation to the displacements represented by subscript "1" (e.g., $v_1(x) = \delta v(x)$). If the column is fixed at $x = 0$

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) \quad \Rightarrow \quad u(0) = 0 \quad \Rightarrow \quad u_1(x) = 0 \\ u(x) &= u_0(x) \quad \forall\, x \in (0, L) \end{aligned}$$

$$\begin{aligned} v(x) &= v_0(x) + v_1(x) \qquad v_0(x) = 0 \\ v(x) &= v_1(x) \quad \forall\, x \in (0, L) \end{aligned}$$

The equilibrium of an element at a distance x is:



Sum of force in the y direction gives

$$^+\!\uparrow \sum F_y = 0, \quad \rightarrow \quad -V_{y_1} + (V_{y_1} + dV_{y_1}) = 0$$

Now divide by dx and let $dx \rightarrow \infty$,

$$\frac{dV_{y_1}}{dx} = 0 \tag{6.25}$$

Take moments at **A**

$$^+ \circlearrowleft \sum M_A = 0, \quad \rightarrow \quad V_{y_1}\,dx - M_{zz_1} + M_{zz_1} + dM_{zz_1} + P\,dv_1 = 0$$

Now divide by dx and let $dx \rightarrow \infty$,

$$\frac{dM_{zz_1}}{dx} + V_{y_1} + \frac{dv_1}{dx}\,P = 0 \tag{6.26}$$

Differentiate Eq. (6.26) with respect to x once

$$\frac{d^2 M_{zz_1}}{dx^2} + \underbrace{\frac{dV_{y_1}}{dx}}_{=0} + \frac{d^2 v_1}{dx^2} P = 0$$

Now use Eq. (6.25) to get

$$\frac{d^2 M_{zz_1}}{dx^2} + \frac{d^2 v_1}{dx^2} P = 0 \tag{6.27}$$

The material law is

$$M_{zz_1} = -EI_{zz} \frac{d\varphi_{z_1}}{dx} = -EI_{zz} \frac{d(-v_1')}{dx} = EI_{zz} v_1'' \tag{6.28}$$

Substituting Eq. (6.28) into Eq. (6.27) we get the governing ordinary differential equation for buckling:

$$\frac{d^2}{dx^2}\left(EI_{zz} \frac{d^2 v_1}{dx^2}\right) + P \frac{d^2 v_1}{dx^2} = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L) \tag{6.29}$$

For a column with EI_{zz} = constant, we can re-write this differential equation as

$$EI_{zz} v_1'''' + P v_1'' = 0 \qquad \Rightarrow \qquad v_1'''' + \underbrace{\frac{P}{EI_{zz}}}_{\lambda^2} v_1'' = 0$$

Hence the homogeneous forth order ordinary differential equation to obtain the buckling load is

$$v_1'''' + \lambda^2 v_1'' = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L), \qquad \lambda^2 = \frac{P}{EI_{zz}} \tag{6.30}$$

General solution for $\lambda^2 > 0$

$$v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4 \tag{6.31}$$

and the buckling load is obtained from Eq. (6.30):

$$\lambda^2 = \frac{P}{EI_{zz}} \qquad \to \qquad P = \lambda^2 EI_{zz}$$

The constants are found by applying the boundary conditions. Before we proceed, the following will be needed

$$\begin{aligned} v_1'(x) &= A_1 \lambda \cos(\lambda x) - A_2 \lambda \sin(\lambda x) + A_3 \\ v_1''(x) &= -A_1 \lambda^2 \sin(\lambda x) - A_2 \lambda^2 \cos(\lambda x) \\ v_1'''(x) &= -A_1 \lambda^3 \cos(\lambda x) + A_2 \lambda^3 \sin(\lambda x) \end{aligned}$$

From the Hooke's law Eq. (6.28)

$$M_{zz_1} = EI_{zz} v_1''(x) = -EI_{zz} [A_1 \lambda^2 \sin(\lambda x) + A_2 \lambda^2 \cos(\lambda x)]$$

From equilibrium Eq. (6.26)

$$V_{y_1} = -M_{zz_1}' - P v_1' = -EI_{zz} [v_1''' + \lambda^2 v_1'] = -EI_{zz} [A_3 \lambda^2]$$

Boundary conditions depend from problem to problem. However, there are three standard boundary condition evaluated at the boundary:

PINNED:	$v_1 = 0$	and	$M_{zz_1} = 0$
FREE:	$M_{zz_1} = 0$	and	$V_{y_1} = 0$
FIXED:	$v_1 = 0$	and	$v_1' = 0$

Example 6.8.

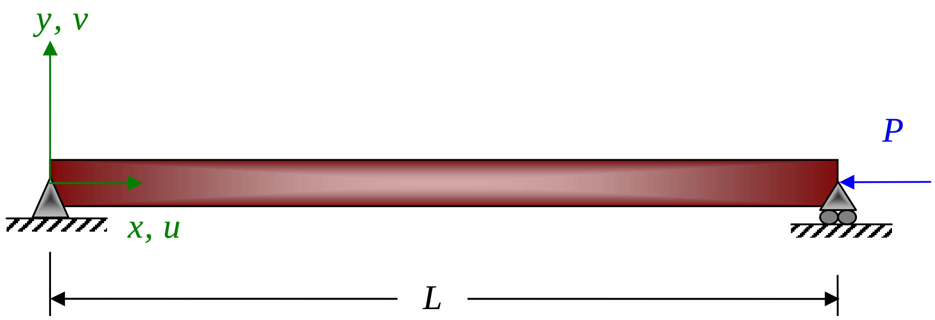


Figure 6.4: A simply-supported beam column subject to an axial load.

The uniform column with bending stiffness EI_{zz}, shown in Fig. 6.4, is pinned at $x = 0$ and pinned at $x = L$. Determine the critical load P_{cr} and the associated buckling mode shape.

(6.8-a) Perturb the system from its equilibrium state.

This leads to the ordinary differential equation given by Eq. (6.30):

$$v_1'''' + \lambda^2 v_1'' = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L), \qquad \lambda^2 = \frac{P}{EI_{zz}}$$

The general solution for $\lambda^2 > 0$ is

$$v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4$$

(6.8-b) Now apply the boundary conditions:

Pinned at $x = 0$: $v_1(0) = 0$ and $M_{zz_1}(0) = EI_{zz} v_1''(0) = 0$

Pinned at $x = L$: $v_1(L) = 0$ and $M_{zz_1}(L) = EI_{zz} v_1''(L) = 0$

Thus

$$v_1(0) = A_2 + A_4 = 0$$

$$v_1''(0) = -A_2\,\lambda^2 = 0$$

$$v_1(L) = A_1\,\sin(\lambda\,L) + A_2\,\cos(\lambda\,L) + A_3\,L + A_4 = 0$$

$$v_1''(L) = -A_1\,\lambda^2\,\sin(\lambda\,L) - A_2\,\lambda^2\,\cos(\lambda\,L) = 0$$

Writing the boundary conditions in a matrix form in terms of the unknown coefficients A_1, A_2, A_3, A_4

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & 0 \\ -\lambda^2\,\sin(\lambda\,L) & -\lambda^2\,\cos(\lambda\,L) & 0 & 0 \\ \sin(\lambda\,L) & \cos(\lambda\,L) & L & 1 \end{bmatrix}}_{\underline{\mathbf{C}}} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.32}$$

(6.8-c) Obtain the characteristic equation.

Non-trivial solution ($\underline{\mathbf{A}} \neq \underline{\mathbf{0}}$) requires det[$\underline{\mathbf{C}}$] =0. The determinant will lead to the characteristic equation:

$$\frac{1}{L^4}\,(\lambda\,L)^4\,\sin(\lambda\,L) = 0 \tag{6.33}$$

(6.8-d) Obtain the buckling load.

We proceed to solve the characteristic equation for λ.

$$\lambda = 0 \quad \Rightarrow \quad P = 0 \quad \text{leads to trivial solution}$$

$$\sin(\lambda\,L) = 0 \quad \Rightarrow \quad \lambda\,L = m\,\pi$$

$$\lambda_m = \frac{m\,\pi}{L} \quad \Rightarrow \quad P_m = \left[\frac{m\,\pi}{L}\right]^2 EI_{zz}$$

The loads is obtained from:

$$P_m = \frac{m^2\,\pi^2\,EI_{zz}}{L^2}$$

Thus critical load is for a simply-supported beam is

$$P_{cr} = P_1 = \frac{\pi^2\,EI_{zz}}{L^2}$$

(6.8-e) Obtain the buckling mode shape.

The buckling mode shapes associated with P_{cr}. Plug-in the value for λ_m in Eq. (6.32) to determine the coefficients. Recall that for our problem $\sin(\lambda_m L) = 0$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda_m^2 & 0 & 0 \\ 0 & -\lambda_m^2 & 0 & 0 \\ 0 & 1 & L & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.34}$$

Therefore the coefficient are

$$\text{row } 2 : A_2 = 0 \qquad \text{row } 1 : A_4 = 0 \qquad \text{row } 4 : A_3 = 0$$

Note that A_1 remains indeterminate. Hence the buckling mode shape associated with P_{cr} is

$$\frac{v_1(x)}{A_1} = \sin\left(\frac{\pi x}{L}\right)$$

End Example □

Example 6.9.

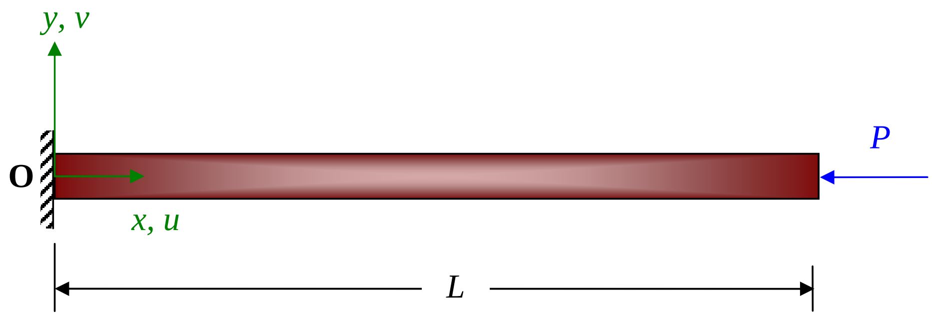


Figure 6.5: Cantilevered beam column subject to an axial load.

The uniform column with bending stiffness EI_{zz}, shown in Fig. 6.5, is clamped at $x = 0$ and free at $x = L$. Determine the critical load P_{cr} and the associated buckling mode shape.

(6.9-a) Perturb the system from its equilibrium state.

This leads to the ordinary differential equation given by Eq. (6.30):

$$v_1'''' + \lambda^2 v_1'' = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L), \qquad \lambda^2 = \frac{P}{EI_{zz}}$$

The general solution for $\lambda^2 > 0$ is

$$v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4$$

(6.9-b) Now apply the boundary conditions:

Clamped at $x = 0$: $v_1(0) = 0$ and $v_1'(0) = 0$

Free at $x = L$: $M_{zz_1}(L) = 0 \Rightarrow v_1''(L) = 0$ and $V_{y_1}(L) = 0$

Thus

$$v_1(0) = A_2 + A_4 = 0$$

$$v_1'(0) = A_1\,\lambda + A_3 = 0$$

$$v_1''(L) = -A_1\,\lambda^2\,\sin(\lambda\,L) - A_2\,\lambda^2\,\cos(\lambda\,L) = 0$$

$$V_{y_1}(L) = A_3\,\lambda^2 = 0$$

Writing the boundary conditions in a matrix form in terms of the unknown coefficients A_1, A_2, A_3, A_4

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & 1 & 0 \\ -\lambda^2\,\sin(\lambda\,L) & -\lambda^2\,\cos(\lambda\,L) & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \end{bmatrix}}_{\underline{\mathbf{C}}} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.35}$$

(6.9-c) Obtain the characteristic equation.

Non-trivial solution ($\underline{\mathbf{A}} \neq \underline{\mathbf{0}}$) requires det[$\underline{\mathbf{C}}$] =0. The determinant will lead to the characteristic equation:

$$\frac{1}{L^5}\,(\lambda\,L)^5\,\cos(\lambda\,L) = 0 \tag{6.36}$$

(6.9-d) Obtain the buckling load.

We proceed to solve the characteristic equation for λ.

$$\lambda = 0 \qquad \Rightarrow \qquad P = 0 \qquad \text{leads to trivial solution}$$

$$\cos(\lambda\,L) = 0 \qquad \Rightarrow \qquad \lambda\,L = (2\,m-1)\frac{\pi}{2}$$

$$\lambda_m = (2\,m-1)\frac{\pi}{2\,L} \qquad \Rightarrow \qquad P_m = \left[(2\,m-1)\frac{\pi}{2\,L}\right]^2 EI_{zz}$$

The loads is obtained from:

$$P_m = \frac{(2\,m-1)^2\,\pi^2\,EI_{zz}}{4\,L^2}$$

Thus critical load is for a cantilevered beam is

$$P_{cr} = P_1 = \frac{\pi^2\,EI_{zz}}{4\,L^2}$$

(6.9-e) Obtain the buckling mode shape.

The buckling mode shapes associated with P_{cr}. Plug-in the value for λ_m in Eq. (6.35) to determine the coefficients. Recall that for our problem $\cos(\lambda_m L) = 0$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda_m & 0 & 1 & 0 \\ -\lambda_m^2 \sin(\lambda_m L) & 0 & 0 & 0 \\ 0 & 0 & \lambda_m^2 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.37}$$

Therefore the coefficient are

$$\text{row } 3: A_1 = 0 \qquad \text{row } 2: A_3 = 0 \qquad \text{row } 1: A_2 = -A_4$$

Now the mode shapes are obtained by substituting these values into

$$v_1(x) = A_1 \sin(\lambda_m x) + A_2 \cos(\lambda_m x) + A_3 x + A_4$$

$$v_1(x) = -A_4 \cos(\lambda_m x) + A_4$$

$$v_1(x) = A_4 \left[1 - \cos\left(\frac{(2m-1)\pi x}{2L} \right) \right]$$

Note that A_4 remains indeterminate. Hence the buckling mode shape associated with P_{cr} is

$$\frac{v_1(x)}{A_4} = 1 - \cos\left(\frac{\pi x}{2L} \right)$$

End Example □

Example 6.10.

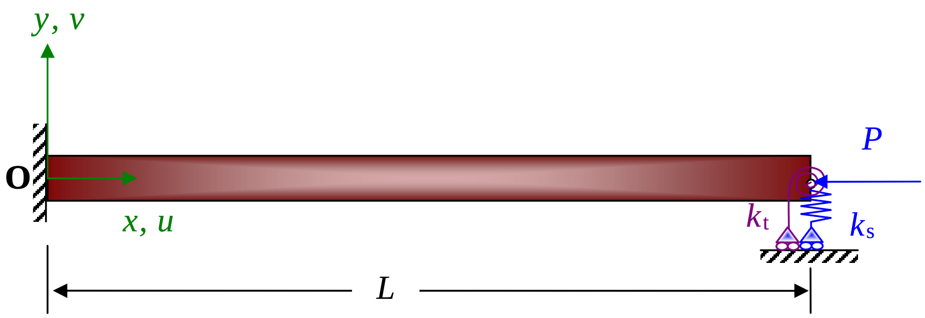


Figure 6.6: A clamped-spring supported beam column subject to an axial load.

The uniform column with bending stiffness EI_{zz}, shown in Fig. 6.6, is clamped at $x = 0$ and pinned to extensional and torsional springs at $x = L$. The linear extensional spring has a stiffness k_s and is unstretched when $v_1(L) = 0$. The linear torsional spring has a stiffness k_t and is unstretched when $v_1'(L) = 0$. Take:

$$k_s = \alpha \frac{EI_{zz}}{L^3}, \qquad k_t = \beta \frac{EI_{zz}}{L}$$

Determine the critical load P_{cr} and the associated buckling mode shape for each of the following cases:

1. $\alpha \to \infty$, $\beta \to \infty$
2. $\alpha \to 0$, $\beta \to 0$
3. $\alpha \to 1$, $\beta \to 1$

(6.10-a) Perturb the system from its equilibrium state.

This leads to the ordinary differential equation given by Eq. (6.30):

$$v_1'''' + \lambda^2 v_1'' = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L), \qquad \lambda^2 = \frac{P}{EI_{zz}}$$

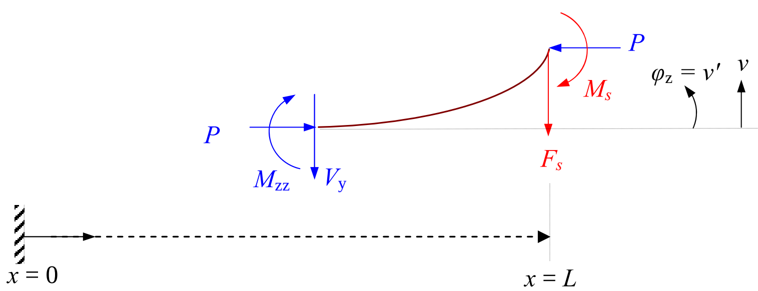
The general solution for $\lambda^2 > 0$ is

$$v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4$$

(6.10-b) Now apply the boundary conditions:

Clamped at $x = 0$: $v_1(0) = 0$ and $v_1'(0) = 0$

If Free at $x = L$: $M_{zz_1}(L) = 0 \ \Rightarrow v_1''(L) = 0$ and $V_{y_1}(L) = 0$. However the springs change the boundary conditions. Replacing the springs by their loads and drawing the free body diagram:



Pinned with a linear linear spring at $x = L$

$$\left.(V_{y_1} + F_s)\right|_{x=L} = 0$$

Pinned with a linear torsional spring at $x = L$

$$\left.(-M_{zz_1} - M_s)\right|_{x=L} = 0$$

Clamped at $x = 0$: $v_1(0) = 0$ and $v_1'(0) = 0$

$$v_1(0) = A_2 + A_4 = 0$$

$$v_1'(0) = A_1\,\lambda + A_3 = 0$$

Extensional spring's force is $F_s = k_s\,v_1$ and the linear spring constant is

$$k_s = \alpha\,\frac{EI_{zz}}{L^3}$$

Now, pinned with a linear extensional spring at $x = L$:

$$\left(V_{y_1} + F_s\right)\Big|_{x=L} = 0$$

$$\left(-EI_{zz}\, A_3\, \lambda^2 + k_s\, v_1\right)\Big|_{x=L} = 0$$

$$\alpha\left(\frac{EI_{zz}}{L^3}\right) A_1\, \sin(\lambda\, L) + \alpha\left(\frac{EI_{zz}}{L^3}\right) A_2\, \cos(\lambda\, L) + \left[\alpha\, L - (\lambda\, L)^2\, L\right]\left(\frac{EI_{zz}}{L^3}\right) A_3 + \alpha\left(\frac{EI_{zz}}{L^3}\right) A_4 = 0$$

$$\alpha\, A_1\, \sin(\lambda\, L) + \alpha\, A_2\, \cos(\lambda\, L) + \left[\alpha\, L - (\lambda\, L)^2\, L\right] A_3 + \alpha\, A_4 = 0$$

Torsional spring's force is $M_s = k\,\theta = k\, v_1'$ and the torsional spring constant is

$$k_t = \beta\,\frac{EI_{zz}}{L}$$

Now, pinned with a linear torsional spring at $x = L$:

$$\left(-M_{zz_1} - M_s\right)\Big|_{x=L} = 0$$

$$\left(-EI_{zz}\, v_1'' - k_t\, v_1'\right)\Big|_{x=L} = 0$$

$$\left(\frac{EI_{zz}}{L^2}\right)\left[(\lambda\, L)^2\, \sin(\lambda\, L) - \beta\,(\lambda\, L)\, \cos(\lambda\, L)\right] A_1 + \left(\frac{EI_{zz}}{L^2}\right)\left[(\lambda\, L)^2\, \cos(\lambda\, L) + \beta\,(\lambda\, L)\, \sin(\lambda\, L)\right] A_2 - \left(\frac{EI_{zz}}{L^2}\right)\beta\, L\, A_3 = 0$$

$$\left[(\lambda\, L)^2\, \sin(\lambda\, L) - \beta\,(\lambda\, L)\, \cos(\lambda\, L)\right] A_1 + \left[(\lambda\, L)^2\, \cos(\lambda\, L) + \beta\,(\lambda\, L)\, \sin(\lambda\, L)\right] A_2 - \beta\, L\, A_3 = 0$$

Writing the boundary conditions in a matrix form in terms of the unknown coefficients

A_1, A_2, A_3, A_4

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & 1 & 0 \\ \left[(\lambda L)^2 \sin(\lambda L) - \beta (\lambda L) \cos(\lambda L)\right] & \left[(\lambda L)^2 \cos(\lambda L) + \beta (\lambda L) \sin(\lambda L)\right] & -L\beta & 0 \\ \alpha \sin(\lambda L) & \alpha \cos(\lambda L) & [\alpha L - (\lambda L)^2 L] & \alpha \end{bmatrix}}_{[C]}$$

$$\times \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.38}$$

(6.10-c) Obtain the characteristic equation.

Non-trivial solution ($\underline{\mathbf{A}} \neq \underline{\mathbf{0}}$) requires det[$\underline{\mathbf{C}}$] =0. The determinant will lead to the characteristic equation:

$$-(\lambda L)^5 \cos(\lambda L) - \beta (\lambda L)^4 \sin(\lambda L) + \alpha (\lambda L)^3 \cos(\lambda L) - \alpha (\lambda L)^2 \sin(\lambda L) + \alpha \beta (\lambda L)^2 \sin(\lambda L) - 2 \alpha \beta (\lambda L) + 2 \alpha \beta (\lambda L) \cos(\lambda L) = 0$$

(6.10-d) Obtain the buckling load.

We proceed to solve the characteristic equation for λ.

$$\lambda L = 0 \qquad \Rightarrow \qquad P = 0 \qquad \text{leads to trivial solution}$$

Now, let us check for the three suggested cases:

(a) $\alpha \to \infty$, $\beta \to \infty$

$$-\frac{1}{\alpha \beta}(\lambda L)^5 \cos(\lambda L) - \frac{1}{\alpha} (\lambda L)^4 \sin(\lambda L) + \frac{1}{\beta} (\lambda L)^3 \cos(\lambda L) - \frac{1}{\beta} (\lambda L)^2 \sin(\lambda L) + (\lambda L)^2 \sin(\lambda L) - 2 (\lambda L) + 2 (\lambda L) \cos(\lambda L) = 0$$

Hence, when $\alpha \to \infty$, $\beta \to \infty$

$$(\lambda L)^2 \sin(\lambda L) - 2 (\lambda L) + 2 (\lambda L) \cos(\lambda L) = 0$$

The nontrivial solution is:

$$\lambda L = 6.28319 \quad (= 2\pi) \qquad \to \qquad \lambda = \frac{2\pi}{L}$$

The buckling load is:

$$P_{cr} = P_1 = \lambda^2 EI_{zz} = \frac{4\pi^2 EI_{zz}}{L^2} = \frac{39.4784\, EI_{zz}}{L^2}$$

which is the solution for the case of clamped-clamped! The buckling mode shape associated with P_{cr} is

$$\frac{v_1(x)}{A_4} = 1 - \cos\left(\frac{2\pi x}{L}\right) = 1 - \cos\left(6.28319\frac{x}{L}\right)$$

(b) $\alpha \to 0$, $\beta \to 0$

$$-(\lambda L)^5 \cos(\lambda L) - \beta(\lambda L)^4 \sin(\lambda L) + \alpha(\lambda L)^3 \cos(\lambda L) - \alpha(\lambda L)^2 \sin(\lambda L) + \alpha\beta(\lambda L)^2 \sin(\lambda L) - 2\alpha\beta(\lambda L) + 2\alpha\beta(\lambda L)\cos(\lambda L) = 0$$

Hence, when $\alpha \to 0$, $\beta \to 0$

$$-(\lambda L)^5 \cos(\lambda L) = 0$$

Hence, the solution is that of clamped-free, see Example 6.9.

(c) $\alpha \to 1$, $\beta \to 1$

$$-(\lambda L)^5 \cos(\lambda L) - \beta(\lambda L)^4 \sin(\lambda L) + \alpha(\lambda L)^3 \cos(\lambda L) - \alpha(\lambda L)^2 \sin(\lambda L) + \alpha\beta(\lambda L)^2 \sin(\lambda L) - 2\alpha\beta(\lambda L) + 2\alpha\beta(\lambda L)\cos(\lambda$$

Hence, when $\alpha \to 1$, $\beta \to 1$

$$-(\lambda L)^5 \cos(\lambda L) - (\lambda L)^4 \sin(\lambda L) + (\lambda L)^3 \cos(\lambda L) - (\lambda L)^2 \sin(\lambda L) + (\lambda L)^2 \sin(\lambda L) - 2(\lambda L) + 2(\lambda L)\cos(\lambda L) = 0$$

The nontrivial solution is:

$$\lambda L = 2.2267 \quad (= 0.708782\,\pi) \qquad \rightarrow \qquad \lambda = \frac{0.708782\,\pi}{L}$$

The buckling load is:

$$P_{cr} = P_1 = \lambda^2 EI_{zz} = \frac{0.502372\,\pi^2 EI_{zz}}{L^2} = \frac{4.95821\, EI_{zz}}{L^2}$$

Let us determine the buckling mode shapes associated with P_{cr}. Plug-in the value for $\lambda_1 = 2.2267/L$ in Eq. (6.38) to determine the coefficients.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2.2267/L & 0 & 1 & 0 \\ 5.28738 & -1.25926 & -L & 0 \\ 0.792495 & -0.609879 & -3.95821\,L & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (6.39)$$

Therefore the coefficient are

$$A_1 = -0.167586\,A_4, \quad A_2 = -A_4, \quad A_3 = \frac{0.373165}{L}\,A_4$$

Now the mode shapes are obtained by substituting these values into

$$v_1(x) = A_1\,\sin(\lambda_1\,x) + A_2\,\cos(\lambda_1\,x) + A_3\,x + A_4$$

$$v_1(x) = A_4 - A_4\,\cos\left(2.2267\,\frac{x}{L}\right)$$

Note that A_4 remains indeterminate. Hence the buckling mode shape associated with P_{cr} is

$$\frac{v_1(x)}{A_4} = 1 - \cos\left(2.2267\,\frac{x}{L}\right)$$

(6.10-e) What does the buckling load represent?

The buckling load is the largest load for which the stability of the equilibrium of a structure exists in its original equilibrium configuration.

End Example □

6.4.7 Several Type of Column End Constraint

If the column ends are not pinned, the critical load and stress would be different from the pinned-pinned case. Often we use the critical load from the pinned-pinned case and substitute the actual length for an effective length:

$$P_{\mathrm{cr}} = \frac{\pi^2 \, E I_{\mathrm{zz}}}{L_{\mathrm{e}}^2} \tag{6.40}$$

The effective length L_{e} of any column is defined as the length of a pinned-pinned column that would buckle at the same critical load as the actual column: that is the effective length is the overall column length minus the portion that takes into account the end conditions. Thus end conditions affect the effective length of the column. The critical load and critical stresses can be expressed in terms the effective length L_{e} depending on the boundary conditions and is its values can be found in Table 6.1:

$$L_{\mathrm{e}} = C_{\mathrm{L}} \, L \tag{6.41}$$

Table 6.1: Effective length coefficient C_{L} for several type of column end constraints

Column End Conditions	Theoretical	Recommended*
pinned-pinned	1	1
pinned-fixed	0.7	0.8
fixed-fixed	0.5	0.65
fixed-free	2	2.1

*Recommended effective column length by AISC (American Institute of Steel Construction, 1989)

Commonly two different equations are widely used, depending on the slender ratio of the beam. Let us define the critical slender ratio R_c as

$$R_c = \sqrt{\frac{2 \, E \, \pi^2}{S_{\mathrm{y}}}} \tag{6.42}$$

where E is the Young's Modulus and S_{y} is the yield strength of the material. The actual effective slender ratio R_a of the column is:

$$R_a = \frac{L_{\mathrm{e}}}{r_g}$$

In the above equation r_g is the radius of gyration and is defined as

$$r_g = \sqrt{\frac{I_{\mathrm{zz}}}{A}}$$

where I_{zz} is the area moment of inertia and A the cross-sectional area. The critical buckling stress will depend on the slender ratio of the column. Three cases need to be considered:

Long Columns: Euler equation. If the slender ratio $R_a \geq R_c$ then the column will buckle elastically, and the Euler equation can be used to obtain the stress at buckling:

$$\sigma_{\text{cr}} = \frac{P_{\text{cr}}}{A} = \frac{\pi^2 E}{\left(\frac{L_{\text{e}}}{r_g}\right)^2} = E \left(\frac{\pi}{R_a}\right)^2 \tag{6.43}$$

Intermediate Columns: J. B. Johnson Equation. If the slender ratio $10 \leq R_a \leq R_c$ then the column will buckle inelastically, and the J. B. Johnson equation should be used to obtain the critical stress at buckling:

$$\sigma_{\text{cr}} = S_{\text{y}} - \frac{S_{\text{y}}^2}{4\,\pi^2\,E} \left(\frac{L_{\text{e}}}{r_g}\right)^2 = S_{\text{y}} - \frac{1}{E} \left(\frac{S_{\text{y}}\,R_a}{2\,\pi}\right)^2 \tag{6.44}$$

For design we take

$$\sigma_{\text{max}} = \frac{S_{\text{y}}}{2}$$

Short Columns: Yielding. If the slender ratio $R_a < 10$, in practice it is assumed that the column will fail due to yielding and the critical stress at buckling would be:

$$\sigma_{\text{cr}} = S_{\text{y}} \tag{6.45}$$

However, one may use the J. B. Johnson equation as well. For design we take

$$\sigma_{\text{max}} = S_{\text{y}}$$

The critical load can be obtained by multiplying the stress by the cross-sectional area

$$P_{\text{cr}} = \sigma_{\text{cr}}\,A$$

Example 6.11.

A column with one end fixed and the other end pinned is made of low-carbon steel. The column's cross-section is rectangular with $h = 0.5$ in and $b = 1.5$ in. Determine the buckling load for the following three lengths:

(a) $L = 0.5\,\text{ft}$

(b) $L = 0.05\,\text{ft}$

(c) $L = 2.5\,\text{ft}$

Using the corresponding tables, for low-carbon steel, the material properties are:

$$S_y = 43 \times 10^3 \quad \text{psi} \quad E = 30 \times 10^6 \quad \text{psi}$$

The important parameters are:

$$A = b\,h = 0.75 \quad \text{in}^2 \qquad \text{Rectangular cross sectional area}$$

$$I_{zz} = \frac{b\,h^3}{12} = 0.01563 \quad \text{in}^4 \qquad \text{Area moment of inertia}$$

$$r_g = \sqrt{\frac{I_{zz}}{A}} = 0.1443 \quad \text{in} \qquad \text{Radius of gyration}$$

$$C_L = \frac{L}{L_e} = 0.7 \qquad \text{Effective length coefficient}$$

$$R_c = \sqrt{\frac{2\,E\,\pi^2}{S_y}} = 117.352 \qquad \text{Critical slender ratio}$$

Now we proceed to solve the problem.

(a) $L = 0.5$ **ft** First calculate the effective length:

$$L_e = 0.7\,L = 0.7\,(6 \text{ in}) = 4.2 \quad \text{in}$$

Now evaluate the actual effective slender ratio

$$R_a = \frac{L_e}{r_g} = 29.0985$$

Since $10 \leq R_a < R_c$, the J. B. Johnson equation should be used:

$$P_{cr} = A\left\{S_y - \frac{1}{E}\left(\frac{S_y\,R_a}{2\,\pi}\right)^2\right\} = 31259 \quad \text{lb}$$

(b) $L = 0.05$ **ft** First calculate the effective length: (we use the recommended effective length)

$$L_{\text{e}} = 0.7\,L = 0.42 \quad \text{in}$$

Now evaluate the actual effective slender ratio

$$R_a = \frac{L_{\text{e}}}{r_g} = 2.909$$

Since $R_a < 0$, we may either use J. B. Johnson equation or equation for short columns. If we use J. B. Johnson:

$$P_{\text{cr}} = A \left\{ S_{\text{y}} - \frac{1}{E} \left(\frac{S_{\text{y}}\, R_a}{2\,\pi} \right)^2 \right\} = 32240 \quad \text{lb}$$

If we use short column assumption:

$$P_{\text{cr}} = A\, \{S_{\text{y}}\} = 32250 \quad \text{lb}$$

As you can see, the difference is very little.

(c) $L = 2.5$ **ft** First calculate the effective length: (we use the recommended effective length)

$$L_{\text{e}} = 0.7\,L = 21 \quad \text{in}$$

Now evaluate the actual effective slender ratio

$$R_a = \frac{L_{\text{e}}}{r_g} = 145.49$$

Since $R_a > R_c$, the Euler equation should be used:

$$P_{\text{cr}} = A \left\{ E \left(\frac{\pi}{R_a} \right)^2 \right\} = 10490.7 \quad \text{lb}$$

Observe that in part (b) the column was short, so that yield strength predominated; in part (a) the columns was intermediate and hence the J. B. Johnson equation was used; and (c) the columns were long enough that the Euler equation was used. Also note as the column becomes larger, the critical load decreases significantly.

End Example □

Example 6.12.

A column with one end fixed and the other end pinned is made of low-carbon steel. The column's cross-section is rectangular with $h = 2\,b$ and b. Determine the dimension b such that the maximum load would be 25 kips. Use a safety factor of 2.

From tables,

$$S_{\text{y}} = 43 \times 10^3 \quad \text{psi} \quad E = 30 \times 10^6 \quad \text{psi}$$

The important parameters are:

$$A = 2\,b^2 \quad \text{in}^2 \qquad \text{Rectangular cross sectional area}$$

$$I_{\text{zz}} = \frac{2\,b^4}{3} \quad \text{in}^4 \qquad \text{Area moment of inertia}$$

$$r_g = \frac{b}{\sqrt{3}} \quad \text{in} \qquad \text{Radius of gyration}$$

$$C_{\text{L}} = \frac{L}{L_{\text{e}}} = 0.7 \qquad \text{Effective length coefficient}$$

$$R_c = \sqrt{\frac{2\,E\,\pi^2}{S_{\text{y}}}} = 117.352 \qquad \text{Critical slender ratio}$$

The effective length:

$$L_{\text{e}} = 0.7\,L = 12.6 \quad \text{in}$$

Now evaluate the actual effective slender ratio

$$R_a = \frac{L_{\text{e}}}{r_g} = \frac{21.8238}{b}$$

The load is

$$P_{cr} = n_{\text{sf}}\,P = 100000 \quad \text{lb}$$

Now we proceed to solve the problem.

Let us assume that $R_a > R_c$,

$$P_{\text{cr}} = A\left\{E\left(\frac{\pi}{R_a}\right)^2\right\} = 10490.7 \quad \text{lb}$$

Since $10 \leq R_a < R_c$, the J. B. Johnson equation should be used:

$$P_{\text{cr}} = A\left\{S_{\text{y}} - \frac{1}{E}\left(\frac{S_{\text{y}}\,R_a}{2\,\pi}\right)^2\right\}$$
$$100000 = 1.24334 \times 10^6\,b^4$$

Hence,

$$b = 0.5325''$$

For this value, we need to verify that the our initial assumption of long column was correct:

$$R_a = 40.98$$

Since $R_a < R_c$, our initial assumption was incorrect.

Assuming now $R_a < R_c$,

$$P_{\text{cr}} = A\left\{S_{\text{y}} - \frac{1}{E}\left(\frac{S_{\text{y}}\,R_a}{2\,\pi}\right)^2\right\}$$
$$100000 = -1487.13 + 86000\,b^2$$

Hence,

$$b = 1.08632''$$

For this value, we need to verify that the our initial assumption of long column was correct:

$$R_a = 20.0898$$

which is correct.

Note that this problem must be solved iteratively. Make an initial guess for b, say $b = 1''$ and find the actual slender ratio and then determine the load P using the corresponding equation. If the load P is greater than 100000 lb, then we want to reduce the value of b, otherwise increase it.

End Example □

Example 6.13.

An engineer is asked to design a safe round tubular column subject to static axial loads. The column has a length L an outside diameter d_o and inside diameter d_i. The diametral ratio is:

$$\alpha = \frac{d_i}{d_o}$$

Determine the outside diameter d_o such that failure of the tube is by buckling.

First note that

$$n_{\text{SF}} = \frac{P_{\text{cr}}}{P} \qquad \rightarrow \qquad P_{\text{cr}} = n_{\text{SF}}\, P$$

or in terms of stress

$$\sigma = \frac{\sigma_{\text{cr}}}{n_{\text{SF}}} \qquad \rightarrow \qquad \sigma_{\text{cr}} = n_{\text{SF}}\, \sigma$$

where the load and stresses are related as follows

$$P_{\text{cr}} = \sigma_{\text{cr}}\, A$$

Also, let

$$\frac{L_{\text{e}}}{L} = C_{\text{L}} \qquad \rightarrow \qquad L_{\text{e}} = C_{\text{L}}\, L$$

where C_{L} refers to the recommended value for the effective length to take into account several type of column end constraints.

For our problem

$$MS = 1.50 \qquad \rightarrow \qquad n_{\text{SF}} = MS + 1 = 2.5$$

It is known that

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) = \frac{\pi}{4}\, d_o^2\, \left(1 - \alpha^2\right)$$

$$I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = \frac{\pi}{64}\, d_o^4\, \left(1 - \alpha^4\right) = \frac{\pi}{64}\, d_o^4\, \left(1 - \alpha^2\right)\, \left(1 + \alpha^2\right)$$

$$r_g^2 = \frac{I}{A} = \frac{d_o^2}{16}\, \left(1 + \alpha^2\right)$$

Thus

$$R_a^2 = \left(\frac{L_{\text{e}}}{r_g}\right)^2 = \frac{C_{\text{L}}^2\, L^2}{\dfrac{d_o^2}{16}\, \left(1 + \alpha^2\right)} = \frac{16\, C_{\text{L}}^2\, L^2}{d_o^2\, \left(1 + \alpha^2\right)}$$

The stress can be expressed as

$$\sigma_{\text{cr}} = n_{\text{SF}}\, \sigma = n_{\text{SF}}\, \frac{P}{A} = n_{\text{SF}}\, \frac{P}{\frac{\pi}{4}\, d_o^2\, \left(1-\alpha^2\right)} = \frac{4\, n_{\text{SF}}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)}$$

1. Assuming the actual slender ratio is smaller than the critical slender ratio [10pts]

   When $R_a \leq 10$ the column is a short column, we can use:

   $$\sigma_{\text{cr}} = S_{\text{y}}$$

   Thus

   $$\sigma_{\text{cr}} = S_{\text{y}}$$

   $$\frac{4\, n_{\text{SF}}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)} = S_{\text{y}}$$

   Rearranging and solving for d_o

   $$d_o^2 = \frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}}$$

   $$d_o = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}}}$$

   When $10 \leq R_a \leq R_c$ the column is an intermediate column, we must use the J. B. Johnson Equation:

   $$\sigma_{\text{cr}} = S_{\text{y}} - \frac{S_{\text{y}}^2}{4\,\pi^2\, E}\, \left(\frac{L_{\text{e}}}{r_g}\right)^2 = S_{\text{y}} - \frac{1}{E}\, \left(\frac{S_{\text{y}}\, R_a}{2\,\pi}\right)^2$$

   Thus

   $$\sigma_{\text{cr}} = S_{\text{y}} - \frac{S_{\text{y}}^2}{4\,\pi^2\, E}\, \left(\frac{L_{\text{e}}}{r_g}\right)^2$$

   $$\frac{4\, n_{\text{SF}}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)} = S_{\text{y}} - \frac{S_{\text{y}}^2}{4\,\pi^2\, E}\, \frac{16\, C_{\text{L}}^2\, L^2}{d_o^2\, \left(1+\alpha^2\right)}$$

   Rearranging

   $$\frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)} = d_o^2\, S_{\text{y}} - \frac{4\, S_{\text{y}}^2\, C_{\text{L}}^2\, L^2}{\pi^2\, E\, \left(1+\alpha^2\right)}$$

   Solving for d_o

   $$d_o^2 = \frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}} + \frac{4\, S_{\text{y}}\, C_{\text{L}}^2\, L^2}{\pi^2\, E\, \left(1+\alpha^2\right)}$$

   $$d_o = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}} + \frac{4\, S_{\text{y}}\, C_{\text{L}}^2\, L^2}{\pi^2\, E\, \left(1+\alpha^2\right)}}$$

2. Assuming the actual slender ratio is bigger than the critical slender ratio [10pts]

When $R_a \geq R_c$ the column is a long column, we must use the Euler Equation:

$$\sigma_{\rm cr} = \frac{P_{\rm cr}}{A} = \frac{\pi^2 E}{\left(\frac{L_{\rm e}}{r_g}\right)^2} = E \left(\frac{\pi}{R_a}\right)^2$$

Thus

$$\sigma_{\rm cr} = \frac{\pi^2 E}{\left(\frac{L_{\rm e}}{r_g}\right)^2}$$

$$\frac{4\, n_{\rm SF}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)} = \frac{\pi^2 E}{\left(\frac{L_{\rm e}}{r_g}\right)^2}$$

$$\frac{4\, n_{\rm SF}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)} = \frac{\pi^2 E}{\dfrac{16\, C_{\rm L}^2\, L^2}{d_o^2\, \left(1+\alpha^2\right)}}$$

Rearranging and solving for d_o

$$d_o^4 = \frac{64\, C_{\rm L}^2\, L^2\, n_{\rm SF}\, P}{\pi^3\, E\, \left(1-\alpha^4\right)} \qquad \rightarrow \qquad d_o = \left[\frac{64\, C_{\rm L}^2\, L^2\, n_{\rm SF}\, P}{\pi^3\, E\, \left(1-\alpha^4\right)}\right]^{\frac{1}{4}}$$

What happens when $\alpha = 0$? Do you expect the column to be more stable when $\alpha > 0$? [10pts]

Note that $0 \leq \alpha < 1$. It cannot take a value of one because there would be no cross-sectional area. When $\alpha = 0$, $d_i = 0$ and the column becomes a solid column and

$$A = \frac{\pi}{4}\, d_o^2\, \left(1-\alpha^2\right) = \frac{\pi}{4}\, d_o^2$$

$$I = \frac{\pi}{64}\, d_o^4\, \left(1-\alpha^4\right) = \frac{\pi}{64}\, d_o^4$$

$$r_g^2 = \frac{I}{A} = \frac{d_o^2}{16}\, \left(1+\alpha^2\right) = \frac{d_o^2}{16}$$

$$R_a^2 = \frac{16\, C_{\rm L}^2\, L^2}{d_o^2\, \left(1+\alpha^2\right)} = \frac{16\, C_{\rm L}^2\, L^2}{d_o^2}$$

$$\sigma_{\rm cr} = \frac{4\, n_{\rm SF}\, P}{\pi\, d_o^2\, \left(1-\alpha^2\right)} = \frac{4\, n_{\rm SF}\, P}{\pi\, d_o^2}$$

It is clear that the slender ratio is smaller and the critical buckling stress decreases in value. This implies the buckling stress would be smaller for a solid column than for a tubular column. Thus the column would be more stable when $\alpha > 0$.

When $R_a \leq 10$

$$d_o = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}}} = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, S_{\text{y}}}}$$

When $10 \leq R_a \leq R_c$

$$d_o = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, \left(1-\alpha^2\right)\, S_{\text{y}}} + \frac{4\, S_{\text{y}}\, C_{\text{L}}^2\, L^2}{\pi^2\, E\, \left(1+\alpha^2\right)}} = \sqrt{\frac{4\, n_{\text{SF}}\, P}{\pi\, S_{\text{y}}} + \frac{4\, S_{\text{y}}\, C_{\text{L}}^2\, L^2}{\pi^2\, E}}$$

When $R_a \geq R_c$

$$d_o = \left[\frac{64\, C_{\text{L}}^2\, L^2\, n_{\text{SF}}\, P}{\pi^3\, E\, \left(1-\alpha^4\right)}\right]^{\frac{1}{4}} = \left[\frac{64\, C_{\text{L}}^2\, L^2\, n_{\text{SF}}\, P}{\pi^3\, E}\right]^{\frac{1}{4}}$$

End Example □

Example 6.14.

It is desired to substitute the tubular column by a solid rectangular one. The column has rectangular cross-section with a height a and a width b ($b = \beta\, a$), determine the width b.

It is known that

$$A = a\, b = \beta\, a^2$$

$$I = \frac{b\, a^3}{12} = \frac{\beta\, a^4}{12}$$

$$r_g^2 = \frac{I}{A} = \frac{a^2}{12}$$

Thus

$$R_a^2 = \left(\frac{L_e}{r_g}\right)^2 = \frac{C_L^2\, L^2}{\frac{a^2}{12}} = \frac{12\, C_L^2\, L^2}{a^2}$$

The stress can be expressed as

$$\sigma_{cr} = n_{SF}\, \sigma = n_{SF}\, \frac{P}{A} = \frac{n_{SF}\, P}{\beta\, a^2}$$

1. Assuming the actual slender ratio is smaller than the critical slender ratio [10pts] When $R_a \leq 10$ the column is a short column, we can use:

$$\sigma_{cr} = S_y$$

Thus

$$\sigma_{cr} = S_y$$

$$\frac{n_{SF}\, P}{\beta\, a^2} = S_y$$

Rearranging and solving for a

$$a^2 = \frac{n_{SF}\, P}{\beta\, S_y}$$

$$a = \sqrt{\frac{n_{SF}\, P}{\beta\, S_y}}$$

The width b is $b = \beta\, a$:

$$b = \sqrt{\frac{\beta\, n_{SF}\, P}{S_y}}$$

When $10 \le R_a \le R_c$ the column is an intermediate column, we must use the J. B. Johnson Equation:

$$\sigma_{\rm cr} = S_{\rm y} - \frac{S_{\rm y}^2}{4\,\pi^2\,E}\left(\frac{L_{\rm e}}{r_g}\right)^2 = S_{\rm y} - \frac{1}{E}\left(\frac{S_{\rm y}\,R_a}{2\,\pi}\right)^2$$

Thus

$$\sigma_{\rm cr} = S_{\rm y} - \frac{S_{\rm y}^2}{4\,\pi^2\,E}\left(\frac{L_{\rm e}}{r_g}\right)^2$$

$$\frac{n_{\rm SF}\,P}{\beta\,a^2} = S_{\rm y} - \frac{S_{\rm y}^2}{4\,\pi^2\,E}\,\frac{12\,C_{\rm L}^2\,L^2}{a^2}$$

Rearranging

$$n_{\rm SF}\,\frac{P}{\beta} = a^2\,S_{\rm y} - \frac{3\,S_{\rm y}^2\,C_{\rm L}^2\,L^2}{\pi^2\,E}$$

Solving for a

$$a^2 = \frac{n_{\rm SF}\,P}{\beta} + \frac{3\,S_{\rm y}\,C_{\rm L}^2\,L^2}{\pi^2\,E}$$

$$a = \sqrt{\frac{n_{\rm SF}\,P}{\beta} + \frac{3\,S_{\rm y}\,C_{\rm L}^2\,L^2}{\pi^2\,E}}$$

The width b is $b = \beta\,a$:

$$b = \sqrt{\frac{\beta\,n_{\rm SF}\,P}{S_{\rm y}} + \frac{3\,\beta^2\,S_{\rm y}\,C_{\rm L}^2\,L^2}{\pi^2\,E}}$$

2. Assuming the actual slender ratio is bigger than the critical slender ratio [10pts]

When $R_a \ge R_c$ the column is a long column, we must use the Euler Equation:

$$\sigma_{\rm cr} = \frac{P_{\rm cr}}{A} = \frac{\pi^2\,E}{\left(\dfrac{L_{\rm e}}{r_g}\right)^2} = E\left(\frac{\pi}{R_a}\right)^2$$

Thus

$$\sigma_{\rm cr} = \frac{\pi^2\,E}{\left(\dfrac{L_{\rm e}}{r_g}\right)^2}$$

$$\frac{n_{\rm SF}\,P}{\beta\,a^2} = \frac{\pi^2\,E}{\dfrac{12\,C_{\rm L}^2\,L^2}{a^2}}$$

$$\frac{n_{\rm SF}\,P}{\beta\,a^2} = \frac{\pi^2\,E\,a^2}{12\,C_{\rm L}^2\,L^2}$$

Rearranging and solving for a

$$a^4 = \frac{12\, C_{\text{L}}^2\, L^2\, n_{\text{SF}}\, P}{\beta\, \pi^2\, E} \qquad \rightarrow \qquad a = \left[\frac{12\, C_{\text{L}}^2\, L^2\, n_{\text{SF}}\, P}{\beta\, \pi^2\, E}\right]^{\frac{1}{4}}$$

The width b is $b = \beta\, a$:

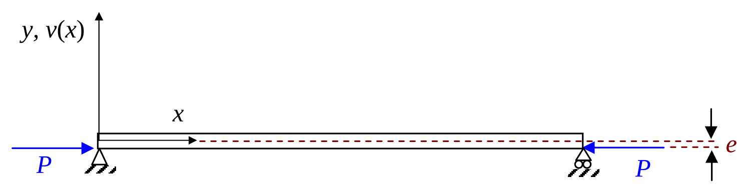
$$b = \left[\frac{12\, \beta^3\, C_{\text{L}}^2\, L^2\, n_{\text{SF}}\, P}{\pi^2\, E}\right]^{\frac{1}{4}}$$

End Example □

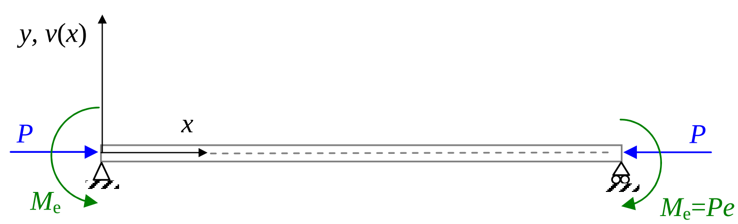
6.4.8 Imperfect Beam-Columns: Eccentric load

In the previous section, we considered the perfect column, one that is initially straight and the axial load is perfectly aligned with the centroidal axis. Imperfect columns consist of geometric imperfection (initial deflection exists) and/or load misalignment (eccentric load). Let us begin by discussion imperfections due to load misalignment.

Columns used in applications rarely have the applied load aligned coincidentally with the centroidal axis of the cross section. The distance between the two axes is called *eccentricity* and is designated by e. Let us derive the equations for a simply-supported straight column subject to an eccentric static compressive force P. The beam column obeys the Euler-Bernoulli Theory, is homogeneous, isotropic, and of uniform cross-section.



It is assumed that the load is always parallel with the centroid of the columns. Figure shows a pinned-end column subjected to forces acting at a distance e from the centerline of the undeformed column. It is assumed that the load is applied to the column at a short eccentric distance from the centroid of the cross section.



This loading on the column is statically equivalent to the axial load and bending moment at the end points is:

$$M_e = -P\,e$$

shown in above Figure. As when one is considering concentrically loaded columns, small deflections and linear elastic material behavior are assumed. The x–y plane is a plane of symmetry for the cross-sectional area.

The eccentric axial load will simultaneously subject the column to compression and bending in the equilibrium state. The analysis for the equilibrium response of the column in compression (axial force P and axial displacement $v(x)$) is identical to the case of the perfect column, since the x-axis passes through the centroid of each cross section (decoupling the axial compression from bending in the material law).

The equilibrium response of the column in bending, which includes the influence of axial compression on bending, will be determined by the same analysis that led to Eq. 6.30, except that we drop the subscript "1" on the transverse displacement, since in the eccentric load case the transverse displacement refers to an equilibrium state and not to a buckling mode.

The ordinary differential equation for a column with EI_{zz} = constant was derived as

$$v_1'''' + \underbrace{\frac{P}{EI_{zz}}}_{\lambda^2} v_1'' = 0 \qquad v_1 = v_1(x) \qquad x \in (0, L) \tag{6.46}$$

General solution for $\lambda^2 > 0$ is

$$v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4 \tag{6.47}$$

and the buckling load is obtained from Eq. (6.46):

$$\lambda^2 = \frac{P}{EI_{zz}} \qquad \rightarrow \qquad P = \lambda^2 EI_{zz}$$

The constants are found by applying the boundary conditions. From the Hooke's law:

$$M_{zz_1} \quad = \quad EI_{zz} v_1''(x) = -EI_{zz} [A_1 \lambda^2 \sin(\lambda x) + A_2 \lambda^2 \cos(\lambda x)] \tag{6.48}$$

and from equilibrium:

$$V_{y_1} \quad = \quad -M_{zz_1}' - P v_1' = -EI_{zz} [v_1''' + \lambda^2 v_1'] = -EI_{zz} [A_3 \lambda^2] \tag{6.49}$$

We want to determine the critical load P_{cr} and the associated buckling mode shape. We first start with the boundary conditions:

Pinned at $x = 0$: $v_1(0) = 0$ and $M_{zz_1}(0) = EI_{zz} v_1''(0) = M_e$

Pinned at $x = L$: $v_1(L) = 0$ and $M_{zz_1}(L) = EI_{zz} v_1''(L) = M_e$

Thus

$$v_1(0) = A_2 + A_4 = 0$$

$$EI_{zz} v_1''(0) = -EI_{zz} A_2 \lambda^2 = -P e$$

$$v_1''(0) = -A_2 \lambda^2 = -\frac{P e}{EI_{zz}} = -\lambda^2 e$$

$$v_1(L) = A_1 \sin(\lambda L) + A_2 \cos(\lambda L) + A_3 L + A_4 = 0$$

$$EI_{zz} v_1''(L) = -EI_{zz} A_1 \lambda^2 \sin(\lambda L) - EI_{zz} A_2 \lambda^2 \cos(\lambda L) = -P e$$

$$v_1''(L) = -A_1 \lambda^2 \sin(\lambda L) - A_2 \lambda^2 \cos(\lambda L) = -\frac{P e}{EI_{zz}} = -\lambda^2 e$$

Write the boundary conditions in a matrix form in terms of the unknown coefficients A, A_2, A_3, A_4:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\lambda^2 & 0 & 0 \\ -\lambda^2 \sin(\lambda L) & -\lambda^2 \cos(\lambda L) & 0 & 0 \\ \sin(\lambda L) & \cos(\lambda L) & L & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\lambda^2 e \\ -\lambda^2 e \\ 0 \end{Bmatrix} \tag{6.50}$$

Note that the boundary conditions are inhomogeneous. Thus these equation do not lead to trivial solution and we proceed to solve in term of the constants. The solution to the above equation is:

$$A_1 = -e\cos(\lambda L) + e\csc(\lambda L) \qquad A_2 = e \qquad A_3 = 0 \qquad A_4 = -e$$

and replacing in Eq. (6.47) and simplifying we get:

$$v_1(x) = e\left\{\tan\left(\frac{L}{2}\lambda\right)\sin(x\lambda) + \cos(\lambda L) - 1\right\} \quad \text{where} \quad \lambda = \sqrt{\frac{P}{EI_{zz}}}$$

The maximum deflection occurs at:

$$x = \frac{L}{2}: \qquad v_{max} = e\left\{\sec\left(\frac{L}{2}\lambda\right) - 1\right\}$$

Note that

$$\frac{\lambda L}{2} = \frac{L}{2 r_g}\sqrt{\frac{P}{EA}} = \left\{\frac{L}{2}\sqrt{\frac{P}{EI_{zz}}}\right\}\sqrt{\frac{\pi^2 E I_{zz}}{P_{cr} L_e^2}} = \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} \qquad \text{where} \qquad P_{cr} = \pi^2 \frac{E I_{zz}}{L_e^2}$$

Thus,

$$v_{max} = e\left\{\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right\}$$

Let us define the dimensionless quantities as

$$\hat{p} = \frac{P}{P_{cr}}$$

Hence,

$$\delta = v_{max} = e\left\{\sec\left(\frac{\pi}{2}\sqrt{\hat{p}}\right) - 1\right\} \tag{6.51}$$

Figure 6.7 shows that as $\delta \to \infty$, $P \to P_{cr}$ for all $e \neq 0$. That is, no matter the magnitude of e, δ gets very large as $P \to P_{cr}$ of the perfect structure. Failure by excessive displacement or loss of structural stiffness.

The maximum compressive stress in the column is caused by an axial load and a moment:

$$\sigma_{max} = \frac{P}{A} - \frac{M c}{I}$$

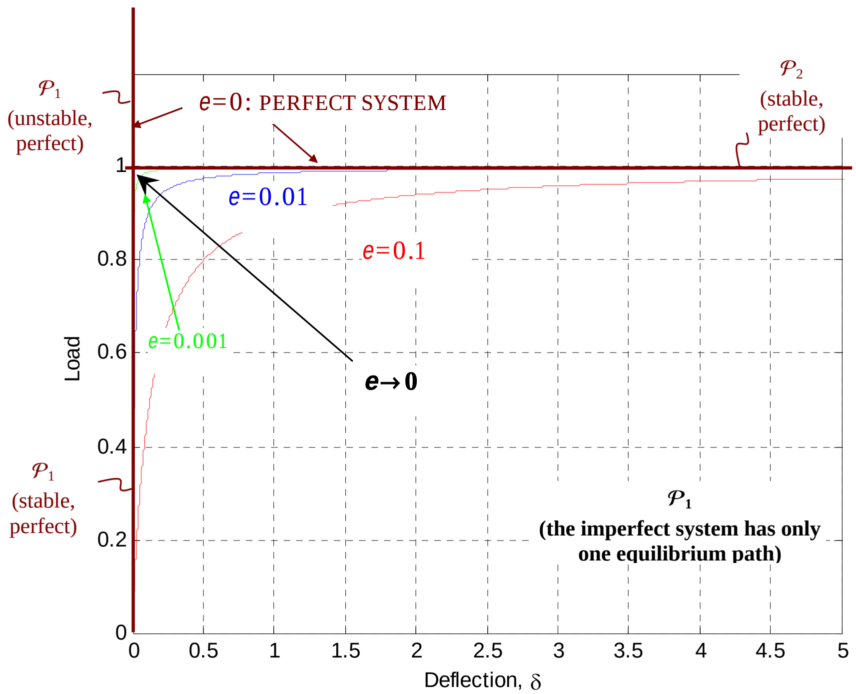


Figure 6.7: Response for various levels of load imperfection.

where the maximum bending moment for the column is

$$M = -P\,(e+\delta) = -P\,e\,\sec\left(\frac{L}{2}\,\lambda\right) = -P\,e\,\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{\rm cr}}}\right)$$

Thus, the maximum compressive stress for the column (possibly away from the ends where there is stress concentration) is

$$\begin{aligned}\sigma_{\max} &= \left\{\frac{P}{A} + \frac{P\,e\,c}{I}\,\sec\left(\frac{L}{2}\,\lambda\right)\right\}\\ &= \frac{P}{A}\left\{1 + \frac{e\,c}{r_g^2}\,\sec\left(\frac{L}{2\,r_g}\sqrt{\frac{P}{E\,A}}\right)\right\}\end{aligned}$$

For different end conditions other than pinned, the length may be replaced with the effective length:

$$\begin{aligned}\sigma_{\max} &= \frac{P}{A}\left\{1 + \frac{e\,c}{r_g^2}\,\sec\left(\frac{L_{\rm e}}{2\,r_g}\sqrt{\frac{P}{E\,A}}\right)\right\}\\ &= \frac{P}{A}\left\{1 + e_r\,\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{\rm cr}}}\right)\right\}\\ &= \frac{P}{A}\left\{1 + e_r\,\sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\}\end{aligned}$$

where R_a is the actual slender ratio and e_r is known as the eccentricity ratio and it is defined as

$$e_r = \frac{c\,e}{r_g^2}$$

Thus,

$$\begin{aligned} v_{\max} &= e\left\{\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{\text{cr}}}}\right) - 1\right\} \\ \sigma_{\max} &= \frac{P}{A}\left\{1 + e_r\,\sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\} \end{aligned} \tag{6.52}$$

In the above equations:

P = critical load where buckling will occur in eccentrically loaded column
A = cross-sectional area of the column
e = eccentricity of load measured from neutral axis of the column's cross-sectional area to the load's line of action
e_r = eccentricity ratio
R_a = actual slender ratio
c = distance from neutral axis to outer fiber of the column
r_g = radius of gyration for the cross-section of the column
L_{e} = effective length of the column
E = modulus of elasticity of the column's material

Because the derivation of Eq. (6.52) is based on the premise that equal couples exist at the locations of the column and constraints, and that the maximum lateral deflection occurs at midspan, the secant formula is valid only for columns that meet these conditions. In order words, the secant formula is valid for columns with pinned-pinned and fixed-fixed end points, but not valid for columns with other boundary conditions.

The secant formula is not very convenient for calculation purposes because the critical load cannot be explicitly isolated, but with the aid of a computer, or by appropriate graphical techniques, it can be employed satisfactorily. A possible iterative process can be used in obtaining the solution for the buckling load. Assuming that the maximum stress occurs at S_{all}:

$$\begin{aligned} \sigma_{\max} &= \frac{P}{A}\left\{1 + e_r\,\sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\} \\ S_{\text{all}} &= \frac{P}{A}\left\{1 + e_r\,\sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\} \end{aligned}$$

Now, solving the above equation in terms of P:

$$P = \frac{S_{\text{all}}\,A}{1 + e_r\,\sec\left(\dfrac{R_a}{2}\sqrt{\dfrac{P}{E\,A}}\right)} = f(P)$$

If the buckling load is what we want, we start with an initial P as the concentric loading condition and obtain a new value of P. This iterative process is continued until a desired accuracy is achieved. This accuracy ma be obtained using the following condition

$$\left| \frac{P - f(P)}{f(P)} \right| \leq \varepsilon$$

where ε is a small number ($\varepsilon = 10^{-3}$).

We should also note that Eq. (6.52) is undefined for zero eccentricity, but for very small eccentricities it approaches the Euler buckling cruve as a limit for long columns and approaches the simple compressive yield curve as a limit for short columns. For struts (short compressive members) with a critical slender ratio of

$$R_{cs} = 0.282 \sqrt{\frac{E\,A}{P}}$$

the largest compressive stress can be expressed as follows:

$$\sigma_{\max} = \frac{P}{A} \{1 + e_r\}$$

If $R_a > R_{cs}$ then use the secant formula.

Example 6.15.

A 36-in long fixed-fixed hollow steel tube with a 3.0-in outside diameter and a 0.03-in wall thickness is subject to an axial load.

1. If the load is concentric, determine the buckling load.
2. If the load is eccentric with an eccentricity of 0.15-in, determine the buckling load.
3. Determine the dimensions for an equivalent design made of squared-hollow tube with the same thickness subject to concentric loading.

The mechanical properties are:

$$S_y = 11 \text{ ksi}, \qquad E = 10300 \text{ ksi}$$

The effective length is taken as (using recommended values for steel):

$$L_e = 0.65\,L = 23.40''$$

The critical slender ratio for the column is:

$$R_c = \sqrt{\frac{2\,E\,\pi^2}{S_y}} = 223.372''$$

1. If the load is concentric, determine the buckling load.

The inside diameter of the column is

$$d_i = d_o - 2\,t_w = 1.94''$$

The cross-sectional plane area properties are

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) = 0.2799 \text{ in}^2, \qquad I_{zz} = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = 0.3087 \text{ in}^4$$

The radius of gyration of the column is:

$$r_g = \sqrt{\frac{I_{zz}}{A}} = 1.0501''$$

and the actual effective slender ratio is

$$R_a = \frac{L_e}{r_g} = 22.2834''$$

Since the slender ratio is $10 \leq R_a \leq R_c$ then the column will buckle inelastically, and the J. B. Johnson equation should be used to obtain the critical stress at buckling:

$$P_{cr} = A\left\{S_y - \frac{1}{E}\left(\frac{S_y\,R_a}{2\,\pi}\right)^2\right\} = 2849.18 \quad \text{lb}$$

2. If the load is eccentric with an eccentricity of 0.15-in, determine the buckling load. Since the slender ratio is $10 \leq R_a \leq R_c$, we use

$$\sigma_{max} = \frac{S_y}{2} = 5.5 \times 10^3 \text{ psi}$$

The distance from the neutral axis to the outer fiber is

$$c = \frac{d_o}{2} = 1.5''$$

Thus, the eccentricity ratio is

$$e_r = \frac{c\,e}{r_g^2} = 0.20404$$

The secant formula becomes

$$\sigma_{max} = \frac{P}{A}\left\{1 + e_r \sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\}$$
$$\frac{S_y}{2} = \frac{P}{A}\left\{1 + e_r \sec\left(\frac{R_a}{2}\sqrt{\frac{P}{E\,A}}\right)\right\}$$

Now, solving the above equation in terms of P:

$$P = \frac{S_{\text{y}}\, A}{2 + 2\, e_r\, \sec\left(\dfrac{R_a}{2}\sqrt{\dfrac{P}{E\, A}}\right)} = f(P)$$

Now, as the initial guess for P let us use the value from the concentric loading condition and obtain a new value of P. Thus,

$$\begin{aligned} q = 0: &\qquad P = 2849.18 \text{ lb} \\ q = 1: &\qquad P = 1185.16 \text{ lb} \\ q = 2: &\qquad P = 1192.35 \text{ lb} \\ q = 3: &\qquad P = 1192.32 \text{ lb} \end{aligned}$$

Hence, the bucking load is $P_{\text{cr}} = 1192.32$ lb.

3. Determine the dimensions for an equivalent design made of squared-hollow tube with the same thickness subject to concentric loading.

For an equivalent column, the buckling load must be the same:

$$P_{\text{cr}}\Big|_{\text{round}} = P_{\text{cr}}\Big|_{\text{squared}}$$

Note that the mechanical, geometric and length remain unchanged:

$$S_y = 11 \times 10^3 \text{ psi}, \qquad E = 10.3 \times 10^6 \text{ psi}, \qquad L_{\text{e}} = 23.4'', \qquad R_c = 223.372''$$

The plane area properties for the hollow square are:

$$\begin{aligned} a_o &= a_i + 2\, t_w = 0.06 + a_i \\ A &= a_o^2 - a_i^2 = 0.0036 + 0.12\, a_i \\ I_{\text{zz}} &= \frac{a_o^4 - a_i^4}{12} = 1.08 \times 10^{-6} + 0.000072\, a_i + 0.0018\, a_i^2 + 0.02\, a_i^3 \end{aligned}$$

The radius of gyration is

$$r_g = \sqrt{\frac{I_{\text{zz}}}{A}} = \sqrt{\frac{1.08 \times 10^{-6} + 0.000072\, a_i + 0.0018\, a_i^2 + 0.02\, a_i^3}{0.0036 + 0.12\, a_i}}$$

Assuming $10 \leq R_a \leq R_c$,

$$\frac{P_{\text{cr}}}{A} = S_{\text{y}} - \frac{1}{E}\left(\frac{S_{\text{y}}\, R_a}{2\,\pi}\right)^2$$

Solving for a_i we get $a_i = 2.308''$. Now, we need to verify our assumption

$$R_a = 24.5088'' \qquad \rightarrow \qquad 10 \leq R_a \leq R_c \qquad \checkmark$$

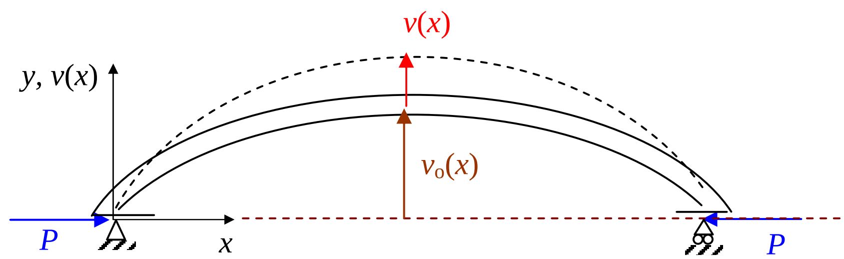
and our assumption was correct. Thus the dimensions are

$$a_i = 2.31'', \quad a_o = 2.37''$$

End Example □

6.4.9 Imperfect Beam-Columns: Geometric Imperfection

Now consider the case of a uniform, pinned-pinned column that is slightly crooked under no load and it is subject to a centric, axial compressive load P. The initial shape under no load is described by the function $v_o(x)$. That is the transverse displacement of the column is such that $v(x) = v_o(x)$ when $P = 0$.



Also, the bending moment in the column is zero under no load. Thus, the material law for bending is

$$M_{zz} = EI_{zz}\left\{v''\right\}$$

Hence, Eq. (6.27) becomes

$$\begin{aligned} \frac{d^2 M_{zz}}{dx^2} + \left(\frac{d^2 v}{dx^2} + \frac{d^2 v_o}{dx^2}\right) P = 0 \\ \frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) + \left(\frac{d^2 v}{dx^2} + \frac{d^2 v_o}{dx^2}\right) P = 0 \end{aligned} \tag{6.53}$$

Hence, the ordinary differential equation for a column with EI_{zz} = constant was derived as

$$v'''' + \underbrace{\frac{P}{EI_{zz}}}_{\lambda^2} v'' = -\lambda^2\, v_o'' \qquad v = v(x) \qquad x \in (0, L) \tag{6.54}$$

Note that we drop the subscript "1" on the transverse displacement, since the transverse displacement refers to an equilibrium state and not to a buckling mode. Let us assume that the initial shape of the bar is that of a sine function with amplitude a_1

$$v_o(x) = a_1 \sin\left(\frac{\pi\, x}{L}\right)$$

where a_1 denotes the amplitude at midspan of the slightly crooked column. hence the nonhomogeneous fourth order differential equation is

$$v'''' + \lambda^2\, v'' = -\lambda^2\, v_o'' = -\left(\frac{\lambda\,\pi}{L}\right)^2 a_1 \sin\left(\frac{\pi\, x}{L}\right) \qquad v = v(x) \qquad x \in (0, L) \tag{6.55}$$

General solution for $\lambda^2 > 0$ is

$$v(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4 + \frac{1}{\left(\frac{\lambda \pi}{L}\right)^2 - 1} a_1 \sin\left(\frac{\pi x}{L}\right) \tag{6.56}$$

The boundary conditions are same as those of a simply-supported beam:

Pinned at $x = 0$: $v(0) = 0$ and $M_{zz}(0) = EI_{zz}\, v''(0) = 0$

Pinned at $x = L$: $v(L) = 0$ and $M_{zz}(L) = EI_{zz}\, v''(L) = 0$

The solution is

$$v(x) = \frac{a_1}{1 - \left(\frac{\lambda \pi}{L}\right)^2} \sin\left(\frac{\pi x}{L}\right) = \frac{a_1}{1 - \hat{p}^2} \sin\left(\frac{\pi x}{L}\right)$$

where

$$\hat{p} = \frac{P}{P_{cr}}$$

Thus,

$$\delta = v_{max} = \frac{a_1}{1 - \hat{p}^2} \tag{6.57}$$

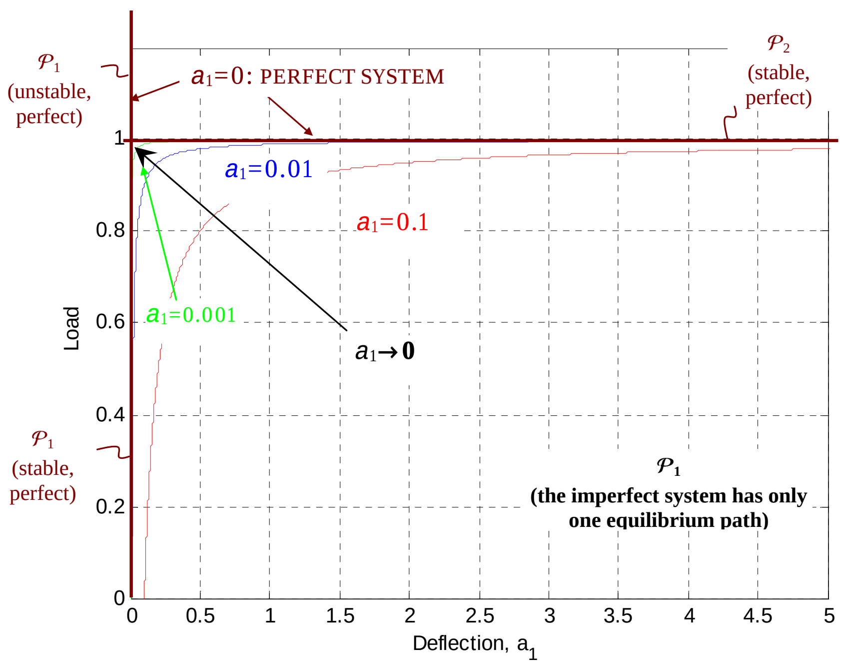


Figure 6.8: Response for various levels of geometric imperfection.

Figure 6.8 shows that as $\delta \to \infty$, $P \to P_{cr}$ for all $a_1 \neq 0$. That is, for a nonzero value of the imperfection amplitude, the displacement gets very large as the axial force approaches the buckling load of the perfect column. Also, the imperfect column deflects in the direction of imperfection; e.g., if $a_1 > 0$, then $\delta > 0$.

Summary of Beam-Column Imperfections

In short, collectively the eccentric load and the geometric shape imperfection are called imperfections. All real columns are imperfect. Even for a well manufactured column whose geometric imperfections are small and with the load eccentricity small, the displacements become excessive as the axial compressive force P approaches the critical load P_{cr} of the perfect column. Hence, the critical load determined from the analysis of the perfect column is meaningful in practice.

6.4.10 Inelastic Buckling

The strength of a compression member (column) depends on its geometry (effective slenderness ratio R_a) and its material properties (stiffness and strength). The Euler formula describes the critical load for elastic buckling and is valid only for long columns. The ultimate compression strength of the column material is not geometry-related and is valid only for short columns.

In between, for a column with intermediate length, buckling occurs after the stress in the column exceeds the proportional limit of the column material and before the stress reaches the ultimate strength. This kind of situation is called inelastic buckling.

Although we have previously discussed two widely accepted theories, in this section we will discuss inelastic buckling theories that fill the gap between short and long columns.

Suppose that the critical stress st in an intermediate column exceeds the proportional limit of the material σ_p. Recall the proportional limit is defined as the stress where the compressive stress-strain curve of the material deviated from a straight line. For some materials the proportional limit is very difficult to obtain.

Thus for intermediate column Young's modulus at that particular stress-strain point is no longer E. Instead, the Young's modulus decreases to the local tangent value, E_t. Replacing the Young's modulus E in the Euler's formula with the tangent modulus E, the critical load becomes,

$$P_{\text{cr}} = \frac{\pi^2 \, E_t \, I_{\text{zz}}}{L_{\text{e}}^2} = \frac{\pi^2 \, E_t}{R_a^2} \qquad \text{where} \qquad E_t = \frac{d\sigma}{d\varepsilon}$$

Few comments on the Tangent-Modulus Theory:

1. The proportional limit σ_p, rather than the yield stress S_{y}, is used in the formula. Although these two are often arbitrarily interchangeable, the yield stress is about equal to or slightly larger than the proportional limit for common engineering materials. However, when the forming process is taken into account, the residual stresses caused by processing can not be neglected and the proportional limit may drop up to 50% with respect to the yield stress in some wide-flange sections.

2. The tangent-modulus theory tends to underestimate the strength of the column, since it uses the tangent modulus once the stress on the concave side exceeds the proportional limit while the convex side is still below the elastic limit.

3. The tangent-modulus theory oversimplifies the inelastic buckling by using only one tangent modulus. In reality, the tangent modulus depends on the stress, which is a function of the bending moment that varies with the displacement v.

6.5 References

Collins, J. A., *Mechanical Design of Machine Elements and Machines*, 2003, John Wiley and Sons, New York, NY.

Hamrock, B. J., Schmid, S. R., and Jacobson, B., *Fundamentals of Machine Elements*, 2005, Second Edition, Mc-Graw Hill, New York, NY.

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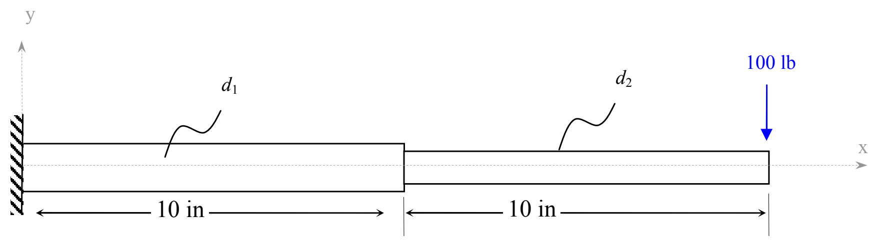
6.6 Suggested Problems

Problem 6.1.

A circular-steel solid shaft is loaded by a vertical force (a downward vertical force of 100 lb at $x = 20$ in as shown in Figure. The shaft is composed of two different diameter cross-sections. The shaft has a diameter of d_1 for $0 \leq x \leq 10$ and a diameter of d_2 for $10 \leq x \leq 20$. All loads are applied at the shafts neutral axis. Take:

$$d_2 = 3\, d_1$$



(a) Determine the slope equations: $v'(x)$ for $0 < x < 20$.

(b) Determine the deflection equations: $v(x)$ for $0 < x < 20$.

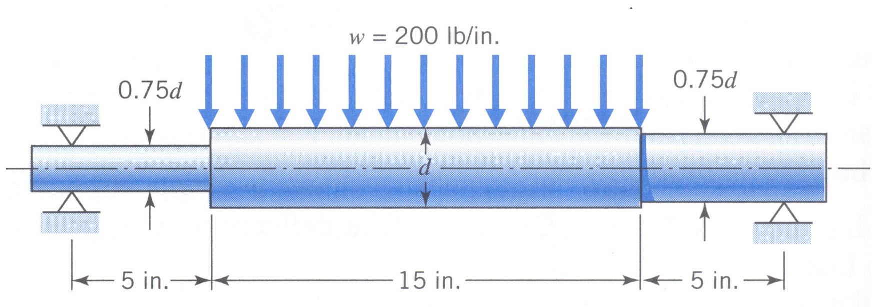
□

Problem 6.2.

Figure below shows a steel shaft supported by self-aligning bearings and subjected to a uniformly distributed loads. If $d = 2$ in,

(a) Determine the slope equations: $v'(x)$ for $0 < x < 25$.

(b) Determine the deflection equations: $v(x)$ for $0 < x < 25$.



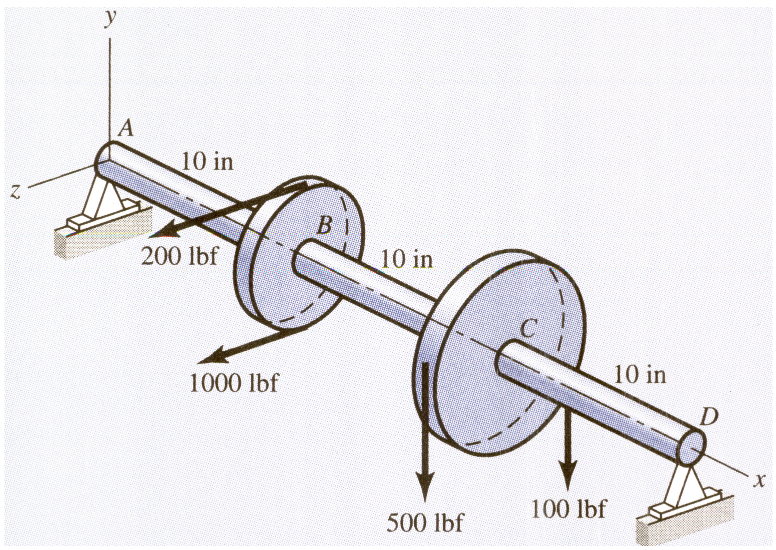
□

Problem 6.3.

Illustrated in the figure is a 1.25 in diameter steel countershaft that supports two pulley. Two pulleys are keyed to the shaft where pulley B is of diameter 4.0 in and pulley C is of diameter 8.0 in. Pulley C delivers power to a machine causing a tension of 500 lb in the tight side of the belt and 100 lb–in the loose side, as indicated. Pulley C receives power from a motor. The belt tensions on pulley C have a tension of 1000 lb in the tight side of the belt and 200 lb–in the loose side, as indicated. Assume that the bearings constitute simple supports:

(a) Find the deflection of the shaft in the y direction at pulleys A and B ($v(x)$).

(b) Find the deflection of the shaft in the z direction at pulleys A and B ($w(x)$).



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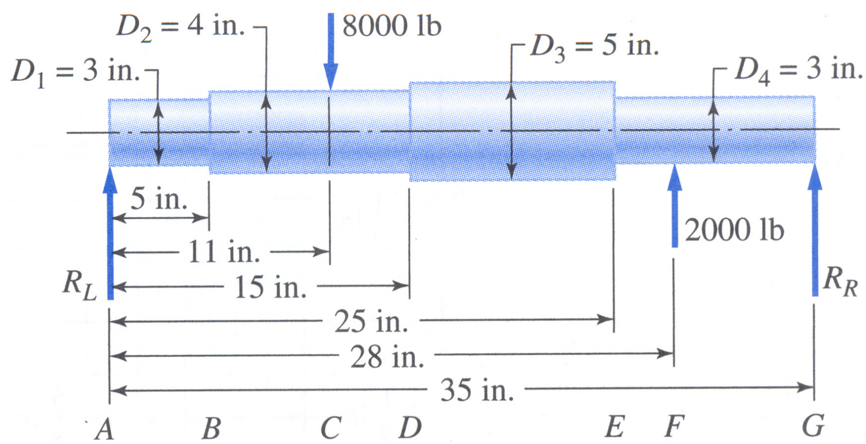
Problem 6.4.

Figure below shows a steel shaft supported by self-aligning bearings and subjected to a point loads.

(a) Determine the slope equations: $v'(x)$ for $0 < x < 35$.

(b) Determine the deflection equations: $v(x)$ for $0 < x < 35$.



□

Problem 6.5.

Problem 4.45 of textbook.

□

Problem 6.6.

Problem 4.24 of textbook. (No need to find the deflection using Castigliano's Theorem) Using methods learned in class, determine the maximum von Mises stress at the built-in cross-section.

□

Problem 6.7.

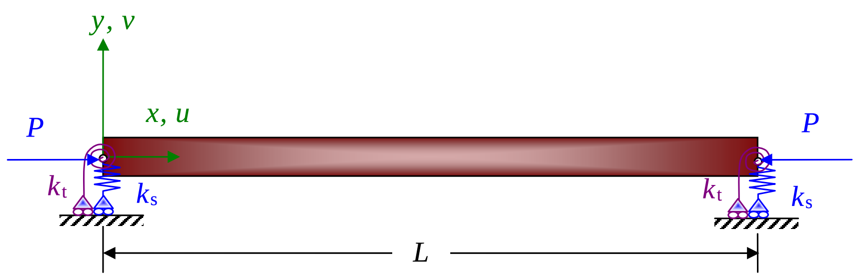


Figure 6.9: A spring-supported beam column subject to an axial load.

The uniform column with bending stiffness EI_{zz}, shown in Fig. 6.9, is spring supported at $x = 0$ and $x = L$. Has both extensional linear springs and torsional springs. Take

$$k_s = \alpha \frac{EI_{zz}}{L^3} \qquad k_t = \beta \frac{EI_{zz}}{L}$$

1. Solve the differential equations to obtain the exact critical load P_{cr} and associated buckling mode.
2. Using the principle of virtual work, determine the approximate critical load P_{cr} and associated buckling mode.
3. What happens when $\alpha \to \infty$? What happens when $\beta \to \infty$? What happens when both $\alpha, \beta \to \infty$?

□

Problem 6.8.

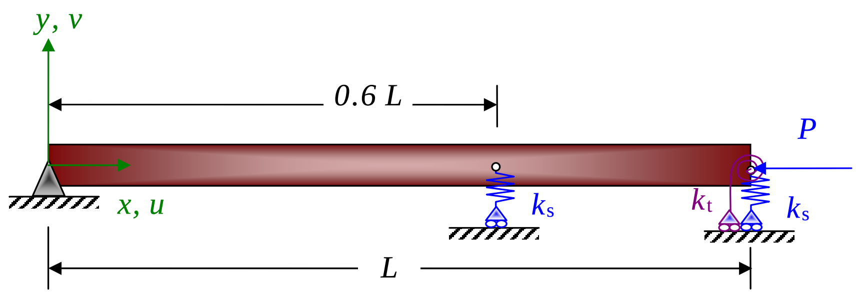


Figure 6.10: A simply-supported beam column subject to an axial load.

The uniform column with bending stiffness EI_{zz}, shown in Fig. 6.10, is pinned at $x = 0$. Has to extensional linear springs located at $x = 0.6\,L$ and at $x = L$. Take

$$k_s = \alpha\,\frac{EI_{zz}}{L^3} \qquad k_t = \beta\,\frac{EI_{zz}}{L}$$

1. Solve the differential equations to obtain the exact critical load P_{cr} and associated buckling mode.
2. What happens when $\alpha \to \infty$? What happens when $\beta \to \infty$? What happens when both $\alpha, \beta \to \infty$?

□

Chapter 7
Uncertainties in Design

The effects of uncertainties have been recognized using traditional approaches. These approaches simplify the problem by considering the uncertain parameters as deterministic and account for uncertainties using empirical design safety factors. However, the conventional methods of design and analysis may not be appropriate for problems involving innovative design because the factors of safety are based on experience and there may be no experience available for these problems.

In fact, structures have inherent uncertainties involved in the manufacturing process, and the end product may have significant variations in properties around the mean values. Thus the uncertainties in material and geometric properties should be taken as random in the analysis. In this context, the safety factors cannot be properly considered. Moreover, these safety factors do not provide any information on how the different parameters influence the overall behavior of the structure.

Thus nowadays probabilistic approaches are becoming more popular when designing for safety. Here we shall begin our discussion with the commonly known deterministic approach and complete our discussion with simple reliability analysis.

7.1 Design Safety factors

Design safety factors refers to system design features and operating characteristics that serve to minimize the potential for human or machine errors or failures that cause injurious accidents or death.

7.1.1 Definition of Safety factor

Safety factor is a simple ratio and is defined

$$n_{\text{SF}} = \frac{\sigma_{\text{all}}}{\sigma_{\text{req}}} \tag{7.1}$$

where n_{SF} is the design safety factor. The allowable stress σ_{all} is the stress limit to which a material can be stressed without damage of any kind:

$$S_{\text{yield}}, \quad \sigma_{\text{fatigue limit}}, \quad \sigma_{\text{buckling}}, \quad \sigma_{\text{fracture}}, \ldots$$

The required stress σ_{req} is the stress calculated for maximum service load condition (the necessary stress to support the loads). Since

$$\sigma_{\text{all}} > \sigma_{\text{req}} \qquad \rightarrow \qquad \frac{\sigma_{\text{all}}}{\sigma_{\text{req}}} > 1$$

Thus the design safety factor is a ratio greater than one. That is, capacity must be greater than load and strength must be greater than stress.

A large safety factor usually means a safer design, however, more material is used in the design with a corresponding increase in cost and weight. Therein lies one of the fundamental trade-offs in engineering design: cost vs. safety. Reducing cost is always a business goal, while the costumer demands increased safety. Thus it is highly important for the design engineer to choose an adequate safety factor to safeguard public safety at an affordable cost.

7.1.2 Factors in the Selection of a Safety Factor

The selection of an appropriate safety factor is based primarily on the following five factors:

1. **Degree of uncertainty about loading:** In some situations loads can be determined with virtual certainty. The centrifugal forces in the rotor of an alternating-current motor cannot exceed those calculated for synchronous speed. The loads acting on an engine valve spring are definitely established by the *valve open* and *valve closed* positions (however, in a later chapter we will mention *spring surge*, which could introduce a degree of uncertainty). But what loads should be used for the design of automotive suspension components, whose loads can vary tremendously depending on the severity of use and abuse? And what about a comparable situation in a completely new kind of machine for which there is no previous experience to serve as a guide? The greater the uncertainty, the more conservative the engineer must be in selecting an appropriate design overload or safety factor.

2. **Degree of uncertainty about material strength.:** Ideally, the engineer would have available extensive data pertaining to the strength of the material as fabricated into the actual (or very similar) parts, and tested at temperatures and in environments similar to those actually encountered. But this is seldom the case. More often, the available material strength data pertain to samples smaller than the actual part, which have not experienced any cold working in part fabrication, and which have been tested at room temperature in ordinary air. Moreover, there is bound to be some variation is strength from one test specimen to another. Sometimes the engineer must work with material test data for which such information as specimen size and degree of data scatter (and the relationship between the reported single value and the total range of scatter) are unknown. Furthermore, the material properties may sometimes change significantly over the service life of the part, The greater the uncertainty about all these factors, the larger the safety factor that must be used.

3. **Uncertainties in relating applied loads to material strength via stress analysis:** At this point the reader is already familiar with a number of possible uncertainties, such as (a) validity of the assumptions involved in the standard equations for calculating nominal stresses, (b) accuracy I in determining the effective stress concentration factors, (c) accuracy in estimating residual stresses,

if any, introduced in fabricating the I part, and (d) suitability of any failure theories and other relationships I used to estimate *significant strength* from available laboratory strength test data.

4. **Need to conserve:** The need to conserve material, weight, space, or dollars

5. **Consequences of failurehuman safety and economics:** If the consequences of failure are catastrophic, relatively large safety factors must, of course, be used. In addition, if the failure of some relatively inexpensive part could cause extensive shutdown of a major assembly line, simple economics dictates increasing the cost of this part severalfold (if necessary) in order to virtually eliminate the possibility of its failure An important item is the nature of a failure. If failure is caused I by ductile yielding, the consequences are likely to be less severe than I if caused by brittle fracture. Accordingly, safety factors recommended in handbooks are invariably larger for brittle materials.

6. **Cost of providing a large safety factor:** This cost involves a monetary I consideration and may also involve important consumption of re-1 sources. In some cases, a safety factor larger than needed may have serious consequences. A dramatic example is a hypothetical aircraft with excessive safety factors making it too heavy to fly! With respect to the design of an automobile, it would be possible to increase safety factors on structural components to the point that a *maniac* driver could hardly cause a failure even when trying. But to do so would penalize *sane* drivers by requiring them to pay for stronger components than they can use. More likely, of course, it would motivate them to buy competitor's cars! Consider this situation. Should an I automotive engineer increase the cost per car by $10 in order to avoid I 100 failures in a production run of a million cars, where the failures would not involve safety, but would entail a $100 repair? That is,

7.1.3 Selection of Design Safety Factor

Selection of a design safety factor must be undertaken with care since there are unacceptable consequences associated with selected values that are either too low or too high. If the selected value is too small, the probability of failure will be too great. If the selected value is too large, the size, weight, or cost may be too high. Proper safety factor selection requires a good working knowledge of the limitations and assumptions in the calculation models or simulation programs used, the pertinent properties of the proposed materials, and operational details of the proposed application. Design experience is extremely valuable in selection of an appropriate design safety factor, but a rational selection may be made even with limited experience.

The method suggested by Collins breaks the selection down into a series of semiquantitative smaller decisions that may be weighted and empirically recombined to calculate an acceptable value for the design safety factor, tailored to the specific application. To implement the selection of a design safety factor, consider separately each of the following eight rating factors:

a) **Accuracy of loads knowledge:** The accuracy with which the loads, forces, deflections, or other failure-inducing agents can be determined

b) **Accuracy of stress calculation:** The accuracy with which the stresses or other loading severity parameters can be determined from the forces or other failure-inducing agents

c) **Accuracy of strength knowledge:** The accuracy with which the failure strengths or other measures of failure can be determined for the selected material in the appropriate failure mode

d) **Need to conserve:** The need to conserve material, weight, space, or dollars

e) **Seriousness of failure consequences:** The seriousness of the consequences of failure in terms of human life and/or property damage

f) **Quality of manufacture:** The quality of workmanship in manufacture

g) **Conditions of operation:** The conditions of operation

h) **Quality of inspection:** The quality of inspection and maintenance available or possible during operation

A semiquantitative assessment of these rating factors may be made by assigning a *rating number*, ranging in value from -4 to +4, to each one. These rating numbers (RNs) have the following meanings:

RN=1 *mild need* to modify n_{SF}

RN=2 *moderate need* to modify n_{SF}

RN=3 *strong need* to modify n_{SF}

RN=4 *extreme need* to modify n_{SF}

Further, if there is a need to increase the safety factor, the selected rating number is assigned a positive (+) sign; if it is to decrease the safety factor, the selected rating number is to assign a negative (−) sign. The next step is to calculate the algebraic sum of the eight rating numbers:

$$t = \sum_{i=1}^{8} RN_i$$

Now using the above result, the design safety factor may be empirically estimated from:

$$n_{\text{SF}} = \begin{cases} 1 + \dfrac{(10+t)^2}{100}, & \text{for } t \geq -6 \\ 1.15, & \text{for } t < -6 \end{cases} \tag{7.2}$$

In general, the design safety factor will be bound by:

$$1.15 \leq n_{\text{SF}} \leq 5 \tag{7.3}$$

although for lightweight structures the safety factor is small as possible and for some machinery we might have safety factors higher than 5.

7.1.4 Recommended Values for a Safety Factor

Having read through this much philosophy of safety factor selection, the reader is entitled to have, at least as a guide, some suggestions values of safety factor that have been found useful. For this purpose, the following recommendations of Joseph Vidosic are suggested. These safety factors are based on yield strength.

1. SF = 1.25 to 1.5 for exceptionally reliable materials used under controllable conditions and subjected to loads and stresses that can be determined with certaintyused almost invariably where low weight is a particularly important consideration.
2. SF = 1.5 to 2 for well-known materials, under reasonably constant environmental conditions, subjected to loads and stresses that can be determined readily.
3. SF = 2 to 2.5 for average materials operated in ordinary environments and subjected to loads and stresses that can be determined.
4. SF = 2.5 to 3 for less tried materials or for brittle materials under average conditions of environment, load, and stress.
5. SF = 3 to 4 for untried materials used under average conditions of environment, load, and stress.
6. SF = 3 to 4 should also be used with better-known materials that are to be used in uncertain environments or subjected to uncertain stresses.
7. Repeated loads: the factors established in items 1 to 6 are acceptable but must be applied to the endurance limit rather than to the yield strength of the material.
8. Impact forces: the factors given in items 3 to 6 are acceptable, but an impact factor should be included.
9. Brittle materials: where the ultimate strength is used as the theoretical maximum, the factors presented in items 1 to 6 should be approximately doubled.
10. Where higher factors might appear desirable, a more thorough analysis of the problem should be undertaken before deciding on their use.

7.1.5 Example

You have been asked to propose a value for the design safety factor to be used in determining the dimensions for the main landing gear support for a new executive jet aircraft. It has been determined that the application may be regarded as “average” in many respects, but the material properties are known a little better than for the average design case, the need to conserve weight and space is strong, there is a strong concern about threat to life and property in the event of a failure, and the quality of inspection and maintenance is regarded as excellent. What value would you propose for the design safety factor?

Solution:

Based on the information given, the *rating numbers* assigned to each of the eight rating factors might be Thus,

Table 7.1: Semiquantitative assessment of rating factors

Rating Factor		Selected Rating Number (RN)
1. Accuracy of loads knowledge	:	0
2. Accuracy of stress calculation	:	0
3. Accuracy of strength knowledge	:	-1
4. Need to conserve	:	-3
5. Seriousness of failure consequences	:	+3
6. Quality of manufacture	:	0
7. Conditions of operation	:	0
8. Quality of inspection	:	-4

$$t = 0 + 0 - 1 - 3 + 3 + 0 + 0 - 4 = -5$$

Since $t \geq -6$,

$$n_{\text{SF}} = 1 + \frac{(10+t)^2}{100} = 1 + \frac{(10-5)^2}{100} = 1.25$$

The recommended value for a design safety factor appropriate to this application would be:

$$n_{\text{SF}} = 1.25$$

7.2 Margin of Safety

Margins of Safety is an index indicating the amount beyond the minimum necessary; in other words, the margin of safety is the strength of the material minus the anticipated stress and is defined as:

$$MS = \frac{\text{excess strength}}{\text{required strength}} = \frac{\sigma_{\text{all}} - \sigma_{\text{req}}}{\sigma_{\text{req}}} = \frac{\sigma_{\text{all}}}{\sigma_{\text{req}}} - 1 = n_{\text{SF}} - 1 \tag{7.4}$$

Lightweight structures have MS small as possible. A margin of safety of zero, implies no safety was considered in the design:

$$n_{\text{SF}} = 1 \quad \rightarrow \quad MS = 0 \quad \text{onset of failure}$$

Negative margin of safety means failure has occurred:

$$n_{\text{SF}} < 1 \quad \rightarrow \quad MS < 0 \quad \text{failure occurred}$$

Thus when designing for safety using deterministic approaches, we must always ensure

$$n_{\text{SF}} > 1 \qquad \rightarrow \qquad MS > 0 \qquad \text{safe}$$

Example 7.1.

It is desired to design a shaft with a 20% margin of safety. What is the safety factor?
The margin of safety is 20% or 0.20. Thus,

$$n_{\text{SF}} = MS + 1 = 0.20 + 1 = 1.20$$

The factor of safety is 1.20.

End Example □

7.3 Probabilistic Approach

The analysis in the previous section is only valid for *perfect* structures, those structures without imperfections. However, uncertainties lead to imperfections in the structure and the deterministic analysis may be no longer valid. Thus here describe a method to perform a more accurate analysis.

7.3.1 Random Variables

In a problem involving uncertainty, one first conducts statistical analysis on the random variables. This can be obtained experimentally or using sampling techniques. Then using this information one calculates the influence of the randomness of the random variables on the wanted response.

A random variable is defined as an uncertain parameter, for example, modulus of elasticity, length of the beam, width, etc. The independent random variables are denoted as

$$\boldsymbol{r} = \{r_1, r_2, \ldots, r_n\} \tag{7.5}$$

where r_i's are the different random variables in our problem. As for an example, the random variables could be

1. axial Young's modulus, E_{xx}

2. yield strength, S_{yield}

3. loads, P

4. cross-sectional properties, d (diameter)

7.3.2 Probability Density Function

For independent random variables, the probability density function can be expressed as follows:

$$F(r_1, r_2, \ldots, r_n) = \prod_{i=1}^{n} f_i(r) \tag{7.6}$$

The probability density function (PDF) does not provide information on the probability but only indicates the nature of the randomness. Among the used density functions in the analysis of structures are the Beta distribution, Normal or Gaussian distribution, Lognormal distribution, and Weibull distribution. From these, the most commonly used distribution is Gaussian:

$$f_i(r) = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{r-\mu_i}{\sigma_i}\right)^2\right] \tag{7.7}$$

where σ_i^2 and μ_i are the variance and the mean value of the i^{th} random variable, respectively.

7.3.3 Reliability Analysis

In progress...

7.4 References

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Hamrock, B. J., Schmid, S. R., and Jacobson, B., *Fundamentals of Machine Elements*, 2005, Second Edition, Mc-Graw Hill, New York, NY.

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7.5 Suggested Problems

Problem 7.1.

See problem end of chapter 5 of your textbook.

□

Chapter 8
Failure Theories for Static Loading

Instructional Objectives of Chapter 8

After completing this chapter, the student should be able to:

1. Understand and explain the purpose of safety factors and margins of safety.
2. Perform failure analysis under yielding.
3. Perform failure analysis under fracture.

In the past few chapters, we have learned how to obtain the principal stresses for the design of various structures. Our main focus in this chapter is a safe design and safe failure, a main requirement for aircraft design. Here we will explore the various failure criterions subject to static loading. Design for static loading dictates that all loading variables remain constant:

a) Magnitude of load is constant

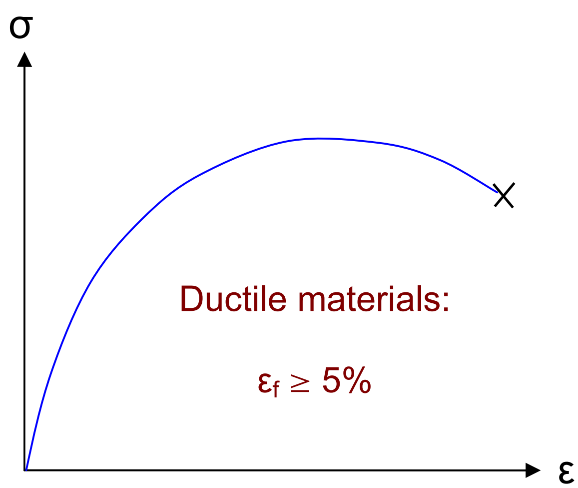
b) Direction of load is constant

c) Point of application of the load is fixed.

Let us define a static load as a slowly varying load applied at a point.

8.1 Ductile and Brittle Failure Theories

Now let us see what governs a three-dimensional state of stress in yielding. Before we do so it is important to understand the behavior of structural metals. Structural metal behavior is typically classified as being ductile or brittle, although under special situations, a material normally considered ductile can fail in a brittle manner.



Ductile materials	Brittle materials
The true strain at fracture is $\varepsilon_f \geq 5$ % in 2 inches.	The true strain at fracture is $\varepsilon_f < 5$ % in 2 inches.
The yield strength is identifiable and is often the same in compression as in tension ($S_{\rm yt} \approx S_{\rm yc} = S_{\rm yield}$).	The yield strength is not identifiable. Typically classified by ultimate tensile strength $S_{\rm ut}$ and ultimate compressive strength $S_{\rm uc}$, where $S_{\rm uc}$ is a positive quantity.
A single tensile test is sufficient to characterize the material behavior of a ductile material, $S_{\rm y}$ and $S_{\rm ut}$.	Two material tests, a tensile test and a compressive test, are required to characterize the material behavior of a brittle material, $S_{\rm uc}$ and $S_{\rm ut}$. The compressive strength is significantly higher than its tensile strength ($S_{\rm uc} \gg S_{\rm ut}$).
Governed by Yielding Criteria	Governed by Fracture Criteria
Generally accepted theories for yielding criteria are: a) Distortion Energy Criterion b) Maximum-Shear-Stress Criterion ($S_{\rm yt} \doteq S_{\rm yc}$) c) Ductile Coulomb-Mohr Criterion ($S_{\rm yt} < S_{\rm yc}$)	Generally accepted theories for fracture criteria are: a) Maximum-Normal-Stress Criterion b) Brittle Coulomb-Mohr Criterion

8.2 3-D Stress State Failure Theories: Brittle Materials

An important attribute in design with brittle materials is they do not possess defense against stress concentration opposed to ductile materials. Thus even small scratches and cracks as naturally occur in their fabrication can lead to brittle fracture. For this reason, brittle materials have to be used with extreme caution in tension structures. For such material one should evaluate the stress concentration factors.

8.2.1 Maximum Normal Stress Criterion

Postulate: *Failure occurs when one of the three principal stresses equals the strength.*

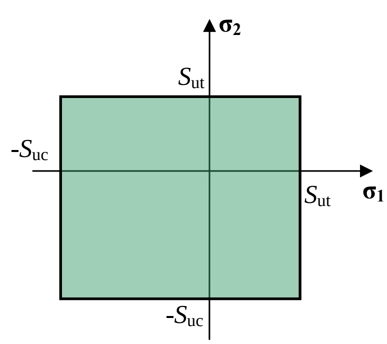
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials. The maximum stress criterion states that failure occurs when the maximum (normal) principal stress reaches either the uniaxial tension strength S_{ut}, or the uniaxial compression strength S_{uc},

$$-S_{\text{uc}} < (\sigma_1, \sigma_2, \sigma_3) < S_{\text{ut}}$$

where σ_1, σ_2, and σ_3 are the principal stresses for 3-D stress. Recall:

$$\sigma_1 > \sigma_2 > \sigma_3$$

To better grasp this Criterion, consider the plane stress problem ($\sigma_3 = 0$). Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone depicted below,



The factor of safety can be obtain by:

$$\sigma_1 = \frac{S_{\text{ut}}}{n_{\text{SF}}} \quad \sigma_1 \geq \sigma_3 \geq 0$$
$$\sigma_1 \geq 0 \geq \sigma_3 \quad \text{and} \quad \left|\frac{\sigma_3}{\sigma_1}\right| \leq \frac{S_{\text{uc}}}{S_{\text{ut}}} \tag{8.1}$$

$$\sigma_3 = -\frac{S_{uc}}{n_{SF}} \quad \sigma_1 \geq 0 \geq \sigma_3 \quad \text{and} \quad \left|\frac{\sigma_3}{\sigma_1}\right| > \frac{S_{uc}}{S_{ut}} \qquad (8.2)$$
$$0 \geq \sigma_1 \geq \sigma_3$$

where S_{uc} is the uniaxial ultimate compression strength and S_{ut} the uniaxial ultimate tensile strength.

In short, according to the Maximum-Normal-Stress Theory, as long as stress state falls within the box, the material will not fail. For a general three-dimensional state of stress, the design criteria is governed by

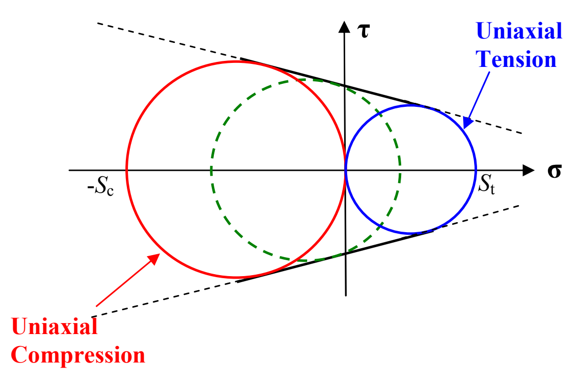
$$\frac{\sigma_1}{S_{ut}} \leq \frac{1}{n_{SF}} \quad \text{for} \quad \begin{cases} \text{(a)} \quad \sigma_1 \geq \sigma_3 \geq 0 \\ \text{(b)} \quad \sigma_1 \geq 0 \geq \sigma_3 \quad \text{and} \quad \left|\frac{\sigma_3}{\sigma_1}\right| \leq \frac{S_{uc}}{S_{ut}} \end{cases}$$

$$-\frac{\sigma_3}{S_{uc}} \leq \frac{1}{n_{SF}} \quad \text{for} \quad \begin{cases} \text{(c)} \quad \sigma_1 \geq 0 \geq \sigma_3 \quad \text{and} \quad \left|\frac{\sigma_3}{\sigma_1}\right| > \frac{S_{uc}}{S_{ut}} \\ \text{(d)} \quad 0 \geq \sigma_1 \geq \sigma_3 \end{cases}$$

where $n_{SF} = 1$ at onset of failure (fracture begins). One should evaluate the corresponding case(s) and choose the design that falls inside the box.

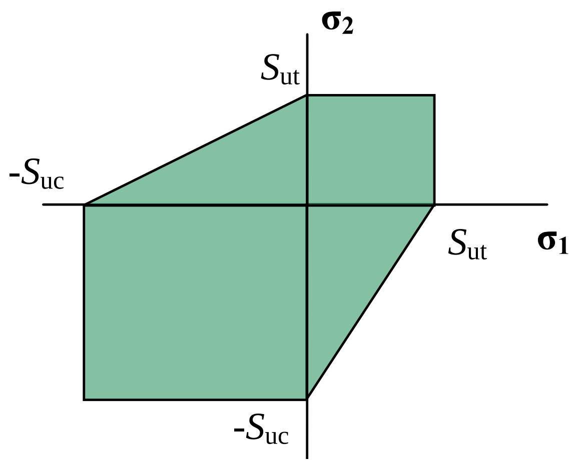
8.2.2 Brittle Coulomb-Mohr Criterion

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous Mohr's Circle. Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of 2-D stress. Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown in the figure below,



The left circle is for uniaxial compression at the limiting compression stress S_{uc} of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress S_{ut}.

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state. Each case defines the maximum allowable values for the two principal stresses to avoid failure. For the plane stress problem:



$$\sigma_1 = \frac{S_{ut}}{n_{SF}} \qquad \sigma_1 \geq \sigma_3 \geq 0$$

$$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} = \frac{1}{n_{SF}} \qquad \sigma_1 \geq 0 \geq \sigma_3 \tag{8.3}$$

$$\sigma_3 = -\frac{S_{uc}}{n_{SF}} \qquad 0 \geq \sigma_1 \geq \sigma_3$$

where S_{uc} is the uniaxial ultimate compression strength and S_{ut} the uniaxial ultimate tensile strength.

For a general three-dimensional state of stress, the design criteria is governed by

$$\left| \frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} \right| \leq \frac{1}{n_{SF}}$$

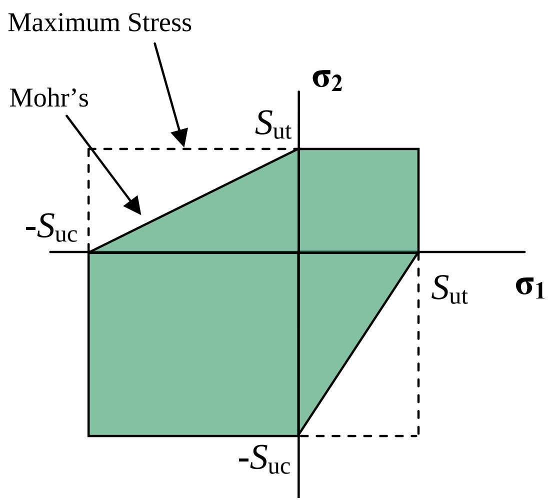
where $n_{SF} = 1$ at onset of failure (fracture begins).

8.2.3 Comparison of MNS and BCM Criterions

Also shown on the figure is the maximum stress criterion (dashed line). The MNS theory is less conservative than Mohr's theory since it lies outside Mohr's boundary. The design safety factors are compared as

$$n_{SF}\Big|_{\text{MNS}} \geq n_{SF}\Big|_{\text{BCM}}$$

Thus BCM Theory is the most conservative one.



8.3 3-D Stress State Failure Theories: Ductile Materials

An important attribute in design with ductile materials is their capacity to accommodate stress concentrations through plastic deformation and hence to redistribute the stresses more evenly. Stress concentrations occur at stress raisers which are either geometric discontinuities (e.g., holes, sharp corners, cracks, fillets, etc.) and/or material discontinuities (notches). The capacity to redistribute stresses at stress riser make's a ductile material "tough", giving the material a defense mechanism against stress concentrations. Thus for static loads with high stress concentrations, we usually take the stress concentration factor as:

$$K_t \rightarrow 1$$

Ductile engineering materials are those for which static strength in engineering applications is limited by yielding and not fracture.

The yield stress S_{yield} is determined from the tensile test data, but the tensile test is designed to produce a uniaxial state of stress. However, we would like to know what governs yielding under combined states of stress that occur in structural components under service loads. Although no theoretical way to correlate yielding in a three-dimensional state of stress with yielding in the uniaxial tensile test exists, three empirical equations have been proposed:

1. Aka Distortion Energy Criterion. Also known as Mises Yield Criterion or Octahedral Shear-stress Criterion. *Good for ductile materials but should not be used for brittle materials.*

2. Maximum Shear Stress Criterion. Also known as Aka Tresca Criterion. *Good for ductile materials with tensile yield strength approximately equal to compressive yield strength and should not be used for brittle materials.*

3. Ductile Coulomb-Mohr. *Good for ductile materials with tensile yield strength different to compressive yield strength and should not be used for brittle materials.*

Aka Distortion Energy Criterion and Maximum Shear Stress Criterion are based on:

1. State of stress can be completely described by the magnitude and direction of the principal stresses. For an isotropic material the principal stress directions are unimportant.

2. Experiments show that hydrostatic state of stress does **NOT** effect yielding.

8.3.1 Aka Distortion Energy Criterion

Postulate: *Yielding will occur when the distortion-energy per unit volume equals the distortion-energy per unit volume in a uniaxial tension specimen stressed to its yield strength.*

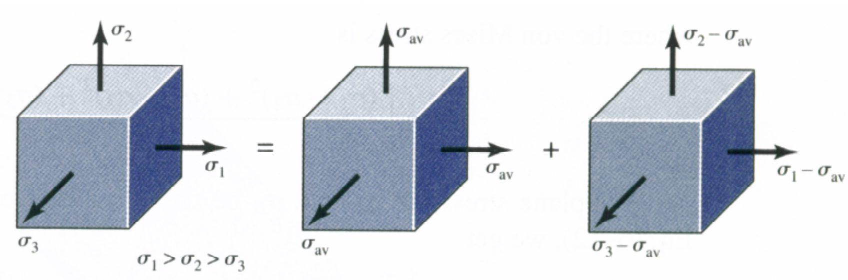
$$\tau_{\text{oct}}\Big|_{\text{3-D}} = \tau_{\text{oct}}\Big|_{\text{1-D}} \tag{8.4}$$

The basis for the maximum distortion energy theory of failure is that the overall strain energy is composed of two parts. The first part is the energy associated with merely changing the volume of the part while the second part is associated with the distortion of the part. Thus, the total strain energy per unit volume u can be written as

$$u = u_v + u_d \tag{8.5}$$

where u_v is the energy of volume change per unit volume and u_d is the energy of distortion per unit volume. It is this distortion part of the strain energy that is the basis for this failure theory. The hypothesis for this theory is that failure will occur in the complex part when the distortion energy per unit volume exceeds that for a simple uniaxial tensile test at failure.

For purposes of describing this failure theory, the principal normal stresses can be thought of as being composed of two parts that are superimposed as follows



(*a*) Triaxial stresses (*b*) Hydrostatic component (*c*) Distortional component

where σ_1, σ_2 and, σ_3 are the principal stresses. For this superposition, the relationships will be

$$\sigma_1 = \sigma_1' + \sigma_{av}$$

$$\sigma_2 = \sigma_2' + \sigma_{av}$$

$$\sigma_3 = \sigma_3' + \sigma_{av}$$

and the state of stress can be written as:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_{av} & 0 & 0 \\ 0 & \sigma_{av} & 0 \\ 0 & 0 & \sigma_{av} \end{bmatrix} + \begin{bmatrix} \sigma_1' & 0 & 0 \\ 0 & \sigma_2' & 0 \\ 0 & 0 & \sigma_3' \end{bmatrix}$$

Here σ_{av} represents the portion of the stress that causes volume change and σ_i' represents the portion of the principal normal stresses that cause distortion:

$$\sigma_1' = \sigma_1 - \sigma_{av}$$

$$\sigma_2' = \sigma_2 - \sigma_{av}$$

$$\sigma_3' = \sigma_3 - \sigma_{av}$$

Now the hydrostatic state of stress can be shown to be:

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{8.6}$$

The distortional element is subject to pure angular distortion, that is no volume change.

The total strain energy density for an element subject to the three principal stresses is:

$$u = \frac{1}{2}\left\{\sigma_1\,\varepsilon_1 + \sigma_2\,\varepsilon_2 + \sigma_3\,\varepsilon_3\right\} = \frac{1}{2\,E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\,\nu\,\left(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_1\,\sigma_3\right)\right\}$$

The total strain energy density for an element subject to the hydrostatic stresses is:

$$u_v = \frac{1}{2\,E}\left\{3\sigma_{av}^2 - 2\,\nu\,\left(3\sigma_{av}^2\right)\right\} = \frac{3\sigma_{av}^2}{2\,E}\left\{1 - 2\,\nu\right\}$$

Now substitute Eq. (8.6) into the above expression to obtain:

$$u_v = \frac{1 - 2\,\nu}{6\,E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\,\left(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_1\,\sigma_3\right)\right\}$$

Now the distortion energy is given by Eq. (8.5)

$$u_d = u - u_v = \frac{1+\nu}{3\,E}\left\{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_2)^2 + (\sigma_3 - \sigma_1)^2}{2}\right\}$$

Note that the distortion energy would be zero if $\sigma_1 = \sigma_2 = \sigma_3$. Now for a simple tensile test, at yield,

$$u_d = \frac{1+\nu}{3\,E} S_{\text{yield}}^2 \qquad (\sigma_1 = S_{\text{yield}},\ \sigma_2 = \sigma_3 = 0)$$

Further recall that von Mises stress was defined as:

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_2)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

Thus von Mises stress is a method to predict yielding in 3-D state of stress.

$$u_d\Big|_{\text{3-D}} = \frac{1+\nu}{3\,E}\,\sigma_{\text{eq}}^2$$

and it was defined as:

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} \tag{8.7}$$

Now, the Aka Distortion Energy Criterion can be expressed in terms of the von Mises stress:

$$\sigma_{\text{eq}} < S_{\text{yield}}$$

The safety factor can be derived as:

$$\sigma_{\text{eq}} = \frac{S_{\text{yield}}}{n_{\text{SF}}} \qquad \rightarrow \qquad n_{\text{SF}} = \frac{S_{\text{yield}}}{\sigma_{\text{eq}}}$$

To better explain the physical meaning of the DE Criterion let us consider a plane stress problem ($\sigma_3 = 0$). Yielding will begin when

$$\sigma_{\text{eq}} = S_{\text{yield}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = \sqrt{\sigma_1^2 - \sigma_1\,\sigma_2 + \sigma_2^2}$$

Rewriting the above:

$$S_{\text{yield}}^2 = \sigma_1^2 - \sigma_1\,\sigma_2 + \sigma_2^2$$

Now taking a transformation of $\theta = 45°$:

$$\bar{\sigma}_1 = \frac{\sigma_1 + \sigma_2}{\sqrt{2}} \qquad \bar{\sigma}_2 = \frac{\sigma_2 - \sigma_1}{\sqrt{2}}$$

Solving for $\bar{\sigma}_1$ and $\bar{\sigma}_2$,

$$\sigma_1 = \frac{\bar{\sigma}_1 - \bar{\sigma}_2}{\sqrt{2}} \qquad \sigma_2 = \frac{\bar{\sigma}_1 + \bar{\sigma}_2}{\sqrt{2}}$$

Then the criterion becomes

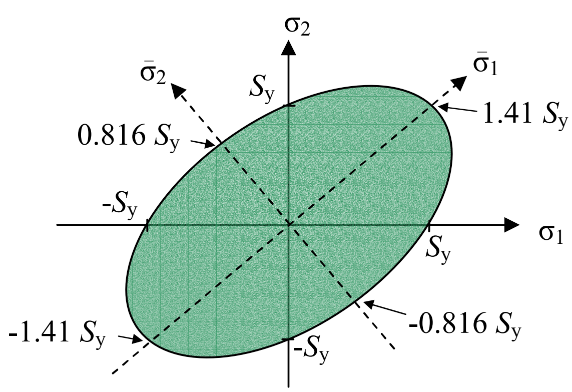
$$S_{\text{yield}}^2 = \frac{\bar{\sigma}_1^2}{2} + \frac{3\,\bar{\sigma}_2^2}{2}$$

$$1 = \frac{\bar{\sigma}_1^2}{2\,S_{\text{yield}}^2} + \frac{3\,\bar{\sigma}_2^2}{2\,S_{\text{yield}}^2}$$

$$1 = \left(\frac{\bar{\sigma}_1}{\sqrt{2}\,S_{\text{yield}}}\right)^2 + \left(\frac{\bar{\sigma}_2}{\sqrt{\frac{2}{3}}\,S_{\text{yield}}}\right)^2$$

$$1 = \left(\frac{\bar{\sigma}_1}{1.4142\,S_{\text{yield}}}\right)^2 + \left(\frac{\bar{\sigma}_2}{0.8165\,S_{\text{yield}}}\right)^2$$

The above is an ellipse rotated at 45°:



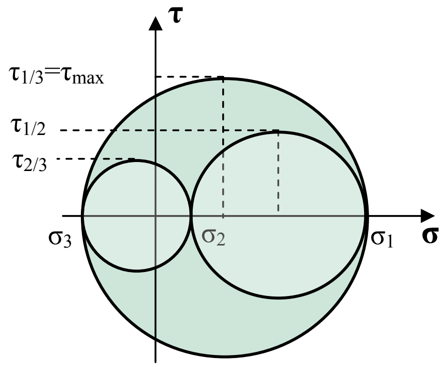
In short, according to the DE Theory, as long as stress state falls within the ellipse, the material will not fail. For a general three-dimensional state of stress, the design criteria is governed by

$$\frac{\sigma_{\text{eq}}}{S_{\text{yield}}} \le \frac{1}{n_{\text{SF}}}$$

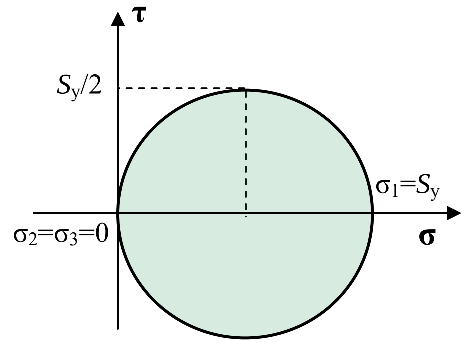
where $n_{\text{SF}} = 1$ at onset of failure (yielding begins).

8.3.2 Maximum Shear Stress Criterion

Postulate: *Yielding begins whenever the maximum shear stress in a part becomes equal to the maximum shear stress in a tension test specimen that begins to yield.*



3-D Stress State in Part



Tensile Test Specimen

Yielding begins in a 3-D stress state when the maximum shear stress τ_{max} is equal to its value at the initiation of yielding in the tension test:

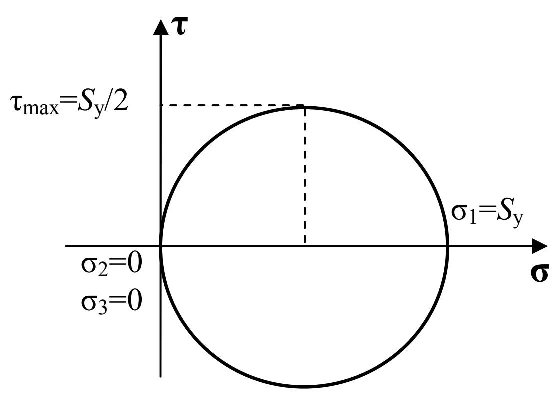
$$\tau_{\text{max}}\Big|_{\text{3-D}} = \tau_{\text{max}}\Big|_{\text{1-D}} \tag{8.8}$$

It is known that

$$\tau_{\text{max}}\Big|_{\text{1-D}} = \frac{S_{\text{yield}}}{2}$$

$$\tau_{\text{max}}\Big|_{\text{3-D}} = \left|\frac{\sigma_1 - \sigma_3}{2}\right|$$

Note that YIELDING BEGINS when (8.8) is true. In other words,



$$\tau_{\text{max}} = 0.5\, S_{\text{yield}}$$

However for design purposes this should not happen. It is wanted that

$$\tau_{\text{max}}\Big|_{\text{3-D}} < \frac{S_{\text{yield}}}{2} \tag{8.9}$$

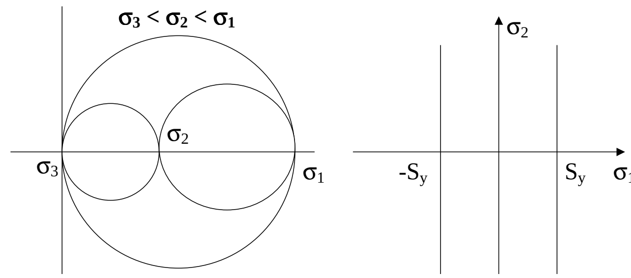
Maximum Shear Stress is another method to predict yielding in 3-D state of stress and it is defined as:

$$\sigma_1 - \sigma_3 = \pm S_{\text{yield}} \tag{8.10}$$

To better explain the physical meaning of the DE Criterion let us consider plane stress problem ($\sigma_3 = 0$). Thus let us consider three different cases with $\sigma_3 = 0$.

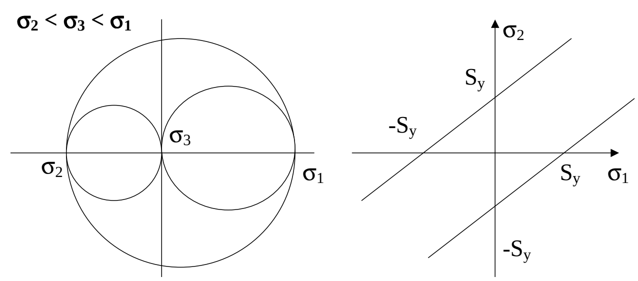
First Case:

$$\sigma_1 - \sigma_3 = \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_1 - \sigma_3 \quad = \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_1 = \quad \pm S_{\text{yield}}$$



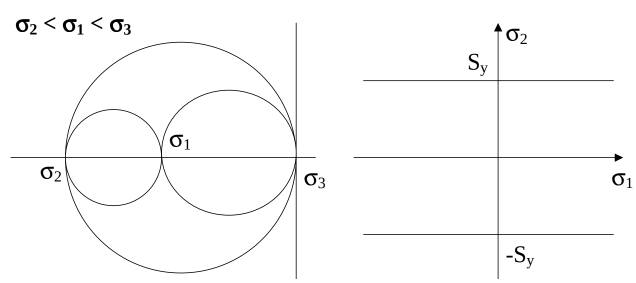
Second Case:

$$\sigma_1 - \sigma_3 = \quad \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_1 - \sigma_2 = \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_1 \quad = \sigma_2 \pm S_{\text{yield}}$$

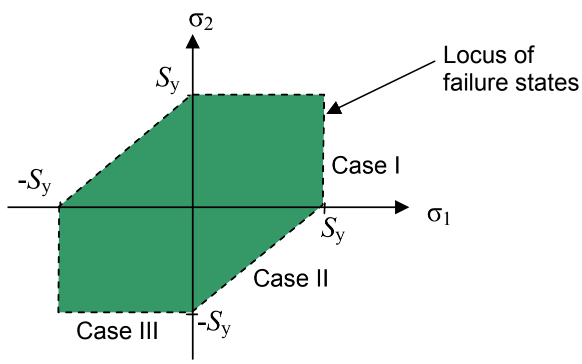


Third Case

$$\sigma_1 - \sigma_3 = \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_3 - \sigma_2 \quad = \pm S_{\text{yield}} \quad \rightarrow \quad \sigma_2 = \mp S_{\text{yield}}$$



Thus for a biaxial representation of the yield, the Maximum-Shear-Stress Theory can be represented as



In short, according to the MSS Theory, as long as stress state falls within the green area, the material will not fail. For a general three-dimensional state of stress, the design criteria is governed by

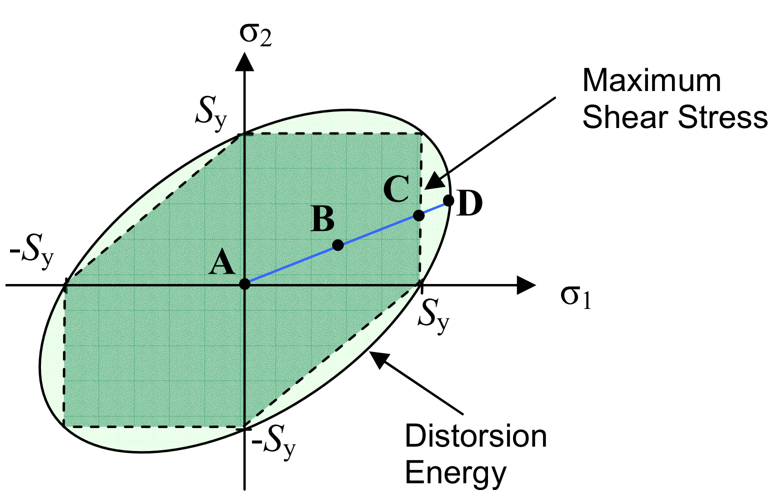
$$\tau_{\max}\Big|_{3\text{-D}} = \frac{S_{\text{yield}}}{2\,n_{\text{SF}}} \qquad \rightarrow \qquad n_{\text{SF}} = \frac{S_{\text{yield}}}{2\,\tau_{\max}}$$

or

$$\frac{2\,\tau_{\max}}{S_y} = \left|\frac{\sigma_1}{S_{\text{yield}}} - \frac{\sigma_3}{S_{\text{yield}}}\right| \leq \frac{1}{n_{\text{SF}}}$$

where $n_{\text{SF}} = 1$ at onset of failure (yielding begins).

8.3.3 Comparison of DE and MSS Criterions



The blue line is the loading line. Let point **B** represent the actual loading condition. Then the design safety factor can be calculate as follows

$$n_{\text{SF}} = \frac{\overline{AC}}{\overline{AB}} \quad \text{Maximum Shear Stress Criterion}$$

$$n_{\text{SF}} = \frac{\overline{AD}}{\overline{AB}} \quad \text{Distortion Energy Criterion}$$

From the above it is clear that, in general,

$$n_{\text{SF}}\Big|_{\text{Maximum Shear Stress Criterion}} \leq n_{\text{SF}}\Big|_{\text{Distortion Energy Criterion}}$$

Thus MSS Theory is the most conservative one.

The distortion-energy theory predicts no failure under hydrostatic stress and agrees well with all data for ductile behavior. Hence, it is the most widely used theory for ductile materials and is recommended for design purposes, although some engineers also apply the MSS Theory because of its simplicity and conservative nature.

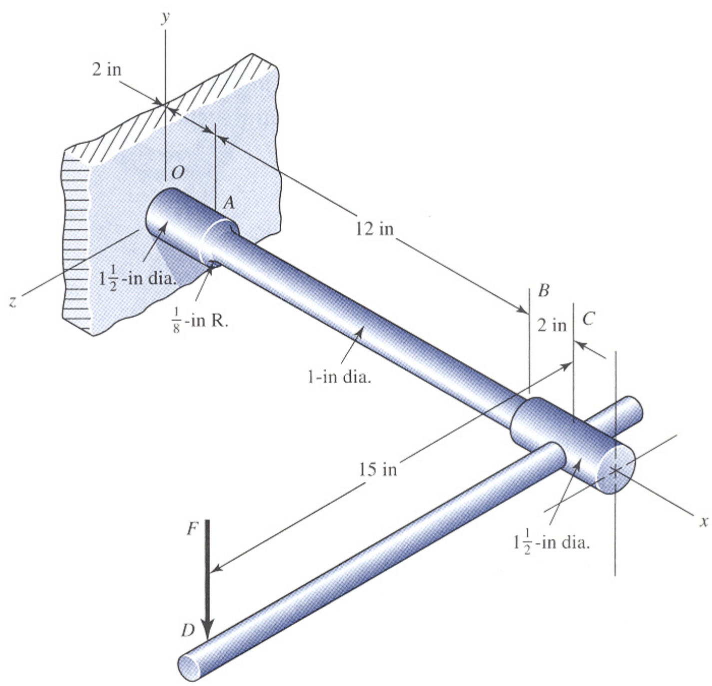
8.3.4 Ductile Coulomb-Mohr Criterion

When the tensile yield strength and compressive yield strength are significantly different for ductile materials, a variation of the brittle Coulomb-Mohr Criterion is commonly used. The criterion for a general three-dimensional state of stress is governed by

$$\left| \frac{\sigma_1}{S_{\text{yt}}} - \frac{\sigma_3}{S_{\text{yc}}} \right| \leq \frac{1}{n_{\text{SF}}}$$

where $n_{\text{SF}} = 1$ at onset of failure (fracture begins), S_{yc} the uniaxial compressive yield strength and S_{yt} the uniaxial ultimate tensile strength.

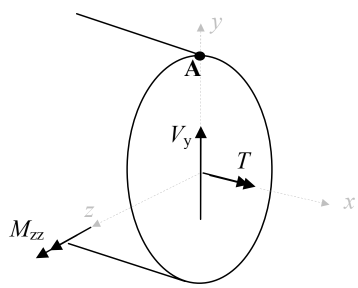
Example 8.1.



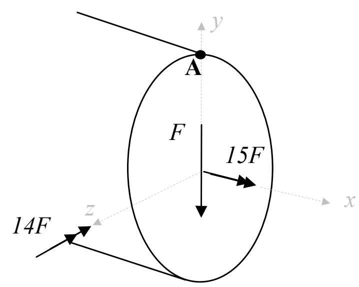
A certain force F applied at **D** near the end of the 15-in lever shown in Figure, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar **OABC**. This bar (**OABC**) is of AISI 1035 steel and it has a minimum (ASTM) yield strength of 81 ksi. We presume that this component would be of no value after yielding. Thus the force F required to initiate yielding can be regarded as the strength of the component part. Find the force F.

Solution: We will assume that lever **DC** is strong enough and hence not a part of the problem. Note that point **O** is the place of maximum bending moment but not necessarily maximum bending stress due to stress concentration at **A**. Points **A** and **O** have the same shear and torsional loads but not necessary the same shear stresses due to torque because of stress concentration at **A**.

Thus a stress element at **A** will be most critical.



Sign Convention



Free Body Diagram at A

For our sign convention:

$$M_{\rm zz} = -14\,F \qquad V_{\rm y} = -F \qquad T = 15\,F$$

Let us consider the stress element on the top surface will be subjected to a tensile bending stress and a torsional stress (note that there is no shear stress due to shear load at **A**. This point is the weakest section, and governs the strength of the assembly. For a static load acting on a ductile material, the stress concentration factor is not important.

The two stresses are

$$\sigma_{\rm xx} = -K_{\rm t_b}\,\frac{M_{\rm zz}\,y}{I_{\rm zz}} \qquad \tau_{\rm xz} = K_{\rm t_s}\,\frac{T\,r}{J_{\rm xx}}$$

At the top:

$$\sigma_{\rm xx} = -K_{\rm t_b}\,\frac{M_{\rm zz}\,(c)}{I_{\rm zz}} = -K_{\rm t_b}\,\frac{M_{\rm zz}}{Z} \qquad \tau_{\rm xz} = K_{\rm t_s}\,\frac{T\,(c)}{J_{\rm xx}} = K_{\rm t_s}\,\frac{T}{Q}$$

Using Tables:

$$\sigma_{\rm xx} = -K_{\rm t_b}\,\frac{M_{\rm zz}}{Z} = -K_{\rm t_b}\,\frac{32\,M_{\rm zz}}{\pi\,d^3}$$

$$\tau_{\rm xz} = K_{\rm t_s}\,\frac{T}{Q} = K_{\rm t_s}\,\frac{16\,T}{\pi\,d^3}$$

Since the stress concentration factor are not important here

$$K_{\rm t_b} = 1.0 \qquad K_{\rm t_s} = 1.0$$

Thus (for $d = 1''$)

$$\sigma_{\rm xx} = -(1.0)\,\frac{32\,(-14\,F)}{\pi\,(1)^3} = 142.6\,F$$

$$\tau_{\rm xz} = (1.0)\,\frac{16\,(15\,F)}{\pi\,(1)^3} = 76.4\,F$$

Now we proceed to find the principal stresses. The state of stress is

$$\underline{\boldsymbol{\sigma}}_{\rm A} = \begin{bmatrix} \sigma_{\rm xx} & \tau_{\rm xy} & \tau_{\rm xz} \\ \tau_{\rm xy} & \sigma_{\rm yy} & \tau_{\rm yz} \\ \tau_{\rm xz} & \tau_{\rm yz} & \sigma_{\rm zz} \end{bmatrix} = \begin{bmatrix} 142.6\,F & 0 & 76.4\,F \\ 0 & 0 & 0 \\ 76.4\,F & 0 & 0 \end{bmatrix} \text{ psi} \tag{8.11}$$

The stress invariants are

$$
\begin{aligned}
I_{\sigma_1} &= \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = 142.6\,F \\
I_{\sigma_2} &= -\tau_{\text{xz}}^2 = -\left(76.4\,F\right)^2 = -5836.1\,F^2 \\
I_{\sigma_3} &= 0 \qquad \text{(Plane stress)}
\end{aligned}
$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = \lambda\,\left(\lambda^2 - 142.6\,F\,\lambda - 5836.1\,F^2\right) \quad = 0$$

The principle stresses are obtained analytically as follows

$$
\begin{aligned}
\lambda_1 &= \frac{I_{\sigma_1}}{2} + \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = 76.4\,F + 104.50\,F = 175.8\,F \\
\lambda_2 &= \frac{I_{\sigma_1}}{2} - \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = 76.4\,F - 104.50\,F = -33.20\,F \\
\lambda_3 &= 0
\end{aligned}
$$

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 175.8\,F \qquad \sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -33.20\,F \qquad \sigma_2 = 0$$

The maximum stresses are

$$\sigma_1 = \max[\sigma_1, \sigma_2, \sigma_3] = 175.8\,F \quad \sigma_3 = \min[\sigma_2, \sigma_2, \sigma_3] = -33.20\,F$$

$$\tau_{\text{max}} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 104.50\,F$$

Distortion energy criterion: If we employ the DE criterion, we need to calculate the von Mises stress:

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 194.50\,F$$

The yielding criteria for DE criterion is

$$\frac{\sigma_{\text{eq}}}{S_{\text{y}}} \leq \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad \frac{194.50\,F}{81000} \quad \leq 1 \qquad (n_{\text{SF}} = 1)$$

Thus a force of $F = 416.38$ lb will cause yielding.

Maximum Shear Stress Criterion: If we employ the MSS criterion, we need to calculate the maximum overall shear stress:

$$\tau_{\text{max}} = \quad \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 104.50\,F$$

The yielding criteria for MSS criterion is

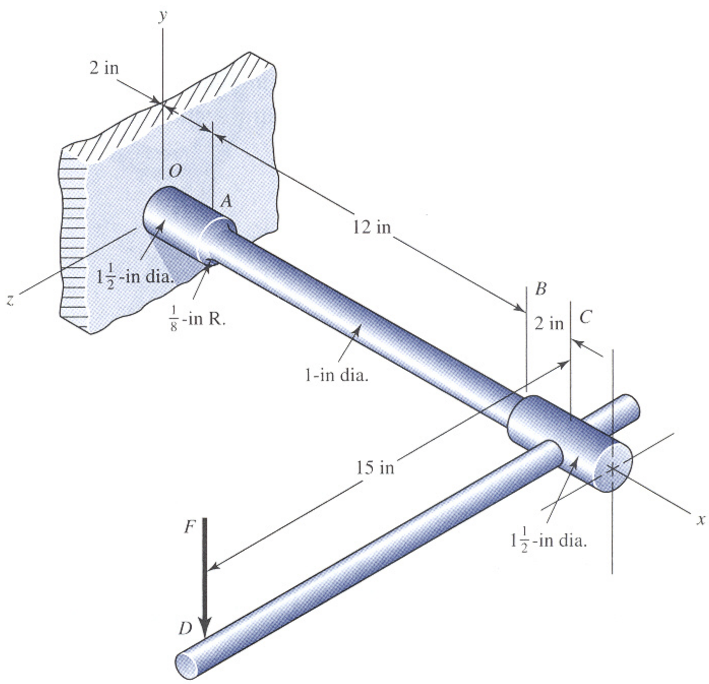
$$\frac{2\,\tau_{\text{max}}}{S_{\text{y}}} \leq \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad \frac{104.50\,F}{40500} \quad \leq 1 \qquad (n_{\text{SF}} = 1)$$

Thus a force of $F = 387.56$ lb will cause yielding.

We can see that the force F required found by MSS is about 6.9% less than the one found for the DE. As stated earlier, the MSS theory is more conservative than the DE theory.

End Example □

Example 8.2.



A certain force F applied at **D** near the end of the 15-in lever shown in Figure, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar **OABC**. This bar (**OABC**) is of AISI 1035 steel and it has a minimum (ASTM) yield strength of 81 ksi. We presume that this component would be of no value after yielding. Thus the force F required to initiate yielding can be regarded as the strength of the component part.

1. If $F = 400$ lbs, will the component fail?
2. Certain safety factor was used to ensure that the structure would not fail. If $F = 300$ lbs was assumed to cause yielding, what is the realized margin of safety?

Solution: We proceed as before.

(1) If $F = 400$ lbs, will the component fail?

The goal is to ensure that the margin of safety is positive and/or safety factor is greater than one. The two yielding theories will give slightly different solutions.

Distortion energy criterion: If we employ the DE criterion, we need to calculate the von Mises stress:

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 194.50\,F$$

The yielding criteria for DE criterion is

$$\frac{\sigma_{\text{eq}}}{S_{\text{y}}} \leq \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad 194.50\,F = \frac{81000}{n_{\text{SF}}} \qquad \rightarrow \qquad 77814.0 = \frac{81000}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 1.041$ was used. Since the safety factor is greater than one (and $MS = 0.041 > 0$) the structure is likely to not fail. Although the design engineer should consider $n_{\text{SF}} = 1.15$ for design, this design is acceptable since failure is not predicted.

Maximum Shear Stress Criterion: If we employ the MSS criterion, we need to calculate the maximum overall shear stress:

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 104.5\,F$$

The yielding criteria for MSS criterion is

$$\frac{2\,\tau_{\max}}{S_{\text{y}}} \leq \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad 104.5 = \frac{40500}{n_{\text{SF}}} \qquad \rightarrow \qquad 41799.5 = \frac{40500}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 0.97$ was used. Since the safety factor is less than one (and $MS = -0.03 < 0$) the design will fail.

Note that the safety factor required found by MSS is smaller than the one found for the DE. As stated earlier, the MSS theory is more conservative than the DE theory.

(2) Certain safety factor was used to ensure that the structure would not fail. If $F = 300$ lbs was assumed to cause yielding, what is the realized margin of safety?

Distortion energy criterion: If we employ the DE criterion, we need to calculate the von Mises stress:

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 194.50\,F$$

The yielding criteria for DE criterion is

$$\frac{\sigma_{\text{eq}}}{S_{\text{y}}} \leq \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad 194.50\,F = \frac{81000}{n_{\text{SF}}} \qquad \rightarrow \qquad 58360.50 = \frac{81000}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 1.39$ was used. Since the safety factor is greater than one the design will not fail. The margin of safety is

$$MS = n_{\text{SF}} - 1 = 0.39 = 39\%$$

Maximum Shear Stress Criterion: If we employ the MSS criterion, we need to calculate the maximum overall shear stress:

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 104.5\,F$$

The yielding criteria for MSS criterion is

$$\frac{2\,\tau_{\max}}{S_{\mathrm{y}}} \leq \frac{1}{n_{\mathrm{SF}}} \qquad \rightarrow \qquad 104.5 = \frac{40500}{n_{\mathrm{SF}}} \qquad \rightarrow \qquad 31349.6 = \frac{40500}{n_{\mathrm{SF}}}$$

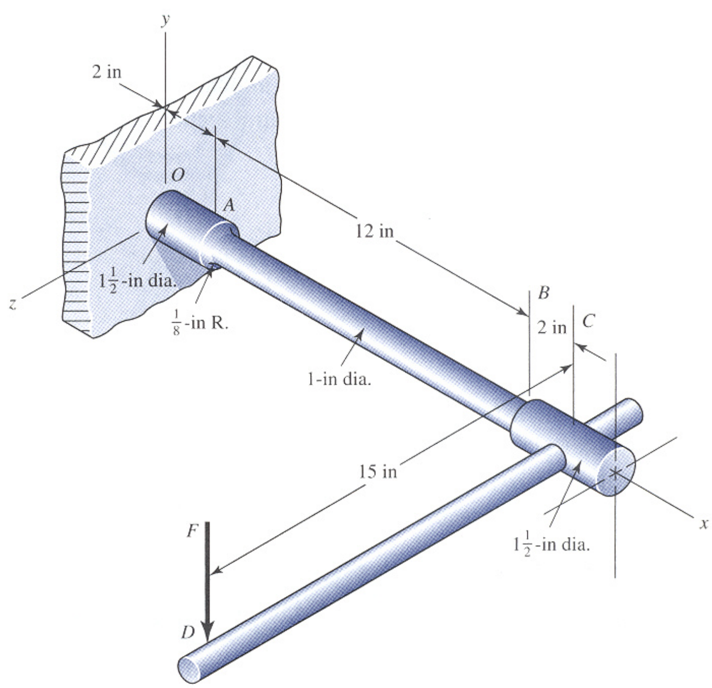
Thus a safety factor of $n_{\mathrm{SF}} = 1.29$ was used. Since the safety factor is greater than one the design will not fail. In addition it is greater than 1.15. The margin of safety is

$$MS = n_{\mathrm{SF}} - 1 = 0.29 = 29\%$$

Note that the safety factor required found by MSS is smaller than the one found for the DE. As stated earlier, the MSS theory is more conservative than the DE theory.

End Example □

Example 8.3.



A certain force F applied at **D** near the end of the 15-in lever shown in Figure, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar **OABC**. This bar (**OABC**) is made of ASTM grade 30 cast iron, machined to dimension. The force F required to fracture this part can be regarded as the strength of the component part. Find the force F.

Solution: We assume that the lever DC is strong enough, and not part of the problem. The tensile ultimate strength is 31 ksi and the compressive ultimate strength is 109 ksi. The stress element at **A** on the top surface will be subjected to a tensile bending stress and a torsional stress (just as before). This location, on the l-in diameter section fillet, is the weakest location, and it governs the strength of the assembly. Since grade 30 cast iron is a brittle material and the load is static, we should find the stress concentration factors. Thus use Charts 4.17 of your textbook for the corresponding stress concentration factors.

The two stresses are at the top are

$$\sigma_{xx} = -K_{t_b} \frac{M_{zz}}{Z} \qquad \tau_{xz} = K_{t_s} \frac{T}{Q}$$

Using Tables:

$$\sigma_{\text{xx}} = -K_{t_b}\,\frac{M_{\text{zz}}}{Z} = -K_{t_b}\,\frac{32\,M_{\text{zz}}}{\pi\,d^3} \qquad \tau_{\text{xz}} = K_{t_s}\,\frac{T}{Q} = K_{t_s}\,\frac{16\,T}{\pi\,d^3}$$

From Charts:

$$\frac{D}{d} = 1.5 \qquad \frac{r}{d} = 0.125 \qquad \Rightarrow \qquad K_{t_b} \approx 1.6 \qquad K_{t_s} \approx 1.33$$

Thus, (for $d = 1''$)

$$\sigma_{\text{xx}} = -(1.6)\,\frac{32\,(-14\,F)}{\pi\,(1)^3} = 228.165\,F$$

$$\tau_{\text{xz}} = (1.33)\,\frac{16\,(15\,F)}{\pi\,(1)^3} = 101.605\,F$$

Now we proceed to find the principal stresses. The state of stress is

$$\underline{\boldsymbol{\sigma}}_{\text{A}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 228.165\,F & 0 & 101.605\,F \\ 0 & 0 & 0 \\ 101.605\,F & 0 & 0 \end{bmatrix} \text{ psi} \tag{8.12}$$

The stress invariants are

$$\begin{aligned} I_{\sigma_1} &= \sigma_{\text{xx}} + \sigma_{\text{yy}} + \sigma_{\text{zz}} = 228.165\,F \\ I_{\sigma_2} &= -\tau_{\text{xz}}^2 = -\left(101.605\,F\right)^2 = -10323.5\,F^2 \\ I_{\sigma_3} &= 0 \qquad \text{(Plane stress)} \end{aligned}$$

The characteristic equation is

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda = \lambda\,\left(\lambda^2 - I_{\sigma_1}\,\lambda + I_{\sigma_2}\right) = \lambda\,\left(\lambda^2 - 228.165\,F\,\lambda - 10323.5\,F^2\right) = 0$$

The principle stresses are obtained analytically as follows

$$\begin{aligned} \lambda_1 &= \frac{I_{\sigma_1}}{2} + \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = 266.16\,F \\ \lambda_2 &= \frac{I_{\sigma_1}}{2} - \frac{1}{2}\sqrt{I_{\sigma_1}^2 - 4\,I_{\sigma_2}} = -38.69\,F \\ \lambda_3 &= 0 \end{aligned}$$

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 266.16\,F \qquad \sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -38.69\,F \qquad \sigma_2 = 0$$

The maximum stresses are:

$$\sigma_1 = \max[\sigma_1, \sigma_2, \sigma_3] = 266.16\,F \quad \sigma_3 = \min[\sigma_1, \sigma_2, \sigma_3] = -38.69\,F$$

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 152.77\,F \qquad \text{(Not needed for brittle failure criterions)}$$

Maximum Normal Stress criterion: First let us assume that $F > 0$. If we employ the MNS criterion, we need to check what case we are working with:

$$\sigma_1 \geq 0 \geq \sigma_3$$

Thus we need to evaluate:

$$\left|\frac{\sigma_3}{\sigma_1}\right| = \left|\frac{-38.69\,F}{266.16\,F}\right| = 0.145364 \qquad \frac{S_{\text{uc}}}{S_{\text{ut}}} = \frac{109}{31} = 3.51613$$

Since:

$$\left|\frac{\sigma_3}{\sigma_1}\right| < \frac{S_{\text{uc}}}{S_{\text{ut}}} \quad \rightarrow \quad \sigma_1 = \frac{S_{\text{ut}}}{n_{\text{SF}}} \quad \rightarrow \quad 266.85\,F = 31000 \quad (n_{\text{SF}} = 1)$$

Thus a force of $F = 116.17$ lb will cause fracture.

$$F = 116.17 \Rightarrow \begin{cases} \sigma_1 = 266.16\,F = 31000 & \text{psi} \\ \sigma_3 = -38.69\,F = -4494.18 & \text{psi} \end{cases}$$

For F both σ_1 and σ_3 fall on the *box*-limit because:

$$-S_{\text{uc}} < \sigma_1, \sigma_3 = S_{\text{ut}}$$

Thus the load for fracture is:

$$F = 116.17 \quad \text{lb}$$

Brittle Coulomb-Mohr Criterion: If we employ the BCM criterion,

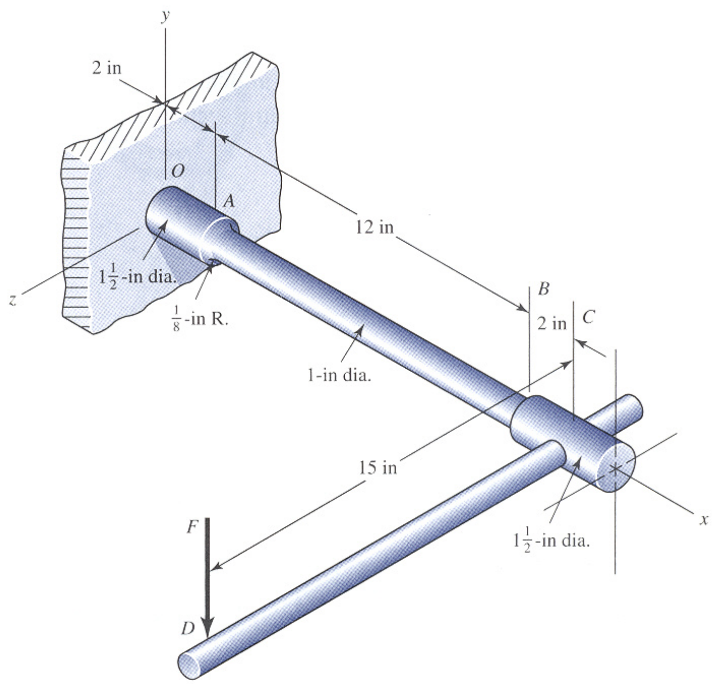
$$\left|\frac{\sigma_1}{S_{\text{ut}}} - \frac{\sigma_3}{S_{\text{uc}}}\right| = \frac{1}{n_{\text{SF}}} \quad \rightarrow \quad 0.008963\,F = 1 \quad (n_{\text{SF}} = 1)$$

Note we took $n_{\text{SF}} = 1$ for failure. Thus a force of $F = 111.57$ lb will cause fracture.

We can see that the force F required found by BCM is about 4.0% less than the one found for the MNS. As stated earlier, the BCM theory is more conservative than the MNS theory.

End Example □

Example 8.4.



A certain force F applied at **D** near the end of the 15-in lever shown in Figure, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar **OABC**. This bar (**OABC**) is made of ASTM grade 30 cast iron, machined to dimension. The force F required to fracture this part can be regarded as the strength of the component part.

1. If $F = 115$ lbs, will the component fail?
2. Certain safety factor was used to ensure that the structure would not fail. If $F = 100$ lbs was assumed to cause fracture, what is the realized margin of safety?

Solution: We proceed as before.

(1) If $F = 115$ lbs, will the component fail?

The goal is to ensure that the margin of safety is positive and/or safety factor is greater than one. The two fracture theories will give slightly different solutions. The principal stresses are:

$$\sigma_1 = 266.16\,F = 30687.8 \qquad \sigma_2 = 0 \qquad \sigma_3 = -38.69\,F = -4448.93$$

Maximum Normal Stress criterion: If we employ the MNS criterion, If we employ the MNS criterion, we need to check what case we are working with:

$$\sigma_1 \geq 0 \geq \sigma_3$$

Thus we need to evaluate and compare:

$$\left|\frac{\sigma_3}{\sigma_1}\right| = \left|\frac{-38.69\,F}{266.16\,F}\right| = 0.145364 \qquad \frac{S_{\text{uc}}}{S_{\text{ut}}} = \frac{109}{31} = 3.51613$$

$$\left|\frac{\sigma_3}{\sigma_1}\right| < \frac{S_{\text{uc}}}{S_{\text{ut}}} \qquad \rightarrow \qquad \sigma_1 = \frac{S_{\text{ut}}}{n_{\text{SF}}} \qquad \rightarrow \qquad 30687.8 = \frac{31000}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 1.012$ was used. Since the safety factor is greater than one (and $MS = 0.012 > 0$) the structure is likely to not fail. Although the design engineer should consider $n_{\text{SF}} = 1.15$ for design, this design is acceptable since failure is not predicted.

Brittle Coulomb-Mohr Criterion: If we employ the BCM criterion,

$$\left|\frac{\sigma_1}{S_{\text{ut}}} - \frac{\sigma_3}{S_{\text{uc}}}\right| = \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad 1.03 = \frac{1}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 0.97$ was used. Since the safety factor is less than one (and $MS = -0.03 < 0$) the design will fail.

We can see that the safety factor required found by BCM is smaller than the one found for the MNS. As stated earlier, the BCM theory is more conservative than the MNS theory.

(2) Certain safety factor was used to ensure that the structure would not fail. If $F = 100$ lbs was assumed to cause yielding, what is the realized margin of safety?

The goal is to ensure that the margin of safety is positive and/or safety factor is greater than one. The two fracture theories will give slightly different solutions. The principal stresses are:

$$\sigma_1 = 266.16\,F = 26685.1 \qquad \sigma_2 = 0 \qquad \sigma_3 = -38.69\,F = -3868.63$$

Maximum Normal Stress criterion: If we employ the MNS criterion, need to check what case we are working with:

$$\sigma_1 \geq 0 \geq \sigma_3$$

Thus we need to evaluate and compare:

$$\left|\frac{\sigma_3}{\sigma_1}\right| = \left|\frac{-38.69\,F}{266.16\,F}\right| = 0.145364 \qquad \frac{S_{\text{uc}}}{S_{\text{ut}}} = \frac{109}{31} = 3.51613$$

$$\left|\frac{\sigma_3}{\sigma_1}\right| < \frac{S_{\text{uc}}}{S_{\text{ut}}} \qquad \rightarrow \qquad \sigma_1 = \frac{S_{\text{ut}}}{n_{\text{SF}}} \qquad \rightarrow \qquad 26685.1 = \frac{31000}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 1.16$ was used. Since the safety factor is greater than one (and $MS = 0.16 > 0$) the structure will not fail and the design is acceptable.

Brittle Coulomb-Mohr Criterion: If we employ the BCM criterion,

$$\left| \frac{\sigma_1}{S_{\text{ut}}} - \frac{\sigma_3}{S_{\text{uc}}} \right| = \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad 0.8963 = \frac{1}{n_{\text{SF}}}$$

Thus a safety factor of $n_{\text{SF}} = 1.12$ was used. Since the safety factor is greater than one (and $MS = 0.12 > 0$) the structure will not fail. However, the design engineer should consider $n_{\text{SF}} = 1.15$ for design, thus this design is not acceptable since $MS \ngeq 0.15$.

We can see that the safety factor required found by BCM is smaller than the one found for the MNS. As stated earlier, the BCM theory is more conservative than the MNS theory.

End Example □

8.4 Introduction to Fracture Mechanics

"every structure contains small flaws whose size and distribution are dependent upon the material and its processing. These may vary from nonmetallic inclusions and micro voids to weld defects, grinding cracks, quench cracks, surface laps, etc."[1]

The objective of a Fracture Mechanics analysis is to determine if these small flaws will grow into large enough cracks to cause the component to fail catastrophically. Fracture Mechanics:

1. is the study of crack propagation in bodies.
2. is the methodology used to aid in selecting materials and designing components to minimize the possibility of fracture.
3. begins with the assumption that all real materials contain cracks of some size—even if only sub-microscopically.
4. is based on three types of displacement modes. As shown in Fig. 8.1.

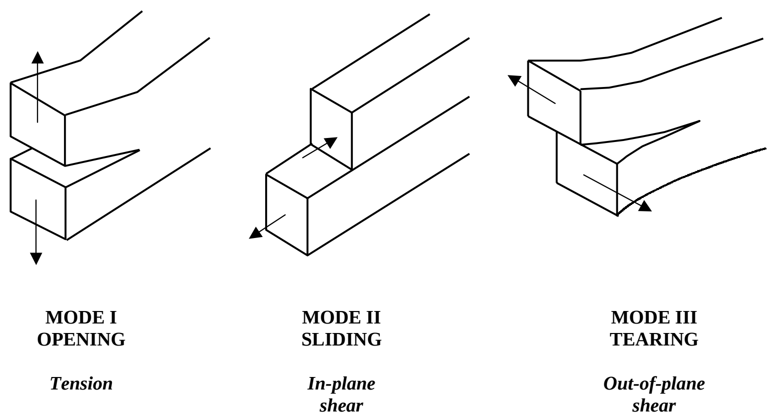


Figure 8.1: Three modes of fracture

MODE I: Opening. The opening (or tensile) mode is the most often encountered mode of crack propagation. The crack faces separate symmetrically with respect to the crack plane.

MODE II: Sliding. The sliding (or in-plane shearing) mode occurs when the crack faces slide relative to each other symmetrically with respect to the normal to the crack plane but asymmetrically with respect to the crack plane.

MODE III: Tearing. The tearing (or antiplane) mode occurs when the crack faces slide asymmetrically with respect to both the crack plane and its normal.

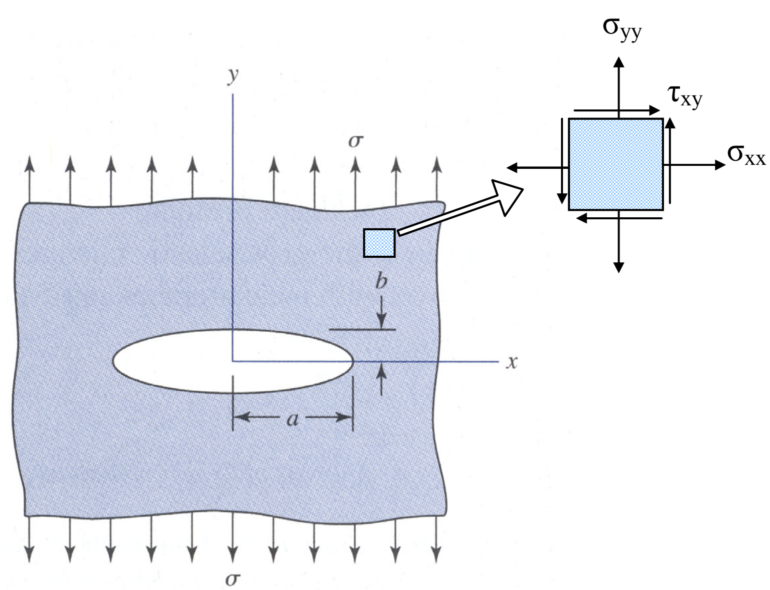
[1]T. J. Dolan, Preclude Failure: A Philosophy for Material Selection and Simulated Service Testing, SESA J. Exp. Mech., Jan. 1970.

8.4.1 Fracture of Cracked Members

1. The presence of a crack in a structure may weaken it so that it fails by fracturing in two or more pieces.
2. Fracture can occur at stresses below the material's yield strength, where failure would not normally be expected.

8.4.2 Cracks as stress raisers

Consider the elliptic hole in an infinite plate loaded by an applied uniaxial stress σ in tension.



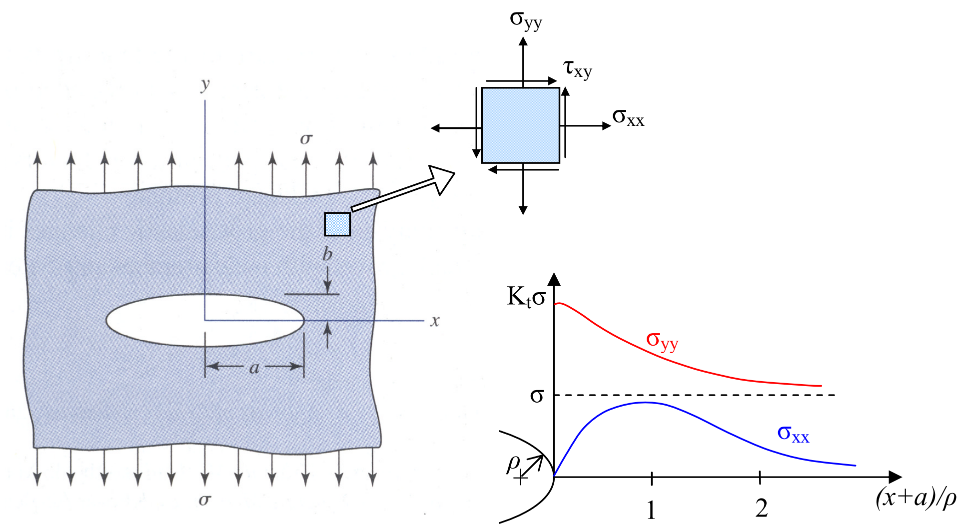
The maximum stress occurs at $(\pm a, 0)$ and has a value of

$$\sigma_{yy}\Big|_{max} = \left(1 + 2\,\frac{a}{b}\right)\,\sigma$$

$$\sigma_{yy}\Big|_{max} = K_t\,\sigma$$

where K_t is the dimensionless stress concentration factor. The radius at the tip of the ellipse can be defined as:

$$\rho = \frac{b^2}{a}$$



Thus the stress concentration factor becomes:

$$K_t = \frac{\sigma_{yy}\Big|_{max}}{\sigma} = 1 + 2\,\frac{a}{b} = 1 + 2\sqrt{\frac{a}{\rho}}$$

and can take values such as

a/b	1	2	3
K_t	3	5	7

Note that when $a = b$ the ellipse becomes a circle can gives a stress concentration factor of 3. When $b \to 0$ or $\rho \to 0$:

$$K_t \to \infty$$

and this geometry is like a crack-like slot. Real materials cannot support infinite stresses.

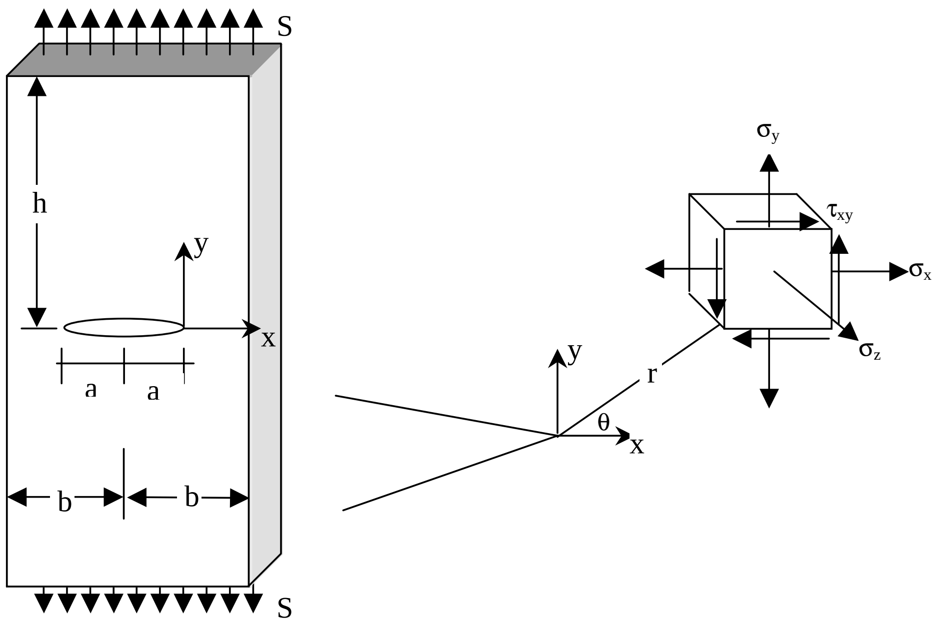
In ductile metals, large plastic deformation exists in the vicinity of the crack-tip. The stress is not ∞ and the sharp crack tip is blunted:

8.4.3 Fracture toughness

1. In fracture mechanics, one does not attempt to evaluate an effective stress concentration, rather a stress intensity factor K
2. After obtaining K, it is compared with a limiting value of K that is necessary for crack propagation in that material, called K_c
3. The limiting value K_c is characteristic of the material and is called fracture toughness
4. Toughness is defined as the capacity of a material to resist crack growth

8.4.4 Fracture Mechanics: MODE I

Stress Intensity Factor



1. Observed that as $a \to b$, the plate fractures into two pieces.
2. The stress intensity factor K_I characterizes the magnitude of the stresses in the vicinity of an ideal sharp crack tip in a linear-elastic and isotropic material under mode I displacement.
3. Near the crack tip the dominant terms in the stress field are:

$$\sigma_{\text{xx}} = \frac{K_\text{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \cdots \tag{8.13}$$

$$\sigma_{\text{yy}} = \frac{K_\text{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \cdots \tag{8.14}$$

$$\tau_{\text{xy}} = \frac{K_\text{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \cdots \tag{8.15}$$

$$\sigma_{\text{zz}} = \begin{cases} 0 & \text{(plane stress)} \\ \nu(\sigma_{\text{xx}} + \sigma_{\text{yy}}) & \text{(plane strain)} \end{cases} \tag{8.16}$$

$$\tau_{\text{yz}} = \tau_{\text{xz}} = 0 \tag{8.17}$$

4. K_{I} measures the severity of the crack and it is generally expressed as:

$$K_{\text{I}} \quad = \quad C_{\text{I}}\,\sigma\,\sqrt{\pi a} \tag{8.18}$$

   C_{I} is a dimensionless quantity accounting for the plate/specimen geometry and relative crack size for mode one,

   σ is the stress (σ_{xx}, σ_{yy}, ...) if no crack were present,

   a is half crack length

5. The dimensional units of K_{I} are: [stress$\sqrt{\text{length}}$], i.e., [MPa$\sqrt{\text{m}}$] or [ksi$\sqrt{\text{in}}$]

In section 8.4.6, we give various expressions for plate with any $\alpha = a/b$. As for an example, the dimensionless geometry constant for a crack-centered plate is

$$C_{\text{I}} = \frac{1 - 0.5\,\alpha + 0.326\,\alpha^2}{\sqrt{1 - \alpha}} \qquad \frac{h}{b} \geq 1.5 \qquad \alpha = a/b \tag{8.19}$$

From the above expression it can be shown that $C_{\text{I}} = 1$ for an infinite plate ($b \to \infty$) and for $0 \ll \alpha \ll 1$. However, for a center-cracked plate with $\alpha \leq 0.4$, when taking $C_{\text{I}} = 1$, the result is accurate within 10%.

Critical Stress Intensity Factor

1. The calculated K_{I} is compared to the critical stress intensity factor or fracture toughness K_{Ic}:

   $K_{\text{I}} < K_{\text{Ic}}$, material will resist crack growth without brittle fracture (safe)

   $K_{\text{I}} = K_{\text{Ic}}$, crack begins to propagate and brittle fracture occurs (fracture)

2. The critical value K_{Ic} is defined as $K_{\text{Ic}} = C\sigma_{\text{c}}\sqrt{\pi a}$

Note that K_{Ic} is a material property but K_{I} is not! The strength-to-ratio $K_{\text{Ic}}/K_{\text{I}}$ can be used to determine the safety factor:

$$n_{\text{SF}} = \frac{K_{\text{Ic}}}{K_{\text{I}}}$$

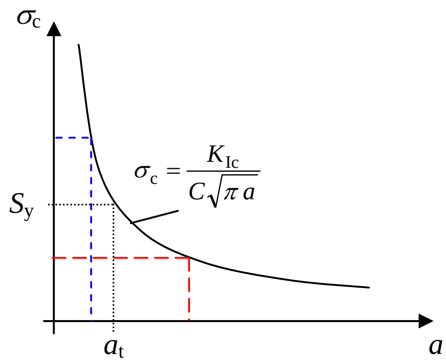


Figure 8.2: Relationship between stress and crack length.

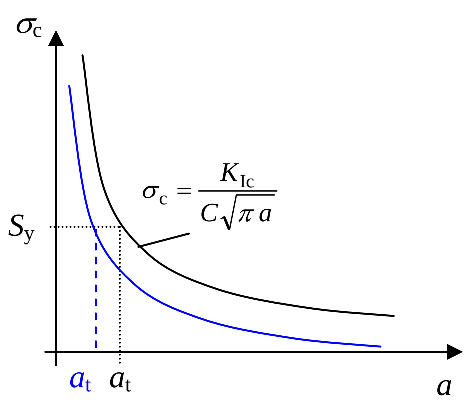
8.4.5 Transition Crack Length

Figure 8.2 shows the relationship between the critical value of the remote stress and the crack length. Here a_t is the transition crack length, and it is defined as the approximate length above which strength is limited by brittle fracture; and $S_y = S_{\text{yield}}$. In other words, a_t is the crack length where $\sigma_c = S_y$:

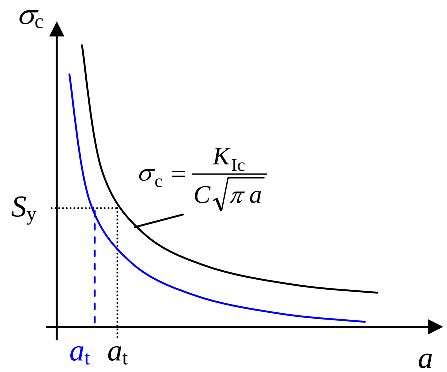
$$a_t = \frac{1}{C^2\pi}\left(\frac{K_{Ic}}{S_y}\right)^2 = \frac{1}{\pi}\left(\frac{K_{Ic}}{S_y}\right)^2 \qquad (C = 1)$$

When $a > a_t$ strength is limited by fracture, and when $0 < a < a_t$ yielding dominates strength. In other words, materials with:

1. high K_{Ic} and low S_y implies long a_t (red line); therefore small cracks are not a problem. In fact, the higher the fracture toughness, the lower the yield strength; and the material has a ductile-like behavior.



2. low K_{Ic} and high S_y implies short a_t (blue line); therefore small cracks can be a problem. In fact, the lower the fracture toughness, the higher the yield strength; and the material has a brittle-like behavior.



Thus when designing one should identify whether the yielding failure is more critical than fracture failure, or fracture failure is more critical than yielding. If yielding failure is more critical, one must ensure safety with the previously discussed three-dimensional theories of yielding failure. Figure 8.3 shows the safety region.

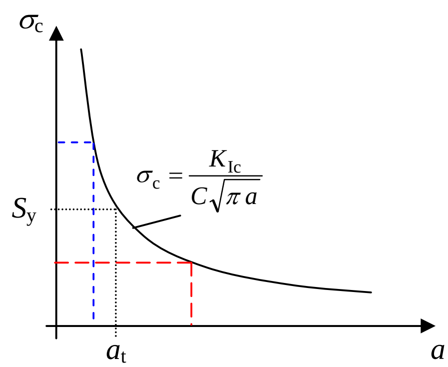


Figure 8.3: Safe region with a structure subject to an initial crack length $2a$.

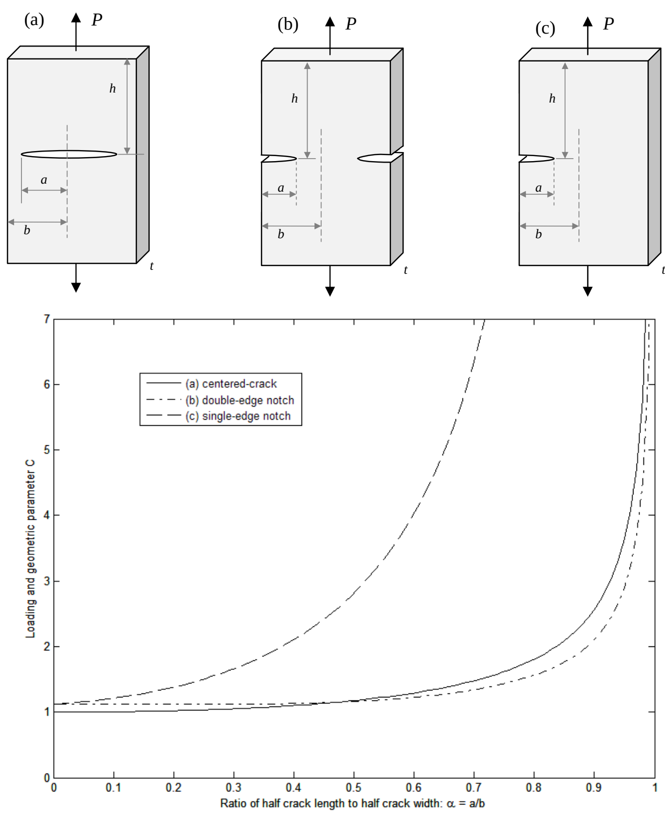
8.4.6 Fracture Mechanics: Tables and Plots

Table 8.1: Plane strain fracture toughness and corresponding tensile properties for representative metals at room temperature.

Material	Toughness K_{Ic} MPa$\sqrt{\text{m}}$ (ksi$\sqrt{\text{in}}$)	Yield Stress S_y MPa (ksi)	Ultimate Stress S_u MPa (ksi)	Elongation $100\varepsilon_f$ %	Reduced Area $\%\,RA$ %
AISI 1144	66 (60)	540 (78)	840 (122)	5	7
ASTM A470-8 (Cr-Mo-V)	60 (55)	620 (90)	780 (113)	17	45
ASTM A517-F	187 (170)	760 (110)	830 (121)	20	66
AISI 4130	110 (100)	1090 (158)	1150 (167)	14	49
18-Ni maraging air melted	123 (112)	1310 (190)	1350 (196)	12	54
18-Ni maraging vacuum melted	176 (160)	1290 (187)	1345 (195)	15	66
300-M 650°C temper	152 (138)	1070 (156)	1190 (172)	18	56
300-M 300°C temper	65 (59)	1740 (252)	2010 (291)	12	48
2014-T651	24 (22)	415 (60)	485 (70)	13	—
2024-T351	34 (31)	325 (47)	470 (68)	20	—
2219-T851	36 (33)	350 (51)	455 (66)	10	—
7075-T651	29 (26)	505 (73)	570 (83)	11	—
7475-T7351	52 (47)	435 (63)	505 (73)	14	—
Ti-6Al-4V (annealed plate)	66 (60)	925 (134)	1000 (145)	16	34
Ti-6Al-4V (annealed bar)	106 (96)	820 (119)	895 (130)	10	

Applications of $K_{\text{I}} = C_{\text{I}}\, S_{\text{yy}}\, \sqrt{\pi\, a}$ to Design and Analysis: Stress intensity factors for three cases of cracked plates under tension. An additional expression for (a) can be found in fracture mechanic books and for (c) the load is centered on the uncracked width. The uncracked stress is defined as:

$$\text{For cases (a) and (b): } S_{\text{yy}} = \frac{P}{2\, b\, t}, \qquad \text{and for case (c): } S_{\text{yy}} = \frac{P}{b\, t}$$



Values for small $\alpha = a/b$ and limits for 10 % accuracy:

(a) $C_{\text{I}} = 1$ $(\alpha \leq 0.4)$ (b) $C_{\text{I}} = 1.12$ $(\alpha \leq 0.6)$ (c) $C_{\text{I}} = 1.12$ $(\alpha \leq 0.13)$

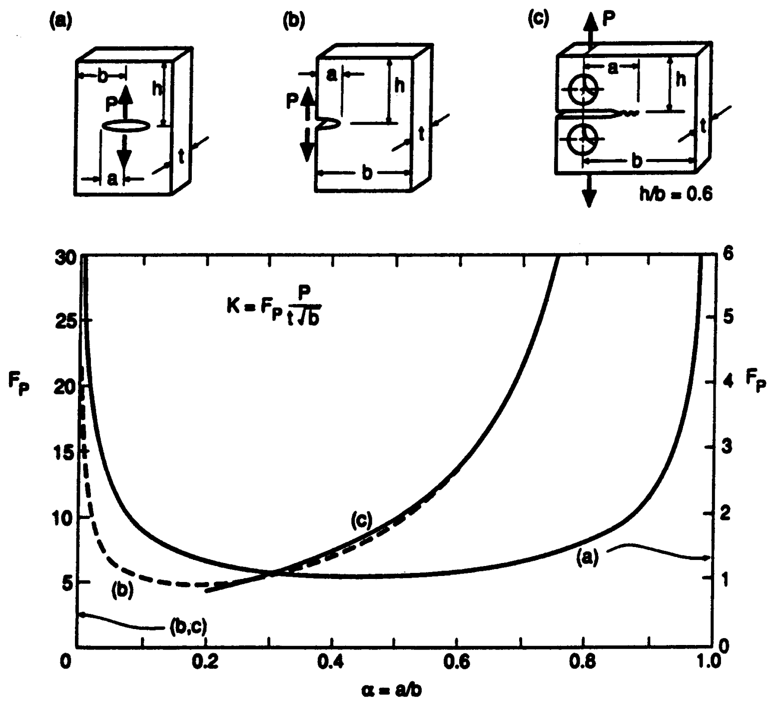
General expressions for C_{I} for any α are (also plotted in figure):

$$\text{Case (a) } C_{\text{I}} = \frac{1 - 0.5\,\alpha + 0.326\,\alpha^2}{\sqrt{1 - \alpha}} \qquad \frac{h}{b} \geq 1.5 \qquad \alpha = a/b$$

$$\text{Case (b) } C_{\text{I}} = \left\{1 + 0.122\,\cos^4\left(\frac{\pi\,\alpha}{2}\right)\right\}\sqrt{\frac{2}{\pi\,\alpha}\,\tan\left(\frac{\pi\,\alpha}{2}\right)} \qquad \frac{h}{b} \geq 2.0 \qquad \alpha = a/b$$

$$\text{Case (c) } C_{\text{I}} = 0.265\,(1 - \alpha)^4 + \frac{0.857 + 0.265\,\alpha}{(1 - \alpha)^{1.5}} \qquad \frac{h}{b} \geq 1.0 \qquad \alpha = a/b$$

Applications of $K_{\rm I}$ to Design and Analysis: Stress intensity factors for three cases of concentrated load. Case (c) is the ASTM standard compact specimen.



$$F_{\rm p} = \frac{C\,\sigma\,t\,\sqrt{\pi\,a\,b}}{P}$$

Values for small $\alpha = a/b$ and limits for 10 % accuracy:

$$\text{(a) } K_{\rm I} = \frac{P}{t\,\sqrt{\pi\,a}} \qquad (\,\alpha \leq 0.3) \qquad \text{(b) } 2.60\frac{P}{t\,\sqrt{\pi\,a}} \qquad (\,\alpha \leq 0.08)$$

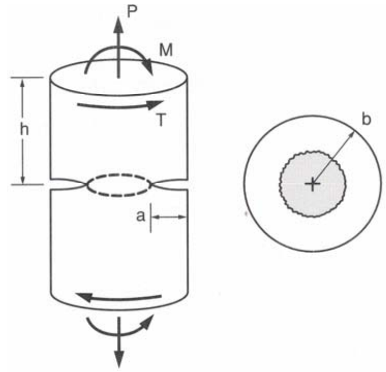
Expressions for $F_{\rm p}$ for any α:

$$\text{Case (a) } F_{\rm p} = \frac{1.297 - 0.297\,\cos\left(\dfrac{\pi\,\alpha}{2}\right)}{\sqrt{\sin\left(\pi\,\alpha\right)}} \qquad \frac{h}{b} \geq 2.0 \qquad \alpha = a/b$$

$$\text{Case (b) } F_{\rm p} = \frac{0.92 + 6.12\,\alpha + 1.68\,(1-\,\alpha)^5 + 1.32\,\alpha^2\,(1-\,\alpha)^2}{\sqrt{\pi\,\alpha}\,(1-\,\alpha)^{1.5}} \qquad \frac{h}{b} \to \text{large} \qquad \alpha = a/b$$

$$\text{Case (c) } F_{\rm p} = \frac{2+\,\alpha}{(1-\,\alpha)^{1.5}}\,\left(0.886 + 4.64\,\alpha - 13.32\,\alpha^2 + 14.72\,\alpha^3 - 5.6\,\alpha^4\right) \qquad \alpha = a/b \geq 0.2$$

Stress intensities for a round shaft with a circumferential crack, including limits on the constant C for 10% accuracy and expressions for any α. For torsion (c), the stress intensity is for the shear Mode III.



$$K_{\rm I} = C_{\rm I}\,\sigma_{\rm xx}\,\sqrt{\pi\,a} \qquad K_{\rm III} = C_{\rm III}\,\tau_{\rm xz}\,\sqrt{\pi\,a} \qquad \alpha = \frac{a}{b} \qquad \beta = 1-\alpha$$

Values for small $\alpha = a/b$ and limits for 10 % accuracy:

(a) axial load P: $\sigma_{\rm xx} = \dfrac{P}{\pi\,b^2}$ $C = 1.12$ $(\alpha \le 0.12)$

(b) bending moment M: $\sigma_{\rm xx} = \dfrac{4\,M}{\pi\,b^3}$ $C = 1.12$ $(\alpha \le 0.12)$

(c) torsion T: $\tau_{\rm xz} = \dfrac{2\,T}{\pi\,b^3}$ $C = 1.00$ $(\alpha \le 0.09)$

Expressions for C for any β:

(a) axial load P: $$C_{\rm I} = \frac{1}{2\,\beta^{1.5}}\left\{1.0 + 0.5\,\beta + 0.375\,\beta^2 - 0.363\,\beta^3 + 0.731\,\beta^4\right\}$$

(b) bending moment M: $$C_{\rm I} = \frac{3}{8\,\beta^{2.5}}\left\{1.0 + 0.5\,\beta + 0.375\,\beta^2 + 0.3125\,\beta^3 + 0.273438\,\beta^4 + 0.537\,\beta^5\right\}$$

(c) torsion T: $$C_{\rm III} = \frac{3}{8\,\beta^{2.5}}\left\{1.0 + 0.5\,\beta + 0.375\,\beta^2 + 0.3125\,\beta^3 + 0.273438\,\beta^4 + 0.208\,\beta^5\right\}$$

or in term of α:

(a) axial load P: $$C_{\rm I} = \frac{1}{(1-\alpha)^{1.5}}\left\{1.1215 - 1.5425\,\alpha + 1.836\,\alpha^2 - 1.2805\,\alpha^3 + 0.3655\,\alpha^4\right\}$$

(b) bending moment M: $$C_{\rm I} = \frac{1}{(1-\alpha)^{2.5}}\left\{1.12423 - 2.23734\,\alpha + 3.12117\,\alpha^2 - 2.54109\,\alpha^3 + 1.10941\,\alpha^4 - 0.201375\,\alpha^5\right\}$$

(c) torsion T: $$C_{\rm III} = \frac{1}{(1-\alpha)^{2.5}}\left\{1.00085 - 1.62047\,\alpha + 1.88742\,\alpha^2 - 1.30734\,\alpha^3 + 0.492539\,\alpha^4 - 0.078\,\alpha^5\right\}$$

8.4.7 Fracture Mechanics: Mixed Modes

Fracture under combined loading

In many cases, the structure is not only subjected to tensile stress σ_{xx} but also a contour shear stress τ_{xy}. Thus, the crack is exposed to tension and shear which leads to mixed mode cracking; i.e., a mixture of mode I and mode II.

Whenever the crack length $2a$ is small with respect to the web length b, the geometric factor C is unity in formulas for the stress intensity factors from LEFM. That is, $K_I = \sigma_{xx}\sqrt{\pi a}$ and $K_{II} = \tau_{xy}\sqrt{\pi a}$, where σ_{xx} and τ_{xy} are the normal and shear stresses in the structure if there were no crack present.

Mixed mode fracture is complicated by the fact that the crack extension takes place at an angle with respect to the original crack direction. If a crack propagates in the direction of the original crack, it is called self-similar crack growth. Under mixed mode fracture the crack growth is, in general, not self-similar. Various mixed mode criteria for crack growth have been proposed based on experiments. A common mixed mode criterion, at the initiation of the fracture, is

$$\left(\frac{K_I}{K_{Ic}}\right)^2 + \left(\frac{K_{II}}{K_{IIc}}\right)^2 = 1 \tag{8.20}$$

where K_{Ic} is the fracture toughness for mode I loading only, and K_{IIc} is the fracture toughness for mode II loading only. The plane strain fracture toughness for mode I loading is usually readily available in the literature, but the mode II fracture toughness is not usually available.

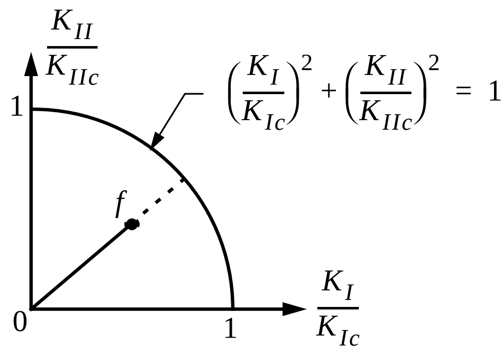
Tests for mode II are more difficult to design than for mode I. To estimate K_{IIc} knowing the value of K_{Ic} we use the maximum principal stress criterion[2]. The maximum principal stress criterion postulates that crack growth will occur in the direction perpendicular to the maximum principal stress in the vicinity of the crack tip. Using this criterion it is possible to estimate K_{IIc} as

$$K_{IIc} = \frac{\sqrt{3}}{2}\,K_{Ic} = 0.866\,K_{Ic} \tag{8.21}$$

The mixed mode criterion given by Eq. (8.20) is plotted in the following figure:
Under proportional loading, the stresses, and in turn the stress intensity factors, are proportional to the magnitude of the total lift acting on the wing. The stress intensity factors at the 80% limit load specified for the damage design condition determine the coordinates of the required strength in the plot, which is represented by the ray $0 - f$.

The quantity f denotes the dimensionless required strength. The excess strength with respect to fracture is represented by $1 - f$, and if it is divided by the required strength we get the margin of safety

[2] Pérez, Néstor, **Fracture Mechanics**, Kluwer Academic Publishers, Boston, USA, 2004 (ISBN 1–4020–7745–9)



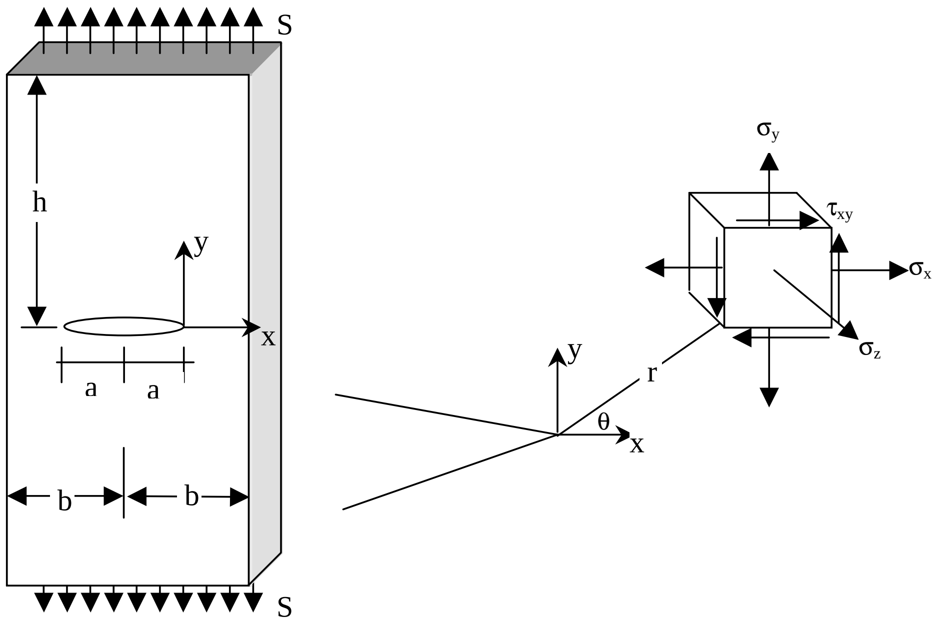
as

$$MS = \frac{1-f}{f} = \frac{\sigma_{\text{allowable}}}{\sigma_{\text{required}}} - 1 = n_{\text{SF}} - 1 \tag{8.22}$$

$$f = \sqrt{\left(\frac{K_{\text{I}}}{K_{\text{Ic}}}\right)^2 + \left(\frac{K_{\text{II}}}{K_{\text{IIc}}}\right)^2} \tag{8.23}$$

The crack is predicted not to propagate if $0 \leq f < 1$, and the initiation of fracture is predicted if $f = 1$. The margin of safety is positive if $0 < f < 1$, and the margin of safety is zero if $f = 1$.

8.4.8 Plastic zone size in cracked metal plates



The in-place stress are

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \text{Higher Order Terms} \tag{8.24}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \text{H.O.T.} \tag{8.25}$$

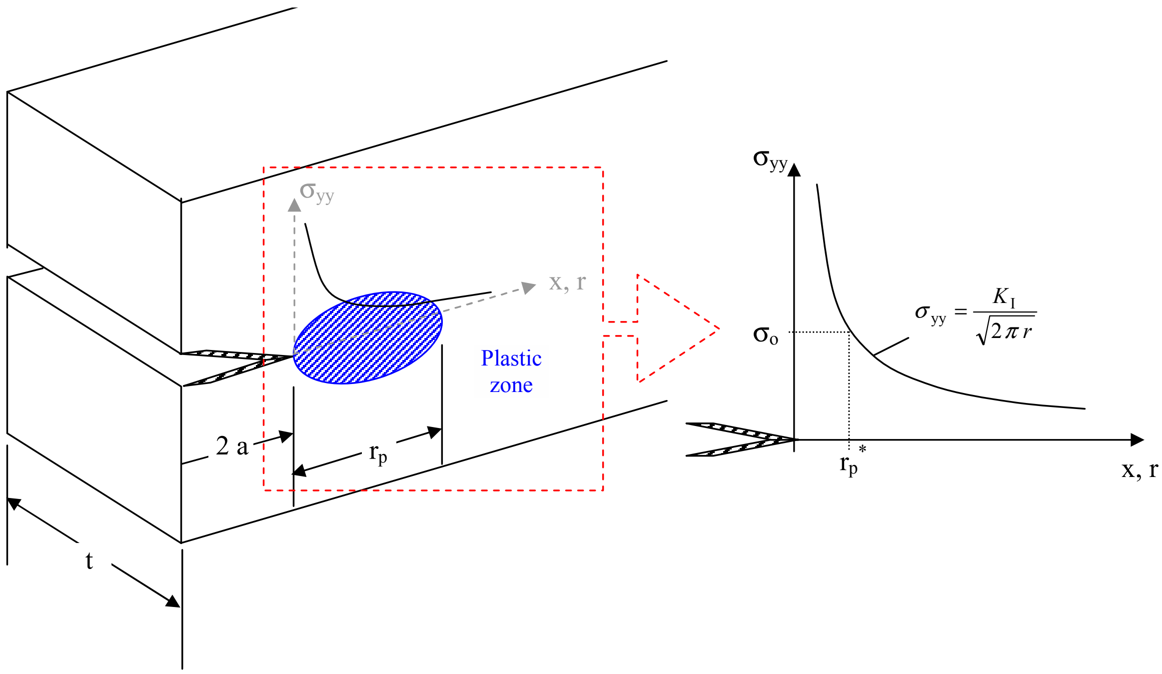
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \text{H.O.T.} \tag{8.26}$$

$$\frac{K_I}{\sqrt{2\pi r}} = \sigma\sqrt{\frac{a}{2r}} \qquad \left(\frac{C\,\sigma\sqrt{\pi\,a}}{\sqrt{2\,\pi\,r}} = C\,\sigma\sqrt{\frac{a}{2\,r}} \qquad \text{and} \qquad C = 1\right) \tag{8.27}$$

$$@\,\theta = 0 \rightarrow x = r, y = 0 \qquad\qquad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

8.4.9 Plastic zone

If we neglecting higher order terms in the above expansion, we can show that the stress distribution near the crack tip is as follows



where r_p^* is the estimate of plastic zone and is defined as

$$r_p^* = \frac{1}{2\,\pi}\left(\frac{K_{\text{Ic}}}{S_{\text{y}}}\right)^2 \tag{8.28}$$

Experiments and analysis show the plastic zone size $r_p > r_p^*$. Furthermore, we define plastic zone size as

$$r_p = c\left(\frac{K_{\text{I}}}{S_{\text{y}}}\right)^2 \qquad c = \text{constant of proportionality} \tag{8.29}$$

The in-plane stresses near crack tip are very large, and ε_{yy} is large. The high stress region near the crack indicates the plastic core at the crack tip wants to contract in the thickness direction due to very large in-plane stresses (σ_{xx}, σ_{yy}, τ_{xy}). The bulk elastic material surrounding plastic core does not contract in the thickness direction, or contracts a lesser amount, than the plastic core. The bulk elastic material constrains the contraction of the plastic core.

Plastic zone is responsible to delay crack propagation, and as soon as the crack hits the elastic region the plane will fracture.

8.4.10 Plane stress and Plane strain

For plane stress $\sigma_{zz} = 0$ and the yielding occurs about $S_y = S_y$

$$r_p^* = r_{o\sigma}$$

$$r_p = 2\, r_{o\sigma} = \frac{1}{\pi}\left(\frac{K_I}{S_y}\right)^2$$

where the constant of proportionality is

$$c_{o\sigma} = \frac{1}{\pi}$$

For plane strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ and the yielding occurs about $S_y = \sqrt{3}\, S_y$:

$$r_p^* = r_{o\varepsilon}$$

$$r_p = 2\, r_{o\varepsilon} = \frac{1}{\pi}\left(\frac{K_I}{\sqrt{3}\, S_y}\right)^2 = \frac{1}{3\,\pi}\left(\frac{K_I}{S_y}\right)^2$$

where the constant of proportionality is

$$c_{o\varepsilon} = \frac{1}{3\,\pi}$$

Note that $r_{o\varepsilon} = \dfrac{r_{o\sigma}}{3}$

8.4.11 Plasticity limitations on LEFM

Plastic zone size is characterized by K_I only if first term in σ_{yy} dominates (recall H.O.T. were neglected in the in-plane stress). If plasticity spreads further then K_I cannot be used to characterize the plastic zone size and the use of Linear Elastic Fracture Mechanics (LEFM) is invalid. In the following situations K_I cannot be used to characterize the stress field because the plastic zone is too large:

1. relative to crack

2. relative to uncracked ligament

3. relative to specimen height

The requirement for plane strain on the use of LEFM is:

$$t, a, (b-a), h \geq 2.5 \left(\frac{K_{\mathrm{I}}}{S_{\mathrm{y}}}\right)^2$$

Since $2\,r_{\mathrm{o}\sigma} > 2\,r_{\mathrm{o}\varepsilon}$, an overall limit for plane stress on the use of LEFM is:

$$a, (b-a), h \geq \frac{4}{\pi} \left(\frac{K_{\mathrm{I}}}{S_{\mathrm{y}}}\right)^2 \tag{8.30}$$

This must be satisfied for all three of $a, (b-a), h$. Note that $K_{\mathrm{I}} \rightarrow K_{\mathrm{Ic}}$ only if it can be considered as plane strain.

8.4.12 Fracture toughness in plane strain and plane stress

Table 2.1 from the course textbook (as well of most books) provides plane strain fracture toughness $K_{\rm Ic}$. However, for plane strain fracture toughness $K_{\rm Ic}$ to be a valid failure prediction criterion, plane strain conditions must exist at the crack tip. In other words, the material must be thick enough to ensure plane strain conditions.

To better understand this concept, consider a plate. The plate thickness is the plastic core whose diameter is equal to $r_{\rm p}$:

$$r_{\rm p} = c\left(\frac{K_{\rm I}}{S_{\rm y}}\right)^2$$

Thick Plate	Thin Plate
Plane Strain	Plane Stress
$\sigma_{\rm zz} = \nu\,(\sigma_{\rm xx} + \sigma_{\rm yy})$	$\sigma_{\rm zz} = 0$
$\varepsilon_{\rm zz} = 0$	$\varepsilon_{\rm zz} = -\dfrac{\nu}{1-\nu}\,(\varepsilon_{\rm xx} + \varepsilon_{\rm yy})$
$\dfrac{t_1}{r_{\rm p}} \rightarrow$ Large The bulk material constrains very thick core to large extent	$\dfrac{t_2}{r_{\rm p}} \rightarrow$ Small The bulk material provides little constraint to core
$\dfrac{t}{\left(\dfrac{K_{\rm I}}{S_{\rm y}}\right)^2} > Q$	$\dfrac{t}{\left(\dfrac{K_{\rm I}}{S_{\rm y}}\right)^2} < Q$

where $Q \simeq 2.5$, $S_{\rm y} = S_{\rm y}$

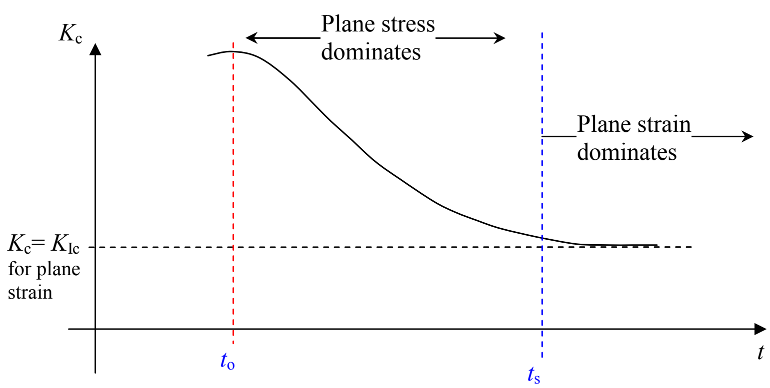
Empirically it has been estimated that the minimum required material thickness for plane strain condition is given by (transition thickness between plane strain and plane stress)

$$t_{\rm s} = 2.5\left(\frac{K_{\rm Ic}}{S_{\rm y}}\right)^2$$

For plane strain conditions the minimum material thickness t must be

$$t \geq t_{\rm s}$$

If the material is not thick enough to meet the above criterion, plane stress better characterizes the state of stress at the crack tip, and $K_{\rm c}$, the critical stress-intensity factor for failure prediction under plane stress conditions, may be estimated using a semiempirical relationship for $K_{\rm c}$ as a function of plane



strain fracture toughness K_{Ic} and thickness t. This relationship is

$$K_c = K_{Ic} \sqrt{1 + \frac{1.4}{t^2} \left(\frac{K_{Ic}}{S_y} \right)^4} = K_{Ic} \sqrt{1 + 0.224 \left(\frac{t_s}{t} \right)^2} \tag{8.31}$$

Note that when $t = t_s$:

$$K_c = K_{Ic} \sqrt{1 + 0.224} = 1.106 \, K_{Ic} \tag{8.32}$$

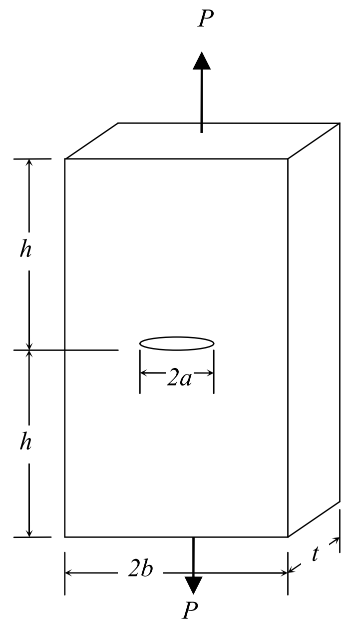
which means that the minimum value of the thickness to assume plane strain problem is within 10 % accuracy for the plane strain fracture toughness, which seems to be commonly acceptable.

As long as the crack-tip plastic zone is in the regime of small-scale yielding, this estimation procedure provides a good design approach. If the plastic zone size ahead of the crack tip becomes so large that the small-scale yielding condition is no longer satisfied, an appropriate elastic-plastic fracture mechanics procedure would give better results.

The plane strain fracture toughness K_{Ic} use in design, of even thin plates, is conservative. In other words, K_I is a minimum value for the material and the actual K_{Ic} may be higher, as a result an over-design of the structure is obtained. **Thus for design purposes, the plane strain fracture toughness K_{Ic} is most commonly used.** Different engineers and researchers decide on whether to use plane strain or plane stress fracture toughness. Here for design purposes we will always use plane strain assumption.

Example 8.5.

The panel of a structure is subject to tensile force $P = 50$ kN and is made of 2024-T351



Aluminum alloy and a crack is being propagated in the center. The length of the panel is 400 mm ($h = 200$ mm), the width 100 mm ($b = 50$ mm), and the thickness is 5 mm ($t = 5$ mm). The panel can be modeled as a plate.

(1) Determine if plane strain is a good approximation for this problem.

Let us determine if the plane strain fracture toughness is valid for our problem. From Table 8.1:

$$K_{Ic} = 34 \text{ MPa}\sqrt{\text{m}} \qquad S_y = 325 \text{ MPa}$$

Then

$$t_s = 2.5 \left(\frac{K_{Ic}}{S_y}\right)^2 = 2.5 \left(\frac{34}{325}\right)^2 \text{ m} = 27.5 \text{ mm}$$

The thickness of the panel is $t = 5$ mm and:

$$t \ngeq t_s \qquad \Rightarrow \qquad (t = 5) < (t_s = 27.5)$$

Since plane strain conditions are not satisfied region, the fracture toughness for our problem is:

$$K_c = K_{Ic}\sqrt{1 + 0.224\left(\frac{t_s}{t}\right)^2}$$

$$= (34)\sqrt{1 + 0.224\left(\frac{27.5}{5}\right)^2} = 94.8 \ \text{MPa}\sqrt{\text{m}}$$

(t_s and t should have same units)

(2) Find the transition half crack length a_t.

For transition length

$$\sigma_c = S_y = 325 \ \text{MPa}$$

Recall that for design purposes we tend to be conservative, thus we use plane strain fracture toughness:

$$K_{Ic} = 34 \ \text{MPa}\sqrt{\text{m}}$$

The first approach is to use the transition crack length approximation:

$$a_t = \frac{1}{\pi}\left(\frac{K_{Ic}}{S_y}\right)^2 = \frac{1}{\pi}\left(\frac{34}{325}\right)^2 = 0.00348 \ \text{m} = 3.48 \ \text{mm}$$

The second approach consists in not assuming $C = 1$. Thus,

$$K_{Ic} = C_I\,\sigma_c\,\sqrt{\pi\,a_t}$$

Let us rearrange:

$$\frac{K_{Ic}}{\sigma_c\sqrt{\pi}} = C_I\sqrt{a_t}$$

Divide both sides by $\sqrt{b}$:

$$\frac{K_{Ic}}{\sigma_c\sqrt{\pi\,b}} = C_I\frac{\sqrt{a_t}}{\sqrt{b}} = C_I\sqrt{\alpha_t}$$

Note $a_t = \alpha_t\,b$. Thus the problem to solve is:

$$\frac{K_{Ic}}{\sigma_c\sqrt{\pi\,b}} = 0.2640 = C_I\sqrt{\alpha_t}$$

Since C_I depends on α_t from Eq. (8.19)

$$C_I = \frac{1 - 0.5\,\alpha + 0.326\,\alpha^2}{\sqrt{1-\alpha}} \qquad \frac{h}{b} \geq 1.5 \qquad \alpha = a/b$$

Thus, solve numerically:

$$0.2640 = \frac{1 - 0.5\,\alpha_t + 0.326\,\alpha_t^2}{\sqrt{1-\alpha_t}}\sqrt{\alpha_t}$$

and choose the positive real solution:

$$\alpha_t = 0.0696$$

Thus the transition half crack length is: $a_t = 0.00348$ m $= 3.48$ mm.
As we can see the both techniques produce the same results. The main reason is that the $\alpha \ll 0.4$.

(3) Determine if the panel will break with a crack length of 20 mm.

Recall that a is half the crack length, thus $a = 10$ mm. Since, $a > a_t$ we know that failure will be governed by fracture and not by yielding. As a proof,

$$\sigma_{yy}\Big|_{true} = \frac{P}{2\,(b-a)\,t} = \frac{50000}{2\,(0.050-0.010)\,(0.005)} = 125 \ \text{MPa} < S_y = 325 \ \text{MPa}$$

$$n_{SF} = \frac{S_y}{\sigma_{req}} = \frac{325}{125} = 2.6$$

Hence, indeed failure will not be predicted by yielding.
In order to verify for brittle fracture, we know that we need to calculate the safety factor, which will tell us if the panel will fail:

$$n_{SF} = \frac{K_{Ic}}{K_I}$$

where

$$K_I = C_I\,\sigma_{yy}\,\sqrt{\pi\,a}$$

The stress with no crack is:

$$\sigma_{yy} = \frac{P}{A} = \frac{P}{2\,b\,t} = \frac{50000}{2\,(0.050)\,(0.005)} = 100 \ \text{MPa}$$

In order to calculate C_I, we need the ratio $\alpha = a/b$ ($a = 10$ mm):

$$\alpha = \frac{a}{b} = \frac{10}{50} = 0.20$$

Now we use Eq. (8.19) or the provided chart:

$$C_I = 1.04$$

(Since $\alpha < 0.4$, we could have used the approximation $C_I \approx 1$.) Thus the apparent fracture toughness is

$$K_I = C_I\,\sigma_{yy}\,\sqrt{\pi\,a} = 18.43 \ \text{MPa}\sqrt{\text{m}}$$

Recall that for design purposes we tend to be conservative, thus we use plane strain fracture toughness:

$$K_{Ic} = 34 \ \text{MPa}\sqrt{\text{m}}$$

Thus the safety factor is

$$n_{SF} = \frac{K_{Ic}}{K_I} = \frac{34}{18.43} = 1.84$$

The hood has a 84 % of margin of safety. Since $MS > 0$, the panel will not break.

If we were to use the plane stress fracture toughness

$$K_c = 94 \ \text{MPa}\sqrt{\text{m}}$$

Thus the safety factor is

$$n_{SF} = \frac{K_c}{K_I} = \frac{94}{18.43} = 5.14$$

The hood has a 414 % of margin of safety. Since $MS > 0$, the panel will not break. It should be clear that the design is over-designed for this particular crack length. However, in practice it is common to stick with plane strain assumption, although each design engineer can make his/her own judgment.

(4) Determine if the panel will break with a crack length of 60 mm.

Recall that a is half the crack length, thus $a = 30$ mm. Since, $a > a_t$ we know that failure will be governed by fracture and not by yielding. As a proof,

$$\sigma_{yy}\Big|_{true} = \frac{P}{2\,(b-a)\,t} = \frac{50000}{2\,(0.050-0.030)\,(0.005)} = 250 \ \text{MPa} < S_y = 325 \ \text{MPa}$$

$$n_{SF} = \frac{S_y}{\sigma_{req}} = \frac{325}{250} = 1.30$$

Hence, indeed failure will not be predicted by yielding.
In order to verify for brittle fracture, we know that we need to calculate the safety factor, which will tell us if the panel will fail:

$$n_{SF} = \frac{K_{Ic}}{K_I}$$

where

$$K_I = C_I\,\sigma_{yy}\,\sqrt{\pi\,a}$$

The stress with no crack is:

$$\sigma_{yy} = \frac{P}{A} = \frac{P}{2\,b\,t} = \frac{50000}{2\,(0.050)\,(0.005)} = 100 \ \text{MPa}$$

In order to calculate C_I, we need the ratio $\alpha = a/b$ ($a = 30$ mm):

$$\alpha = \frac{a}{b} = \frac{30}{50} = 0.60$$

Now we use Eq. (8.19) or the provided chart:

$$C_I = 1.31$$

Thus

$$K_I = C_I\,\sigma_{yy}\,\sqrt{\pi\,a} = 39.8 \ \text{MPa}\sqrt{\text{m}}$$

Recall that for design purposes we tend to be conservative, thus we use plane strain fracture toughness:

$$K_{\text{Ic}} = 34 \ \text{MPa}\sqrt{\text{m}}$$

Thus the safety factor is

$$n_{\text{SF}} = \frac{K_{\text{Ic}}}{K_{\text{I}}} = \frac{34}{39.8} = 0.85$$

The hood has a negative 15 % of margin of safety, thus it will break due to fracture.

If we were to use the plane stress fracture toughness

$$K_{\text{c}} = 94 \ \text{MPa}\sqrt{\text{m}}$$

Thus the safety factor is

$$n_{\text{SF}} = \frac{K_{\text{c}}}{K_{\text{I}}} = \frac{94}{39.8} = 2.38$$

The hood has a 138 % of margin of safety. Since $MS > 0$, the panel will not break. Interestingly, the design apparently has failed under the plain strain assumption. However, the real problem is a plane stress problem which indicated the contrary. Thus the panel in reality has not fractured.

(5) Determine the load P_c for brittle fracture initiation if the critical crack length $2\,a_{\text{c}} = 30$ mm.

The safety factor tells us if the panel will fail:

$$n_{\text{SF}} = \frac{K_{\text{Ic}}}{K_{\text{I}}}$$

For fracture initiation $n_{\text{SF}} = 1$ (at onset of failure or fracture initiation). Thus

$$K_{\text{I}} = \frac{K_{\text{Ic}}}{n_{\text{SF}}} = K_{\text{Ic}}$$

and

$$K_{\text{I}} = C_{\text{I}}\,\sigma\,\sqrt{\pi\,a}$$

Thus

$$K_{\text{Ic}} = C_{\text{I}}\,\sigma_c\,\sqrt{\pi\,a_c}$$

Stress with no crack present:

$$\sigma_{\text{yy}} = \frac{P_c}{A} = \frac{P_c}{2\,b\,t} = \frac{P_c}{2\,(0.050)\,(0.005)} = 0.002\,P_c \ \text{MPa}$$

Recall that for design purposes we tend to be conservative, thus we use plane strain fracture toughness:

$$K_{\text{Ic}} = 34 \ \text{MPa}\sqrt{\text{m}}$$

Here $\sigma_c = \sigma_{\text{yy}} = 0.002\,P_c$ MPa. Recall that a is half the crack length, thus $a = 15$ mm. Let us rearrange:

$$P_c = \frac{K_{\text{Ic}}}{C_{\text{I}}\,0.002\,\sqrt{\pi\,a_c}}$$

Note $\alpha_c = a_c/b$:

$$\alpha_c = a_c/b = 0.3 \qquad \rightarrow \qquad C_{\text{I}} = \frac{1 - 0.5\,\alpha_c + 0.326\,\alpha_c^2}{\sqrt{1-\ \alpha_c}} = 1.051$$

Thus the critical load is $P_c = 74.51$ kN. Which means that a load of $P_c = 74.51$ kN will initiate brittle fracture.

We should check for yielding failure:

$$\sigma_{\text{yy}}\Big|_{\text{true}} = \frac{P}{2\,(b-a)\,t} = \frac{74510}{2\,(0.050 - 0.015)\,(0.005)} = 212.886\ \text{MPa} < S_{\text{y}} = 325\ \text{MPa}$$

Thus failure by yielding is not predicted.

(6) The failure of the panel is governed by yielding or fracture?

Since $a_c > a_{\text{t}}$ it will fail due to fracture and not yielding. Indeed for the first two cases $a > a_{\text{t}}$, suggesting that failure will occur due to fracture and not yielding.

End Example □

8.4.13 Superposition of Combined Loading

Stress intensity solutions for combined loading can be obtained by superposition, that is, by adding the contribution to K from the individual load components:

$$K = K_1 + K_2 = C_1\,\sigma_1\,\sqrt{\pi\,a} + C_2\,\sigma_2\,\sqrt{\pi\,a} = (C_1\,\sigma_1 + C_2\,\sigma_2)\,\sqrt{\pi\,a}$$

Example 8.6.

A 20 mm diameter shaft made of Ti-6Al-4V has a circumferential surface crack of depth $a = 1.5$ mm. The shaft is loaded with an eccentric axial force of P, which produces a bending moment of $P\,e$, combined with a torque of T. Can we make another flight without replacing the shaft? Note that K_{IIIc} is unknown, and a reasonable and probably conservative assumption is to employ a relationship of the form:

$$\sqrt{\left(\frac{K_{\text{I}}}{K_{\text{Ic}}}\right)^2 + \left(\frac{K_{\text{III}}}{K_{\text{IIIc}}}\right)^2} = f \tag{8.33}$$

where K_{Ic} is the fracture toughness for mode I loading only, and K_{IIIc} is the fracture toughness for mode III loading only. The crack is predicted not to propagate if $0 \leq f < 1$, and the initiation of fracture is predicted if $f = 1$. Assume that $K_{\text{IIIc}} = 0.5\,K_{\text{Ic}}$. Take:

$$P = 150 \text{ N} \qquad e = 5 \text{ mm} \qquad T = 300 \text{ N–m}$$

SOLUTION: First, we need to determine if failure is governed by yielding or fracture. However, since it is a mixed more problem there no specific transition crack length equation; hence, we will have to determine both fracture and yielding. Before we proceed let us obtain the material properties from Table 8.1:

$$K_{\text{Ic}} = 106 \text{ MPa}\sqrt{\text{m}} \qquad S_{\text{y}} = 820 \text{ MPa}$$

and from the problem statement we can determine mode III fracture toughness

$$K_{\text{IIIc}} = 0.5\,K_{\text{Ic}} = 53 \text{ MPa}\sqrt{\text{m}}$$

We can continue the operation if the margin of safety bigger then zero. The margin of safety is defined as

$$MS = n_{\text{SF}} - 1$$

and the safety factor is defined as

$$n_{\text{SF}} = \frac{1}{f} \qquad \text{or} \qquad n_{\text{SF}} = \frac{S_{\text{y}}}{\sigma_{\text{req}}}$$

Thus our goal is to find the margin of safety for yielding and fracture and ensure it is a positive quantity.

1. YIELDING:
   The margin of safety is defined as

$$MS = n_{\text{SF}} - 1$$

   and the safety factor is defined as

$$n_{\text{SF}} = \frac{S_{\text{y}}}{\sigma_{\text{req}}}$$

   Note that for the case of yielding we need to use a 3-dimensional theory of failure for yielding. Thus let us use the distortional energy theory.

   The loads at the cross-section are:

$$N_{\text{xx}} = P = 150 \text{ N} \qquad M_{\text{zz}} = -P\,e = -0.75 \text{ N–m} \qquad M_{\text{xx}} = T = 300 \text{ N–m}$$

   and the radius is $b = 0.010$ m. The true stresses at the critical point in the cross-sectional element:

$$\sigma_{\text{xx}} = \sigma_{\text{xx}}\Big|_{\text{bending}} + \sigma_{\text{xx}}\Big|_{\text{axial}} = \frac{4\,M}{\pi\,(b-a)^3} + \frac{P}{\pi\,(b-a)^2} = 2.21579 \times 10^6 \text{ Pa}$$

$$\tau_{\text{xz}} = \tau\Big|_{\text{torsion}} = \frac{2\,T}{\pi\,(b-a)^3} = 3.10989 \times 10^8 \text{ Pa}$$

   The state of stress is

$$\underline{\boldsymbol{\sigma}}_{\text{M}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{xy}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{xz}} & \tau_{\text{yz}} & \sigma_{\text{zz}} \end{bmatrix} = \begin{bmatrix} 2.21579 \times 10^6 & 0 & 3.10989 \times 10^8 \\ 0 & 0 & 0 \\ 3.10989 \times 10^8 & 0 & 0 \end{bmatrix} \text{ Pa} \tag{8.34}$$

   The stress invariants are

$$\begin{aligned} I_{\sigma_1} &= 2.21579 \times 10^6 \text{ Pa} \\ I_{\sigma_2} &= -9.6714 \times 10^{16} \text{ Pa}^2 \\ I_{\sigma_3} &= 0 \end{aligned}$$

   The von Mises stress is

$$\sigma_{\text{eq}} = \sqrt{I_{\sigma_1}^2 - 3\,I_{\sigma_2}} = 5.38653 \times 10^8 \text{ Pa}$$

The yielding criteria for DE criterion is

$$\frac{\sigma_{\text{eq}}}{S_{\text{y}}} = \frac{1}{n_{\text{SF}}} \qquad \rightarrow \qquad n_{\text{SF}} = 1.52232 \qquad \rightarrow \qquad MS = 0.52232$$

Since $MS > 0$, it will not fail due to yielding.

2. FRACTURE:
The margin of safety is defined as

$$MS = n_{\text{SF}} - 1$$

and the safety factor is defined as

$$n_{\text{SF}} = \frac{1}{f}$$

where

$$f = \sqrt{\left(\frac{K_{\text{I}}}{K_{\text{Ic}}}\right)^2 + \left(\frac{K_{\text{III}}}{K_{\text{IIIc}}}\right)^2}, \qquad K_{\text{I}} = C_{\text{I}}\,\sigma_{\text{xx}}\,\sqrt{\pi\,a}, \qquad K_{\text{III}} = C_{\text{III}}\,\tau_{\text{xz}}\,\sqrt{\pi\,a}$$

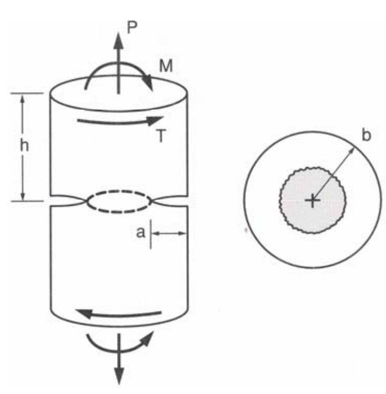
Stress intensity solutions for combined loading can be obtained by superposition:

$$K_{\text{I}} = K_{\text{I}}\Big|_{\text{axial}} + K_{\text{I}}\Big|_{\text{bending}} = C_{\text{I}}\,\sigma_{\text{xx}}\,\sqrt{\pi\,a}\Big|_{\text{axial}} + C_{\text{I}}\,\sigma_{\text{xx}}\,\sqrt{\pi\,a}\Big|_{\text{bending}}$$

$$= \left\{C_{\text{I}}\,\sigma_{\text{xx}}\Big|_{\text{axial}} + C_{\text{I}}\,\sigma_{\text{xx}}\Big|_{\text{bending}}\right\}\,\sqrt{\pi\,a}$$

Thus nominal stress for the combined loading is obtained by superposition of two states of stress for axial force P and moment M_{zz}, is expressed as

$$C_{\text{I}}\,\sigma_{xx} = C_{\text{I}}\Big|_{\text{axial}}\,\sigma_{xx}\Big|_{\text{axial}} + C_{\text{I}}\Big|_{\text{bending}}\,\sigma_{xx}\Big|_{\text{bending}} \tag{8.35}$$



The ratio of the length to bracket width is

$$a = 0.0015 \text{ m} \qquad \alpha = \frac{a}{b} = 0.15 \qquad \beta = 0.85$$

Using expressions for C:

(a) axial load P: $C_{\text{Ia}} = 1.18329$

(b) bending moment M: $C_{\text{Ib}} = 1.27729$

(c) torsion T: $C_{\text{III}} = 1.19511$

Thus nominal stresses at the critical point in the cross-sectional element are

$$\sigma_{\text{xx}}\Big|_{\text{bending}} = \frac{4\,M}{\pi\,b^3} = 954930 \ \text{Pa}$$

$$\sigma_{\text{xx}}\Big|_{\text{axial}} = \frac{P}{\pi\,b^2} = 477465 \ \text{Pa}$$

$$\tau_{\text{xz}} = \tau\Big|_{\text{torsion}} = \frac{2\,T}{\pi\,b^3} = 1.90986 \times 10^8 \ \text{Pa}$$

Thus nominal stress for the combined loading is obtained by superposition of two states of stress for axial force P and moment M_{zz}, is expressed as

$$C_{\text{I}}\,\sigma_{xx} = C_{\text{Ia}}\,\sigma_{xx}\Big|_{\text{axial}} + C_{\text{Ib}}\,\sigma_{xx}\Big|_{\text{bending}} = 1.78471 \times 10^6 \ \text{Pa}$$

$$C_{\text{III}}\,\tau_{xz} = 2.2825 \times 10^8 \ \text{Pa}$$

$$K_{\text{I}} = C_{\text{I}}\,\sigma_{\text{xx}}\,\sqrt{\pi\,a} = 0.122514 \ \text{MPa}\sqrt{\text{m}}, \qquad K_{\text{III}} = C_{\text{III}}\,\tau_{\text{xz}}\,\sqrt{\pi\,a} = 15.6686 \ \text{MPa}\sqrt{\text{m}}$$

Thus

$$f = \sqrt{\left(\frac{K_{\text{I}}}{K_{\text{Ic}}}\right)^2 + \left(\frac{K_{\text{III}}}{K_{\text{IIIc}}}\right)^2} = 0.295637$$

and

$$n_{\text{SF}} = \frac{1}{f} = 3.38253 \qquad \rightarrow \qquad MS = 2.38253$$

Since $MS > 0$, it is save due to fracture.

Hence, it is safe to continue flying.

End Example □

8.5 References

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Juvinall, R. C., and Marsheck, K. A., *Fundamentals of Machine Component Design*, 2000, John Wiley and Sons, New York, NY.

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8.6 Suggested Problems

Problem 8.1.

The state of stress at a point is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} -p & \tau & \tau \\ \tau & -p & \tau \\ \tau & \tau & -p \end{bmatrix} \tag{8.36}$$

where $p > 0$ and $\tau > 0$.

a) If the true strain at fracture is 20 %, find the maximum allowable values of p and τ, according to each of the related failure criteria. Take $S_y = 30$ MPa.

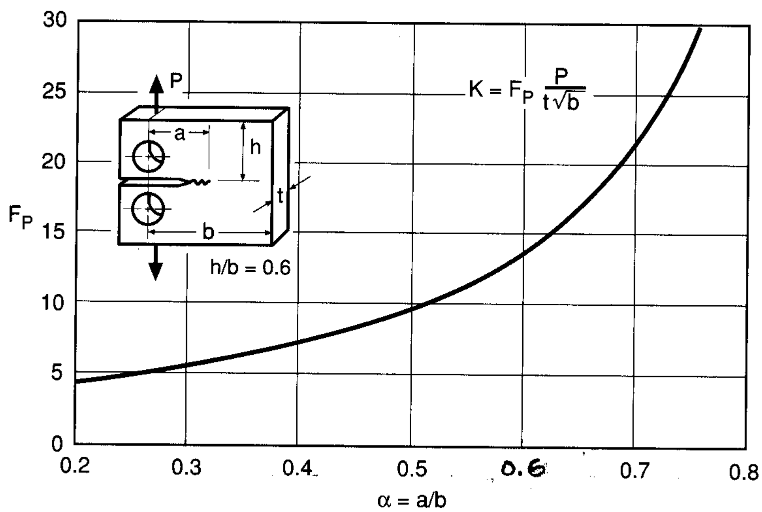
b) If the true strain at fracture is 2 %, find the maximum allowable values of p and τ, according to each of the related failure criteria. Take $S_{ut} = 200$ MPa and $S_{uc} = 850$ MPa.

□

Problem 8.2.

A fracture toughness test was conducted on AISI 4340 steel having a yield strength of 1380 MPa. The standard compact specimen used had dimensions, as defined in the figure below, $b = 50.0$ mm, $t = 15.0$ mm, $h/b = 0.6$ ($h = 30.0$ mm), and a sharp precrack to $a = 26.0$ mm. Failure occurred suddenly at $P_Q = 15.0$ kN.

1. Calculate K_Q at fracture
2. Does this value qualify as a valid (plane strain) K_{Ic} value?
3. Estimate the plastic zone size at fracture



□

Problem 8.3.

The state of stress at the most critical point of a structure is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 10000 & 5000 & -6000 \\ 5000 & 15000 & 8000 \\ -6000 & 8000 & 4000 \end{bmatrix} \quad \text{psi}$$

Calculate the margin of safety based on:

a) The true strain at fracture is 20 %. Take $S_y = 25$ ksi.

b) The true strain at fracture is 2 %. Take $S_{ut} = 30$ ksi and $S_{uc} = 120$ ksi.

□

Problem 8.4.

The state of stress at a point is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 100 & 60 & \tau \\ 60 & -50 & 0 \\ \tau & 0 & 75 \end{bmatrix} \quad \text{MPa}$$

a) If the true strain at fracture is 20 %, find the value of τ for a 12% margin of safety according to each of the related failure criteria. Take $S_{\text{y}} = 30$ MPa.

b) If the true strain at fracture is 2 %, find the value of τ for a 12% margin of safety according to each of the related failure criteria. Take $S_{\text{ut}} = 200$ MPa and $S_{\text{uc}} = 850$ MPa.

□

Problem 8.5.

Static/Quasi-Static Loading on a Shaft:
The fundamental kinematic component of our mechanical universe is the wheel and axle. An essential part of this revolute joint is the shaft. It is a good example of a static, quasi-static, and dynamically loaded body. Application of the information developed to shafts is useful and necessary.

It is left for the student to show that critical state of stress at an element located on the surface of a solid round shaft of diameter d subjected to bending, axial loading, and twisting is

$$\sigma_{\text{xx}} = \frac{32\,M}{\pi\,d^3} + \frac{4\,F}{\pi\,d^2} \qquad \tau_{\text{xz}} = -\frac{16\,T}{\pi\,d^3}$$

1. Determine the principal stresses and von Mises stress.
2. Under many axial force F is either zero or so small that its effect may be neglected. Thus $F = 0$.
   a) If the true strain at fracture is 20 %, find the value of d for a 80% margin of safety according to each of the related failure criteria. Take $S_{\text{y}} = 30$ ksi.
   b) If the true strain at fracture is 2 %, find the value of d for a 80% margin of safety according to each of the related failure criteria. Take $S_{\text{ut}} = 30$ MPa and $S_{\text{uc}} = 120$ MPa.

Take $M = 1925$ lb–in, $T = 3300$ lb–in.

□

Problem 8.6.

A standard compact fracture specimen has the dimensions of $b = 50$ mm and $t = 25$ mm, and it is subjected to an applied load of $P = 22$ KN.

1. Plot the intensity factor K versus crack length a for an interval of crack lengths $a = 15$ to 35 mm.
2. If the material is 2219-T851 aluminum, what is the longest crack that would permit the 22 kN load to be applied without brittle fracture occurring?

□

Problem 8.7.

Data are given below for compact specimens of 7075-T651 aluminum in the same sizes as those photographed in Figure 8.44. All had dimensions, as defined in figure 8.15(c), of $b = 50.8$ mm and $h = 30.5$ mm, and initial sharp precracks and thickness as tabulated below. For each test:

1. Calculate K_Q and determine where or not K_Q qualifies as a valid (plane strain) K_{Ic}
2. Estimate the plastic zone size at K_Q, using $2\,r_{o\sigma}$ or $2\,r_{o\varepsilon}$ as applicable
3. Determine whether analysis by LEFM is applicable
4. Plot K_Q versus thickness t and comment on the trend observed and its relationship to fracture surfaces in Figure 8.44

Test No.	a_i mm	t mm	P_Q kN	Basis of P_Q (See Fig. 8.27.)	P_{max} kN
1	24.1	3.18	2.56	P_5 for Type I	3.96
2	24.7	6.86	4.96	Pop-in, Type II	6.16
3	23.3	19.35	12.00	P_{max} for Type III	12.00

□

Chapter 9
Failure Theories for Dynamic Loading

Instructional Objectives of Chapter 9

After completing this chapter, the student should be able to:

1. Understand and solve structural problems under rapidly moving loads.
2. Understand and solve structural problems under time-dependent loads.

In the previous chapter we dealt exclusively with *static* loadings or, if time-varying, loads that are gradually and smoothly applied, with all parts continually in contact. The fact is that most of the mechanical engineering problems encounter dynamic loading. By dynamic loading we mean both impact and cyclic loading.

What distinguishes static and dynamic loading is the time duration of the applied load:

(i) if the load is applied slowly, it is considered static;

(ii) if the load is applied rapidly, it is considered impact;

(iii) if the load is is time-dependent, it is considered cyclic.

Since for most problems, a fundamental knowledge in vibrations is crucial in the design of machine components, a brief discussion of fundamental natural frequency is included here. This chapter is then followed by impact dynamics and concludes with a throughout discussion of fatigue analysis in the design of machine components.

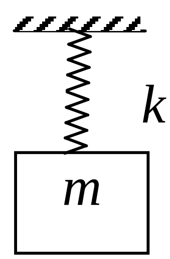
9.1 Vibration Analysis

Vibration may be defined as the oscillation or repetitive motion of a structure about an equilibrium position. The equilibrium position is the position the structure will attain when the force acting on it is zero. If the motion is the result of a disturbing force that is applied once and then removed, the motion is known as natural (or free) vibration. If a force of impulse is applied repeatedly to a system, the motion is known as forced vibration. Within both of the categories of natural and forced vibrations are the subcategories of damped and undamped vibrations. If there is no damping (i.e., no friction), a system will experience free vibrations indefinitely. This is known as free vibration and simple harmonic motion.

Fundamental Natural Frequency

Here we will focus on the most important information free (natural) vibrational analysis provide us. It is the information regarding the natural frequencies. Natural frequencies are frequencies at which the structure's enters into resonance. We have experienced as the washing machine might suddenly and uncontrollably start shaking as a consequence of relocation of clothes within the machine, the automobile starts to shake and as you increase or decrease the speed the shaking disappears. All these are examples of resonances. As the system's frequency enters in resonance with system's natural frequency, it causes loss of structural stiffness.

To better understand this topic, consider a simple spring-mass system:



The frequency at which the system will became in resonance is defined as

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{\rm st}}}$$

where ω is the angular frequency of vibration and has units of radians per second, $\delta_{\rm st}$ is the total static deflection with units of length, and g the gravitational constant with units of acceleration. The simple mass and ideal spring illustrated in the above Figure is an example of free vibration. After the mass is displaced and released, it will oscillate up and down. Since there is no friction (i.e., the vibration is undamped), the oscillations will continue forever.

In design, we want to increase or decrease the natural frequencies to avoid the structures' frequencies enter in resonance with the structure's natural frequency. The problem reduces in trying to express all structures in terms of the above spring-mass system.

In free vibration analysis, besides the fundamental angular frequency of vibration (usually called as the fundamental natural frequency) we define the linear frequency of vibration as:

$$\omega = 2\,\pi\,f \quad \rightarrow \quad f = \frac{\omega}{2\,\pi}$$

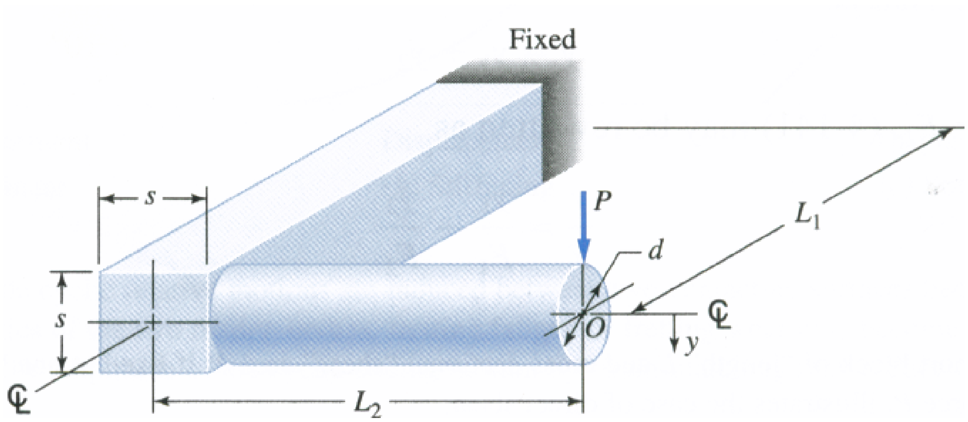
and the period of oscillation, the time to complete one cycle of oscillation, is defined as

$$T = \frac{1}{f}$$

The units for the linear frequency are Hertz (Hz), 1/sec, and the units for the period are seconds.

An important concept used in calculating the behavior of a vibrating system is the static deflection, δ_{st}.

Example 9.1.



The steel right-angle support bracket with bar lengths $L_1 = 10$ inches and $L_2 = 5$ inches, as shown in Figure, is to be used to support the static load $P = 1000$ lb. The load is to be applied vertically at the free end of the cylindrical bar, as shown. Both bracket bar centerlines lie in the same horizontal plane. If the square bar has side $s = 1.25$ inches, and the cylindrical leg has diameter $d = 1.25$ inches.

a) The total static deflection is defined as:

$$\delta_{\text{st}} = \frac{P}{k_{\text{eff}}}$$

The load P is known and the problem reduces to find the overall spring rate of the system. Note the square bar will be subject to both torsional and bending deflections, while the cylindrical bar is subject to bending only. This can be modeled as spring in series. Thus

$$k_{\text{eff}} = \frac{1}{\sum_{i=1}^{3} \frac{1}{k_i}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}}$$

where k_1 is the spring rate caused by bending of the square bar, k_2 the spring rate caused by torsion through of the square bar reflected to point **O** through rigid body rotation of cylinder bar length L_2, and k_3 is the spring rate caused by bending of the cylindrical.

For the bending of the squared cross-section,

$$k_1 = \frac{P}{y_1}$$

Using tables,

$$y_1 = \frac{P\,L_1^3}{3\,EI} \rightarrow k_1 = \frac{P}{y_1} = \frac{3\,EI}{L_1^3}$$

For a squared cross-section:

$$I = \frac{s^4}{12}$$

Thus

$$k_1 = \frac{E\,s^4}{4\,L_1^3}$$

Next, for the torsion of the square cross-section,

$$k_2 = \frac{P}{y_2}$$

where $y_2 = L_2\,\theta$. The total rotation angle is calculated as

$$\theta = \frac{P\,L_2\,L_1}{K_{xx}\,G}$$

Using this information:

$$k_2 = \frac{P}{y_2} = \frac{P}{L_2\,\theta} = \frac{P}{L_2\left(\dfrac{P\,L_2\,L_1}{K_{xx}\,G}\right)} = \frac{K_{xx}\,G}{L_1\,L_2^2}$$

Using tables for a squared cross-section:

$$K_{xx} = 2.25\left(\frac{s}{2}\right)^4 = 0.14\,s^4$$

Thus

$$k_2 = \frac{0.14\,s^4\,G}{L_1\,L_2^2}$$

For the bending of the circular cross-section,

$$k_3 = \frac{P}{y_3}$$

Using tables,

$$y_3 = \frac{P\,L_2^3}{3\,EI} \rightarrow k_3 = \frac{P}{y_3} = \frac{3\,EI}{L_3^3}$$

For a circular cross-section:

$$I = \frac{\pi d^4}{64}$$

Thus

$$k_3 = \frac{3\,\pi\,E\,d^4}{64\,L_2^3}$$

Thus, the overall spring rate is

$$k_{\text{eff}} = \frac{1}{\dfrac{1}{\dfrac{E\,s^4}{4\,L_1^3}} + \dfrac{1}{\dfrac{0.14\,s^4\,G}{L_1\,L_2^2}} + \dfrac{1}{\dfrac{3\,\pi\,E\,d^4}{64\,L_2^3}}}$$

$$= \frac{E\,s^4}{L_1^3}\left(\frac{1}{4 + 0.14\left(\dfrac{E}{G}\right)\left(\dfrac{L_2}{L_1}\right)^2 + \dfrac{64}{3\,\pi}\left(\dfrac{L_2}{L_1}\right)^3\left(\dfrac{s}{d}\right)^4}\right)$$

Using tables,

$$E = 30 \times 10^6 \quad \text{psi} \qquad G = 11.5 \times 10^6 \quad \text{psi}$$

$$k_{\text{eff}} = 7.70 \times 10^3 \quad \frac{\text{lb}}{\text{in}}$$

The total static deflection for the given structure is

$$\delta_{\text{st}} = \frac{P}{k_{\text{eff}}} = 0.13 \quad \text{in} = 0.010833 \quad \text{ft}$$

b) Determine the fundamental natural frequency in rpm (revolutions per minute)

$$\omega = \sqrt{\frac{g}{\delta_{\text{st}}}}$$

where

$$g = 32.2 \quad \frac{\text{ft}}{\text{sec}^2}$$

and the total static deflection for the given structure, is

$$\delta_{\text{st}} = \frac{P}{k_{\text{eff}}} = 0.13 \quad \text{in} = 0.010833 \quad \text{ft}$$

Hence

$$\omega = 54.51887 \quad \frac{\text{rad}}{\text{sec}}\left(\frac{\text{rev}}{2\,\pi\,\text{rad}}\right)\left(\frac{60\,\text{sec}}{\text{min}}\right) = 520.62 \quad \text{rpm}$$

c) Determine the period of oscillation when the structure enters in resonance with the fundamental frequency in seconds.

The period of oscillation is defined as

$$T = \frac{1}{f}$$

where f is the fundamental linear frequency defined as;

$$f = \frac{\omega}{2\,\pi} = 8.676954 \quad \frac{1}{\text{sec}} = 8.676954 \quad \text{Hz}$$

Thus

$$T = 0.12 \quad \text{sec}$$

End Example □

9.2 Impact

Impact loading is also known as shock, sudden, or impulsive loading. We constantly experience this types of loading: driving a nail with a hammer, hitting a baseball with a bat, automobile collisions, wheels dropping into potholes, jumping of a diving board, a bird striking an aircraft jet engine blade and the list goes on. Impact loads may be divided into three categories:

1. Rapidly moving loads of essentially constant magnitude, as produced by a vehicle crossing a bridge;
2. Suddenly applied loads, such as those in an explosion, or from combustion in an engine cycle.
3. direct-impact loads, as produced by a vehicle collision.

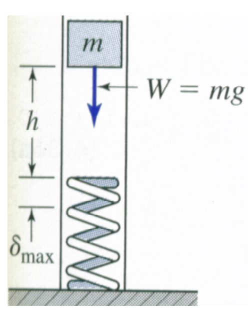
9.2.1 Assumptions

Although impact load causes elastic members to vibrate until equilibrium is reestablished, our concern here is with only the influence of impact or shock force on the maximum stress and deformation within the member. In engineering, the design of structures subject to impact loading may be far more complicated that the approach shown in textbook. However, few approximation greatly simplify the problem providing a qualitative guide in designing these structures. Here typical impact problems will use the energy approach of the mechanics of materials theory along with the following common assumptions:

1. The displacements is proportional to the loads.
2. The material behaves elastically, and a static stress-strain diagram is also valid under impact.
3. The inertia of the member resisting impact may be neglected.
4. No energy is dissipated because of local deformation at the point of impact or at the supports.

Although there are many other types of impact loadings such as torsional loading, here we will limit to loadings that cause axial and bending stresses only.

9.2.2 Freely falling body



Consider the free-standing spring with a spring rate k, on which is dropped a body of mass m from a height h. The total energy in the system may be expressed as

$$E_k + E_p = E_s$$

where E_k, E_p, and E_s is the total change in kinetic, potential, and stored energy from its initial position to the instant of maximum deflection, respectively.

For a freely falling body, the initial velocity is zero and again zero at the instant of maximum deflection of the spring (δ_{max}) and thus the change in kinetic energy of the system is zero. Therefore, the work done by gravity as its falls is equal to the resisting work done by the spring:

$$m\, g\, \eta_{\text{m}}\ (h + \delta_{\text{max}}) = \frac{1}{2}\, k\, \delta_{\text{max}}^2 \tag{9.1}$$

where η_{m} is a correction factor to account for the energy dissipation associated with the particular type of elastic member being struck and may defined for various cases. If the dissipation is negligible, η_{m} will be one. In general,

$$0 < \eta_{\text{m}} \leq 1.0$$

and by taking $\eta_{\text{m}} = 1.0$ we are being conservative.

Let the total weight of the mass be $W = m\, g$, and ignore the dissipation for derivation, and rearranging Eq. (9.1) we get:

$$W\ (h + \delta_{\text{max}}) = \frac{1}{2}\, k\, \delta_{\text{max}}^2$$

$$2\, \frac{W}{k}\, (h + \delta_{\text{max}}) = \delta_{\text{max}}^2$$

From the previous section, the deflection corresponding to a static force is simply the total static deflection, δ_{st}. Thus the above may be expressed as

$$2\, \delta_{\text{st}}\ (h + \delta_{\text{max}}) = \delta_{\text{max}}^2$$

Now let us solve for the maximum deflection of the spring

$$\delta_{\text{max}}^2 - 2\, \delta_{\text{st}}\, \delta_{\text{max}} - 2\, \delta_{\text{st}}\, h = 0$$

Using the quadratic equation as the maximum dynamic deflection is defined as

$$\delta_{\text{max}} = \frac{-\,(-2\, \delta_{\text{st}}) + \sqrt{(-2\, \delta_{\text{st}})^2 - 4\ (-2\, \delta_{\text{st}}\, h)\ (1)}}{2\,(1)} = \delta_{\text{st}} + \sqrt{\delta_{\text{st}}^2 + 2\, \delta_{\text{st}}\, h}$$

$$= \delta_{\text{st}} \left(1 + \sqrt{1 + \frac{2\, h}{\delta_{\text{st}}}}\right)$$

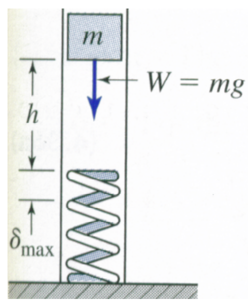
More generally let us define the impact factor as

$$K_{m_1} = 1 + \sqrt{1 + \frac{2\,h}{\delta_{st}}\,\eta_m} \tag{9.2}$$

Thus the maximum dynamic deflection is defined as

$$\delta_{max} = K_{m_1}\,\delta_{st} \tag{9.3}$$

9.2.3 Falling body with a velocity



Consider the free-standing spring with a spring rate k, on which is a body of mass m is approach with a speed v from a height h. The total energy in the system may be expressed as

$$E_k + E_p = E_s$$

where E_k, E_p, and E_s is the total change in kinetic, potential, and stored energy from its initial position to the instant of maximum deflection, respectively.

At impact the energy relationship is:

$$E_k = E_p \rightarrow \frac{1}{2}\,m\,v^2 = m\,g\,h \rightarrow h = \frac{v^2}{2\,g}$$

we can use the relationships for the free falling object and substitute

$$h = \frac{v^2}{2\,g}$$

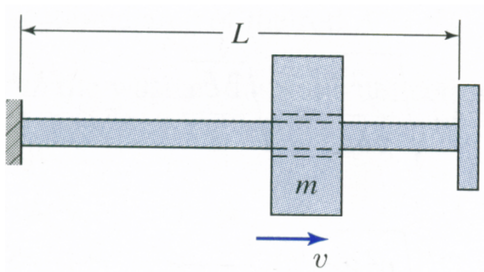
Thus the impact factor will be

$$K_{m_2} = 1 + \sqrt{1 + \frac{v^2}{\delta_{st}\,g}\,\eta_m} \tag{9.4}$$

and the maximum dynamic deflection is defined as

$$\delta_{max} = K_{m_2}\,\delta_{st} \tag{9.5}$$

9.2.4 Horizontally Moving Weight



Consider a mass (m) in horizontal motion with a velocity v, stopped by an elastic body. The total energy in the system may be expressed as

$$E_k + E_p = E_s$$

where E_k, E_p, and E_s is the total change in kinetic, potential, and stored energy from its initial position to the instant of maximum deflection, respectively.

Since the mass is moving horizontally the potential energy is zero and the velocity is zero at the instant of maximum deflection of the spring (δ_{max}). Thus

$$E_k = E_s \rightarrow \frac{1}{2}\, m\, \eta_{\text{m}}\, v^2 = \frac{1}{2}\, k\, \delta_{\text{max}}^2$$

Rearranging

$$\frac{1}{2}\, m\, \eta_{\text{m}}\, g\, v^2 = \frac{1}{2}\, k\, g\, \delta_{\text{max}}^2 \rightarrow \frac{W}{k}\, \eta_{\text{m}}\, v^2 = g\, \delta_{\text{max}}^2 \rightarrow \delta_{\text{st}}^2\, \eta_{\text{m}}\, \frac{v^2}{g\, \delta_{\text{st}}} = \delta_{\text{max}}^2$$

Thus the maximum dynamic deflection can be written as

$$\delta_{\text{max}} = \delta_{\text{st}} \sqrt{\eta_{\text{m}}\, \frac{v^2}{g\, \delta_{\text{st}}}} = K_{\text{m}_2}\, \delta_{\text{st}}$$

where the impact factor may be defined as

$$K_{\text{m}_3} = \sqrt{\eta_{\text{m}}\, \frac{v^2}{g\, \delta_{\text{st}}}}$$

9.2.5 Maximum Dynamic Load and Stress

Since

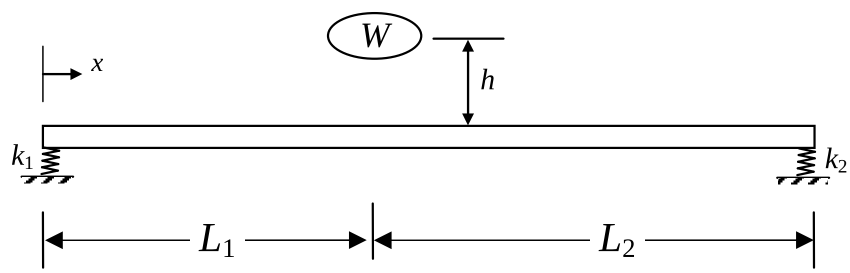
$$\delta_{\text{max}} = \frac{F_{\text{max}}}{k}$$

we can show the following relationships are true

$$F_{\text{max}} = K_{\text{m}}\, F_{\text{st}} \qquad \text{and} \qquad \sigma_{\text{max}} = K_{\text{m}}\, \sigma_{\text{st}}$$

Example 9.2.

Freely falling object



An engineer has designed a machine component that can be modeled as a $S3 \times 5.7$ beam on two identical springs, as shown in Figure. The 5 feet long beam is made of AISI 304 annealed steel. Just after the installation was completed, a 100 lb object 0.245 meters above the structural component suddenly falls at a distance 2 feet from the left spring. The spring rate are 100 lb/in. Using a dissipation correction factor of 0.95, determine if structural component needs to be replaced.

First of all, AISI 304 annealed steel is a ductile material and thus we need to check for ductile failure. Let us consider the distortional energy theory. From tables:

$$S_{\text{yield}} = 35 \times 10^3 \text{ psi} \qquad E = 28.0 \times 10^6 \text{ psi}$$

For a $S3 \times 5.7$ beam:

$$I_{zz} = 2.5 \text{ in}^4$$

From the problem

$$L_1 = 2 \text{ ft} = 24 \text{ in} \qquad L_2 = 3 \text{ ft} = 36 \text{ in} \qquad L = L_1 + L_2 = 60 \text{ in}$$

$$h = 0.245 \text{ m} = 10 \text{ in} \qquad \eta_{\text{m}} = .95$$

Assume that the impact load is uniform at the location of impact at the beam.

The maximum static deflection for the beam only is:

$$\delta_{\text{st,beam}} = \frac{W\, L_1\, L_2\, (L_1 + 2L_2)\, \sqrt{3\, L_1\, (L_1 + 2\, L_2)}}{27\, E\, I_{zz}\, L} = 0.00608098 \text{ in}$$

and occurs at

$$x = \sqrt{\frac{L_1\, (L_1 + 2\, L_2)}{3}} = 27.71 \text{ in}$$

The static deflection for the supporting springs only is:

$$\delta_{\text{st,springs}} = \frac{W}{k_1 + k_2} = 0.50 \text{ in}$$

The total static deflection is:

$$\delta_{\text{st}} = \delta_{\text{st,beam}} + \delta_{\text{st,springs}} = \frac{W\, L_1\, L_2\, (L_1 + 2L_2)\, \sqrt{3\, L_1\, (L_1 + 2\, L_2)}}{27\, E\, I_{\text{zz}}\, L} + \frac{W}{2\, k} = 0.506081 \text{ in}$$

The impact factor for freely falling object is

$$K_{\text{m}_1} = 1 + \sqrt{1 + \frac{2\, h}{\delta_{\text{st}}}\, \eta_{\text{m}}} = 7.20833$$

Thus the maximum deflection is

$$\delta_{\text{max}} = K_{\text{m}}\, \delta_{\text{st}} = 3.65 \text{ in}$$

and the maximum load is:

$$W_{\text{max}} = K_{\text{m}}\, W_{\text{st}} = 720.833 \text{ lb}$$

As we can see from the figure the maximum bending moment will occur at the point of the load:

$$M_{\text{max}} = -\frac{W_{\text{max}}\, L_1\, L_2}{L} = -10380 \text{ lb-in}$$

and the maximum bending stress will occur at an element at the top $y = c = 0.9+0.63 = 1.53$ in (from tables for a $S3 \times 5.7$ beam):

$$\sigma_{\text{xx}} = -\frac{M_{\text{max}}\, c}{I_{\text{zz}}} = 6352.56 \text{ psi}$$

Using the distortional energy,

$$\sigma_{\text{eq}} = 6352.56 \text{ psi}$$

In order to determine if it is safe, we need to find the factor of safety:

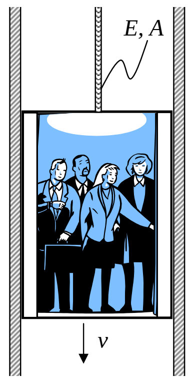
$$n_{\text{SF}} = \frac{S_{\text{yield}}}{\sigma_{\text{eq}}} = 5.50959$$

and the margin of safety is 450%. Thus there is no need to change the structural component.

End Example □

Example 9.3.

Object falling with a speed



A 5-ton elevator is supported by a titanium cable with an effective modulus of elasticity of 18×10^6 psi and a cross-sectional area A. The titanium has a true strain at fracture higher than 5% in 2 inches. As the elevator is descending at a constant speed of 400 fpm, an accident causes the top of the cable, 70 ft above the elevator, to stop suddenly. What area A will ensure a 150% safety? For the design area what will be the maximum elongation the cable will experience. Be conservative.

For a titanium cable:

$$S_{\text{yield}} = 128 \times 10^3 \text{ psi}$$

For a 150% margin of safety,

$$n_{\text{SF}} = 2.5 \quad \rightarrow \quad \sigma_{\text{all}} = \frac{S_{\text{yield}}}{n_{\text{SF}}} = 51200 \text{ psi}$$

From the information provided:

$$W = 5 \text{ ton} = 10000 \text{ lb} \qquad L = 70 \text{ ft} = 840 \text{ in} \qquad v = 400 \text{ ft/min} = 80 \text{ in/s}$$

Assume: the mass of the cable is negligible ($\eta_{\text{m}} = 1$), neglect any stress concentrations, ignore damping due to internal friction within the cable, and the cable responds elastically to the impact.

The static deflection is:

$$\delta_{\text{st}} = \frac{W\,L}{E\,A} = \frac{7}{15\,A}$$

The impact factor is:

$$K_{\text{m}_2} = 1 + \sqrt{1 + \frac{v^2}{\delta_{\text{st}}\,g}\,\eta_{\text{m}}} = 1 + \sqrt{1 + 426.254\,A}$$

The maximum load is

$$W_{\text{max}} = K_{\text{m}}\, W_{\text{st}} = 10000 + 10000\sqrt{1 + 426.254\, A}\ \text{lb}$$

The maximum axial stress is:

$$\sigma_{\text{xx}} = \frac{W_{\text{max}}}{A} = \frac{10000 + 10000\sqrt{1 + 426.254\, A}}{A}\ \text{psi}$$

Since we want to be conservative and the material is ductile, let us use the maximum shear stress theory:

$$\sigma_1 = \frac{10000 + 10000\sqrt{1 + 426.254\, A}}{A}, \qquad \sigma_2 = \sigma_3 = 0$$

$$\tau_{\text{max}} = \frac{5000 + 5000\sqrt{1 + 426.254\, A}}{A}$$

Thus,

$$2\,\tau_{\text{max}} = \frac{S_{\text{y}}}{n_{\text{SF}}} = \sigma_{\text{all}} \quad \rightarrow \quad \frac{10000 + 10000\sqrt{1 + 426.254\, A}}{A} = 51200$$

Solving for area:

$$A = 16.65\ \text{in}^2$$

A cross-sectional area of 16.65 in^2 will ensure a 150% margin of safety. Thus the maximum elongation the cable will experience is:

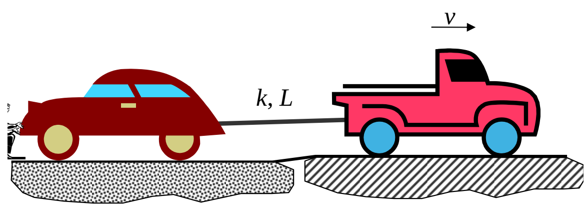
$$K_{\text{m}_2} = 85.25 \qquad \delta_{\text{st}} = 0.028\ \text{in}$$

$$\delta_{\text{max}} = K_{\text{m}}\, \delta_{\text{st}} = 2.39\ \text{in}$$

End Example □

Example 9.4.

Object moving horizontally



A car became stuck in sand at a waterfront. A pickup truck, of 1400-kg mass, has offered to help by attempting to jerk the stuck vehicle back onto the road using a 5-m steel tow cable of stiffness $k = 5000$ N/mm. The traction available to the pickup truck prevented it from exerting any significant force on the cable. With the aid of a push from bystanders, the rescue car was able to back against the stuck car and then go forward and reach a speed of 4 km/h at the instant the cable became taut (stretched tight). If the cable is attached rigidly to the masses of the automobiles, estimate the maximum impact force that can be developed in the cable, and the resulting cable elongation.

From the information provided:

$$k = 5000 \text{ N/mm} = 5 \times 10^6 \text{ N/m} \qquad L = 5 \text{ m}$$

$$W = (1400 \text{ kg})(9.81 \text{ m/s}^2) = 13734 \text{ N} \qquad v = 4 \text{ km/hr} = 1.11 \text{ m/s}$$

Assume: the mass of the rope is negligible ($\eta_{\text{m}} = 1$), neglect any stress concentrations, the rope is attached rigidly to the mass of the cars, ignore damping due to internal friction within the rope, and the rope responds to the impact elastically.

The static deflection is:

$$\delta_{\text{st}} = \frac{W}{k} = 0.00275 \text{ m}$$

The impact factor is:

$$K_{\text{m}_3} = \sqrt{\frac{v^2}{\delta_{\text{st}}\, g}\, \eta_{\text{m}}} = 6.76$$

Thus the maximum impact for is:

$$F_{\text{max}} = K_{\text{m}}\, W = 92.6 \text{ kN}$$

The maximum cable elongation is

$$\delta_{\text{max}} = K_{\text{m}}\, \delta_{\text{st}} = 0.00186 \text{ m}$$

End Example □

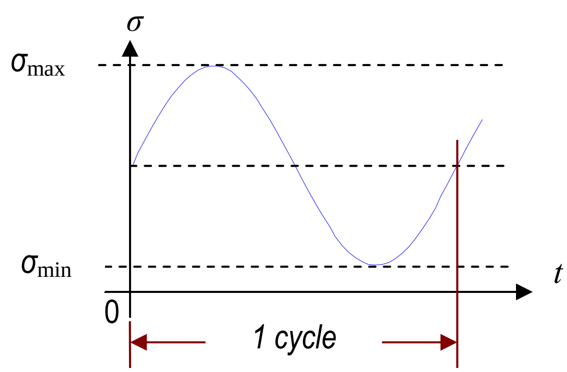
9.3 Fatigue

Fatigue was first introduced in 1839 by Poncelet of France. Fatigue fractures begin with minute cracks at critical areas and propagate. Final fracture is largely "brittle" fracture and results from repeated plastic deformation. Fatigue failue may occur at stress levels far below the yield strength after thousands or millions of cycles. Strengthening vulnerable locations is often as effective as making the entire part from a stronger material. Most of the work depends on experimental data.

9.3.1 Cyclic Stresses

A cyclic stress is a time-dependent function where the variation is such that the stress sequence repeats itself. The cyclic stresses may be axial (compressive or tensile), flexural (bending), or torsional (twisting). There are several parameters used to characterize fluctuating cyclic stresses.

First, let us define the life cycle with N. Note that one stress cycle ($N = 1$) constitutes a single application and removal of a load and then another application and removal of the load in the opposite direction.



Thus $N = 1/2$ means the load is applied once and then removed, which is the case with the simple tension test.

The **mean stress** σ_m is the average of the maximum and minimum stresses in the cycle:

$$\sigma_\mathrm{m} = \frac{\sigma_\mathrm{max} + \sigma_\mathrm{min}}{2}$$

The **stress range** σ_r is the difference range of the maximum and minimum stresses in the cycle:

$$\sigma_\mathrm{r} = \Delta\sigma = |\sigma_\mathrm{max} - \sigma_\mathrm{min}|$$

The **stress amplitude** σ_a is the one-half of the stress range in the cycle:

$$\sigma_\mathrm{a} = \frac{\sigma_\mathrm{r}}{2} = \left|\frac{\sigma_\mathrm{max} - \sigma_\mathrm{min}}{2}\right|$$

The **stress ratio** R_s is the ratio of minimum to maximum stress amplitudes:

$$R_s = \frac{\sigma_{\min}}{\sigma_{\max}}$$

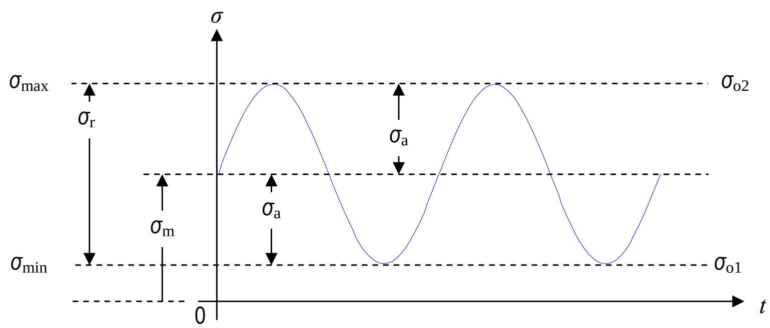
The **amplitude ratio** $\mathbb{A}_a$ is the ratio of the stress amplitude to the mean stress:

$$\mathbb{A}_a = \frac{\sigma_a}{\sigma_m} = \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} = \frac{1 - R_s}{1 + R_s}$$

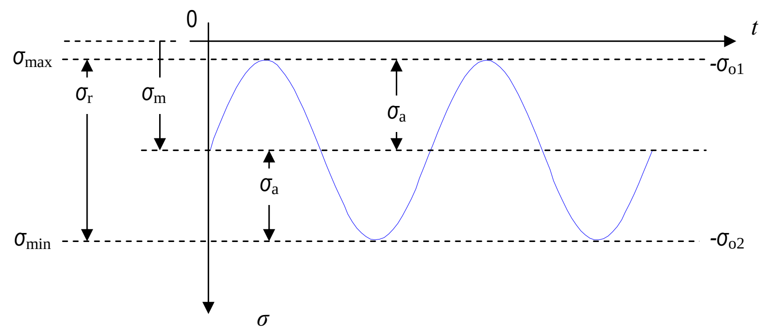
The **maximum and minimum stresses** may also be calculated using:

$$\sigma_{\max} = \sigma_m + \sigma_a \qquad \sigma_{\min} = \sigma_m - \sigma_a$$

Fluctuating



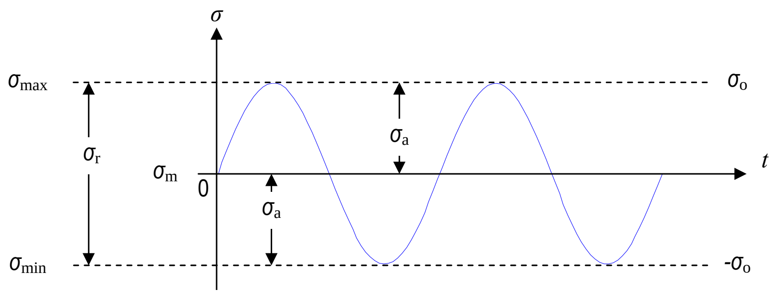
$$\sigma_r = 2\,\sigma_o \qquad \sigma_a = \sigma_o$$



$$\sigma_r = 2\,\sigma_o \qquad \sigma_a = \sigma_o$$

Fully Reversed

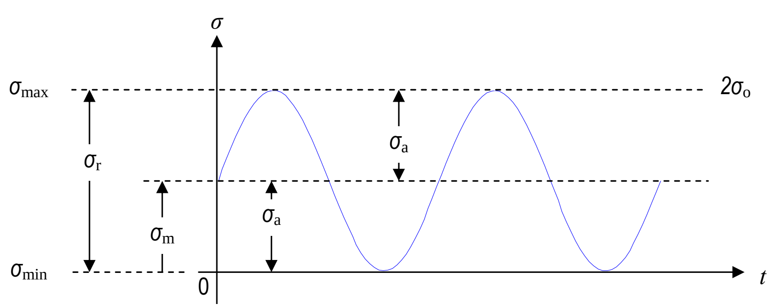
It is also known as zero-mean or completely reversed.



$$\sigma_{\text{max}} = \sigma_{\text{o}} \qquad \sigma_{\text{min}} = -\sigma_{\text{o}} \qquad \sigma_{\text{m}} = 0 \qquad \sigma_{\text{r}} = 2\,\sigma_{\text{o}} \qquad \sigma_{\text{a}} = \sigma_{\text{o}}$$

$$\mathbb{A}_{\text{a}} = \infty \qquad R_{\text{s}} = -1$$

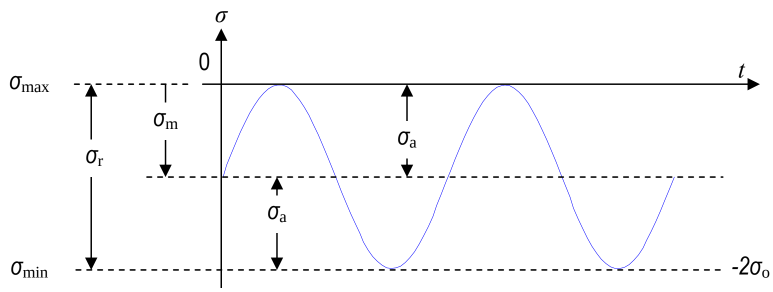
Repeated (Tension)



$$\sigma_{\text{max}} = 2\,\sigma_{\text{o}} \qquad \sigma_{\text{min}} = 0 \qquad \sigma_{\text{m}} = \sigma_{\text{o}} \qquad \sigma_{\text{r}} = 2\,\sigma_{\text{o}} \qquad \sigma_{\text{a}} = \sigma_{\text{o}}$$

$$\mathbb{A}_{\text{a}} = 1 \qquad R_{\text{s}} = 0$$

Repeated (Compression)



$$\sigma_{\rm max} = 0 \qquad \sigma_{\rm min} = -2\,\sigma_{\rm o} \qquad \sigma_{\rm m} = -\sigma_{\rm o} \qquad \sigma_{\rm r} = 2\,\sigma_{\rm o} \qquad \sigma_{\rm a} = \sigma_{\rm o}$$

$$\mathbb{A}_{\rm a} = -1 \qquad R_{\rm s} = \infty$$

9.4 Alternate and mean stresses

Alternate state of stress is found by determining the alternate loads and find the stresses for these loads:

$$T_{\rm a} = \frac{T_{\rm max} - T_{\rm min}}{2} \qquad \rightarrow \qquad \tau_{\rm a}\Big|_{\rm torsion}$$

$$V_{\rm a} = \frac{V_{\rm max} - V_{\rm min}}{2} \qquad \rightarrow \qquad \tau_{\rm xy,a}\Big|_{\rm shear}$$

$$M_{\rm a} = \frac{M_{\rm max} - M_{\rm min}}{2} \qquad \rightarrow \qquad \sigma_{\rm xx,a}\Big|_{\rm bending}$$

$$P_{\rm a} = \frac{P_{\rm max} - P_{\rm min}}{2} \qquad \rightarrow \qquad \sigma_{\rm xx,a}\Big|_{\rm axial}$$

Mean state of stress is found by determining the mean loads and find the stresses for these loads:

$$T_{\rm m} = \frac{T_{\rm max} + T_{\rm min}}{2} \qquad \rightarrow \qquad \tau_{\rm m}\Big|_{\rm torsion}$$

$$V_{\rm m} = \frac{V_{\rm max} + V_{\rm min}}{2} \qquad \rightarrow \qquad \tau_{\rm xy,m}\Big|_{\rm shear}$$

$$M_{\rm m} = \frac{M_{\rm max} + M_{\rm min}}{2} \qquad \rightarrow \qquad \sigma_{\rm xx,m}\Big|_{\rm bending}$$

$$P_{\rm m} = \frac{P_{\rm max} + P_{\rm min}}{2} \qquad \rightarrow \qquad \sigma_{\rm xx,m}\Big|_{\rm axial}$$

Most real design situations involve fluctuating loads that produce multiaxial states of cyclic stress:

$$\underline{\boldsymbol{\sigma}}_{\mathrm{a}} = \begin{bmatrix} \sigma_{\mathrm{xx,a}} & \tau_{\mathrm{xy,a}} & \tau_{\mathrm{xz,a}} \\ \tau_{\mathrm{yx,a}} & \sigma_{\mathrm{yy,a}} & \tau_{\mathrm{yz,a}} \\ \tau_{\mathrm{zx,a}} & \tau_{\mathrm{zy,a}} & \sigma_{\mathrm{zz,a}} \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\mathrm{m}} = \begin{bmatrix} \sigma_{\mathrm{xx,m}} & \tau_{\mathrm{xy,m}} & \tau_{\mathrm{xz,m}} \\ \tau_{\mathrm{yx,m}} & \sigma_{\mathrm{yy,m}} & \tau_{\mathrm{yz,m}} \\ \tau_{\mathrm{zx,m}} & \tau_{\mathrm{zy,m}} & \sigma_{\mathrm{zz,m}} \end{bmatrix}$$

A consensus has not yet reached on the best approach to predict failure under multiaxial cyclic stress. However, the following techniques will be used in this book.

9.4.1 Ductile materials

Although there is little multiaxial fatigue data available, for ductile materials, the distortion energy multiaxial fatigue failure theory is the best theory to use. It consists in determining the von Mises stress for both mean state of stresses, $\underline{\boldsymbol{\sigma}}_{\mathrm{m}}$, and alternate state of stresses, $\underline{\boldsymbol{\sigma}}_{\mathrm{a}}$:

$$\begin{aligned}
\sigma_{\mathrm{a}} = \sigma_{\mathrm{eq,a}} &= \sqrt{\frac{(\sigma_{1,\mathrm{a}} - \sigma_{2,\mathrm{a}})^2 + (\sigma_{2,\mathrm{a}} - \sigma_{3,\mathrm{a}})^2 + (\sigma_{3,\mathrm{a}} - \sigma_{1,\mathrm{a}})^2}{2}} \\
&= \sqrt{\frac{(\sigma_{\mathrm{xx,a}} - \sigma_{\mathrm{yy,a}})^2 + (\sigma_{\mathrm{yy,a}} - \sigma_{\mathrm{zz,a}})^2 + (\sigma_{\mathrm{zz,a}} - \sigma_{\mathrm{xx,a}})^2 + 6\left(\tau_{\mathrm{xy,a}}^2 + \tau_{\mathrm{yz,a}}^2 + \tau_{\mathrm{xz,a}}^2\right)}{2}} \\
&= \sqrt{I_{\sigma_{1,\mathrm{a}}}^2 - 3\, I_{\sigma_{2,\mathrm{a}}}} \\
\sigma_{\mathrm{m}} = \sigma_{\mathrm{eq,m}} &= \sqrt{\frac{(\sigma_{1,\mathrm{m}} - \sigma_{2,\mathrm{m}})^2 + (\sigma_{2,\mathrm{m}} - \sigma_{3,\mathrm{m}})^2 + (\sigma_{3,\mathrm{m}} - \sigma_{1,\mathrm{m}})^2}{2}} \\
&= \sqrt{\frac{(\sigma_{\mathrm{xx,m}} - \sigma_{\mathrm{yy,m}})^2 + (\sigma_{\mathrm{yy,m}} - \sigma_{\mathrm{zz,m}})^2 + (\sigma_{\mathrm{zz,m}} - \sigma_{\mathrm{xx,m}})^2 + 6\left(\tau_{\mathrm{xy,m}}^2 + \tau_{\mathrm{yz,m}}^2 + \tau_{\mathrm{xz,m}}^2\right)}{2}} \\
&= \sqrt{I_{\sigma_{1,\mathrm{m}}}^2 - 3\, I_{\sigma_{2,\mathrm{m}}}}
\end{aligned}$$

The equivalent maximum and minimum stresses may be found by:

$$\sigma_{\max} = \sigma_{\mathrm{m}} + \sigma_{\mathrm{a}} \qquad \sigma_{\min} = \sigma_{\mathrm{m}} - \sigma_{\mathrm{a}}$$

9.4.2 Brittle materials

Although there is little multiaxial fatigue data available, for brittle materials, the maximum normal multiaxial fatigue failure theory is the best theory to use. It consists in determining the maximum stress for both mean and alternate state of stresses:

$$\sigma_a = \sigma_{1,a} \qquad \text{and} \qquad \sigma_m = \sigma_{1,m}$$

The equivalent maximum and minimum stresses may be found by:

$$\sigma_{max} = \sigma_m + \sigma_a \qquad \sigma_{min} = \sigma_m - \sigma_a$$

9.5 Fatigue Stress Concentration Factor

The stress concentration factor is a function of the type of discontinuity (hole, fillet, groove), the geometry of the discontinuity, and the type of loading being experienced. Some materials are not as sensitive to notches as implied by the theoretical stress concentration factor. For these materials a reduced value of K_t maybe used K_f.

Not all ductile materials are ductile under all conditions, many become brittle under some circumstances. The most common cause of brittle behavior in materials normally considered to be ductile is being exposed to low temperatures. For ductile materials subjected to cyclic loading the stress concentration factor has to be included in the factors that reduce the fatigue strength of a component.

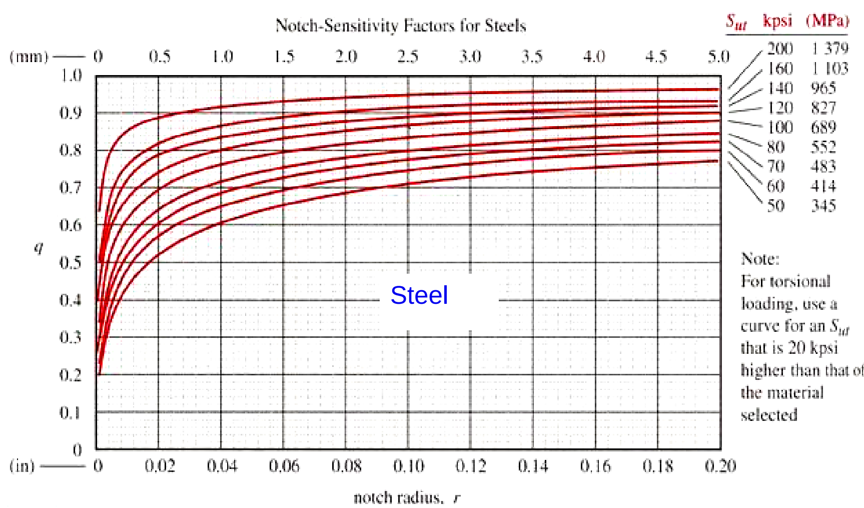


Figure 9.1: Notch-Sensitivity Factors for Steels.

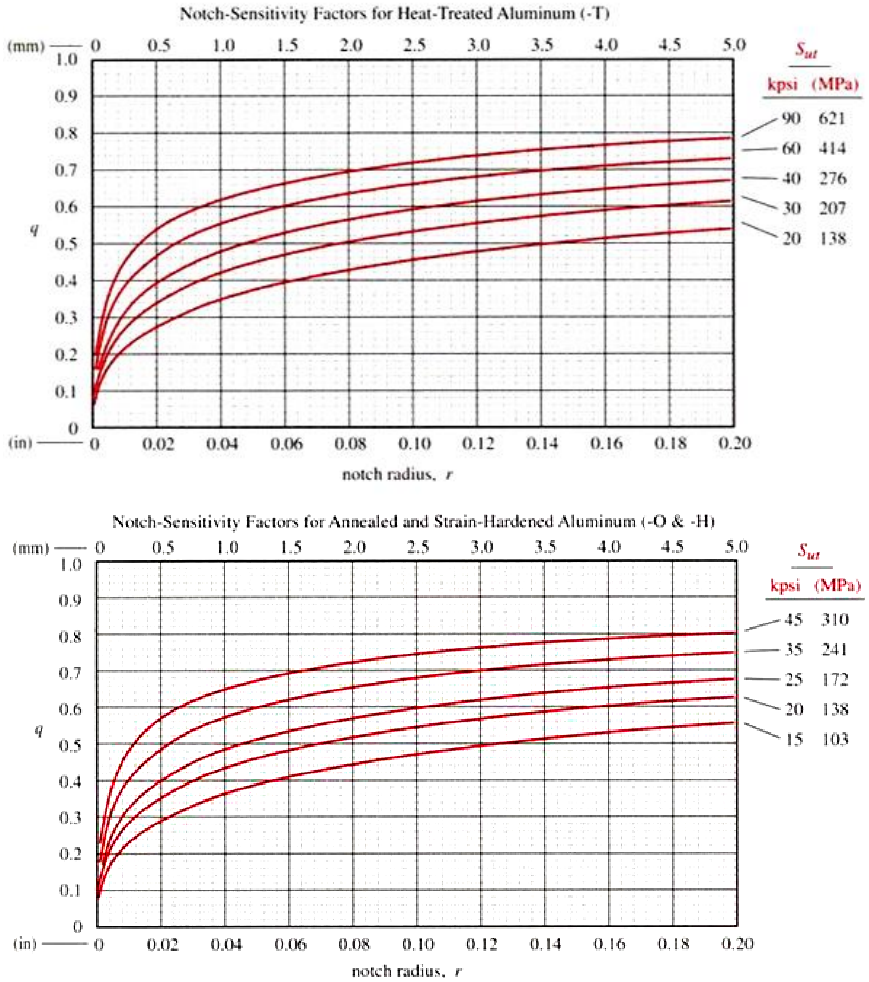


Figure 9.2: Notch-Sensitivity Factors for Aluminum..

Some materials are not as sensitive to notches as implied by the theoretical stress concentration factor. For these materials a reduced value of K_t is used, that is K_f. In these materials the maximum stress is:

$$\sigma_t = K_f \sigma_N \tag{9.6}$$

In the Stress-Life approach, the effect of notches is accounted for by the fatigue notch factor K_f (also known as the fatigue stress concentration factor). The fatigue notch factor relates the unnotched fatigue strength (the endurance limit for ferrous metals) of a member to its notched fatigue strength:

$$K_f = \frac{S_e \text{ (unnotched)}}{S_e \text{ (notched)}}$$

In almost all cases, the fatigue stress concentration is less than the stress concentration factor, and is less than 1:

$$1 \leq K_f \leq K_t$$

The static stress concentration factor K_t can be related to the fatigue notch factor K_f. Unlike the stress concentration factor K_t, the fatigue notch factor K_f is dependent on the type of material and notch size. To account for these additional effects, a notch sensitivity factor q was developed

$$q = \frac{K_f - 1}{K_t - 1} \tag{9.7}$$

where q is the notch sensitivity factor and ranges between 0 ($K_f = 1$) and 1 ($K_f = K_t$). This will be discussed later when working with fatigue analysis. Depending the case we are working with is the stress concentration factor we choose. A number of researchers have proposed analytical relationships for the determination of q, based on correlation to experimental data. The most common relationships are those proposed by Peterson and Neuber. Both the Peterson and Neuber relations are empirical curve fits to data. When used for analysis there is little difference to the approaches. Both methods show that q is related to material, notch geometry, and notch size. These are given in Figs. 9.1 and 9.2.

Different authors have chosen different approaches in solving these problems. Here, we choose the following methodology. Recall, for static loading the geometric stress concentration factor K_t is used for brittle materials but taken as one for ductile materials. For fatigue loading the fatigue stress concentration factor K_f may be used, depending whether it is ductile or brittle.

9.5.1 Ductile materials

Here, the following approach will be used:

a) When the plastic strain at the notch can be avoided, apply the fatigue stress concentration factor to the alternate stresses (K_f) and for the mean stress use the following approach. There is the method by Dowling for ductile materials, which, for materials with a pronounced yield point and approximated by an elastic-perfectly-plastic behavior model, quantitatively expresses the steady stress component stress-concentration factor $K_{\rm fm}$ as

$$\text{No yielding:} \quad K_{\rm fm} = K_{\rm f} \qquad K_{\rm f}\,|\sigma_{\rm max,nom}| < S_{\rm y}$$

$$\text{Initial yielding:} \quad K_{\rm fm} = \frac{K_{\rm f} - K_{\rm f}\,\sigma_{\rm a,nom}}{|\sigma_{\rm max,nom}|} \qquad K_{\rm f}\,|\sigma_{\rm max,nom}| > S_{\rm y}$$

$$\text{Reversed yielding:} \quad K_{\rm fm} = 0 \qquad K_{\rm f}\,|\sigma_{\rm max,nom} - \sigma_{\rm min,nom}| > 2\,S_{\rm y}$$

where $\sigma_{\rm max,nom}$ and $\sigma_{\rm min,nom}$ are the fluctuating maximum and minimum nominal stresses, and $\sigma_{\rm a,nom}$ and $\sigma_{\rm m,nom}$ are the alternate and mean nominal stresses.

b) When the plastic strain at the notch cannot be avoided, apply fatigue stress concentration factor to the alternate stress ($K_{\rm f}$) and take it as one for the mean stresses ($K_{\rm fm} = 1$, conservative approach).

If no information is known on the plastic zone, then we use (a) as our standard.

9.5.2 Brittle materials

For brittle materials a stress raiser increases the likelihood of failure under either steady or alternating stresses, and it is customary to apply a stress concentration factor to both. Thus, apply the fatigue stress concentration factor $K_{\rm f}$ to the alternating component of stress for ductile materials. In brittle materials, apply the geometric stress concentration factor $K_{\rm t}$ to the mean components of stress and fatigue stress concentration factor $K_{\rm f}$ to the alternating components of stress. Thus

$$K_{\rm fm} = K_{\rm t}$$

9.5.3 Summary

In general,

$$\sigma_{\rm t,a}\big|_{\rm axial} = K_{{\rm f}_a}\,\sigma_{\rm N,a}\big|_{\rm axial} \qquad \sigma_{\rm t,a}\big|_{\rm bending} = K_{{\rm f}_b}\,\sigma_{\rm N,a}\big|_{\rm bending} \qquad \tau_{\rm t,a}\big|_{\rm torque} = K_{{\rm f}_s}\,\tau_{\rm N,a}\big|_{\rm torque}$$

$$\sigma_{\rm t,m}\big|_{\rm axial} = K_{{\rm fm}_a}\,\sigma_{\rm N,m}\big|_{\rm axial} \qquad \sigma_{\rm t,m}\big|_{\rm bending} = K_{{\rm fm}_b}\,\sigma_{\rm N,m}\big|_{\rm bending} \qquad \tau_{\rm t,m}\big|_{\rm torque} = K_{{\rm fm}_s}\,\tau_{\rm N,m}\big|_{\rm torque}$$

and only the mean fatigue stress-concentration factor $K_{\rm fm}$ changes for each material type.

9.6 Stress versus Life Curves (S–N Diagrams)

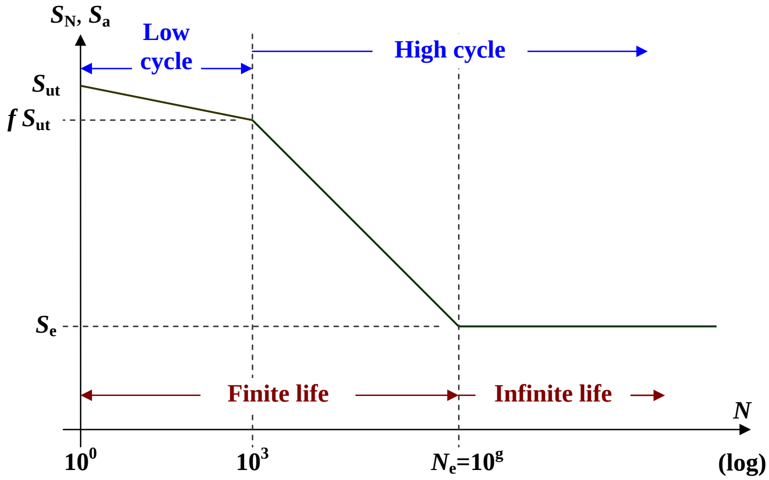


Figure 9.3: Typical S-N diagram for ferrous materials.

Fatigue test data are frequently presented in the form of a plot of fatigue strength S or completely reversed stress versus the number of cycles to failure or fatigue life N with a semi-logarithm scale; that is S–log N, as shown in Fig. 9.3.

9.6.1 Fatigue Regimens

The S–N diagram has two basic regimens and these are: low-cycle fatigue and high-cycle fatigue. The low-cycle fatigue is any loading that causes failure below approximately 1000 cycles:

$$10^0 \leq N \leq 10^3$$

High-cycle fatigue is concerned with failure corresponding to stress cycle greater than 1000 cycles:

$$N > 10^3$$

9.6.2 Endurance Stress and Theoretical Fatigue Strength

Figure 9.4 shows that ferrous and nonferrous materials behave differently. In the case of ferrous materials, a "knee" occurs in the S–N diagram, and beyond this knee failure will not occur, no matter how great the number of cycles. The strength corresponding to this knee is called the endurance limit S'_e, or the fatigue limit. The endurance limit is usually defined as the maximum stress a material can withstand "indefinitely" without fracture:

$$S'_e : \qquad N \geq N_e \qquad \text{where} \quad N_e \to \infty$$

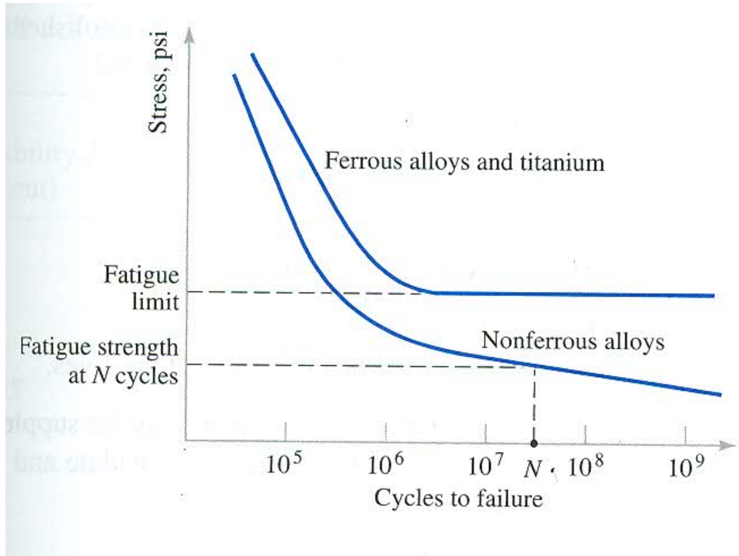


Figure 9.4: Two types of material response to cyclic loading.

The endurance limit is therefore stated with no associated number of cycles to failure. The relationship for the endurance limit for common ferrous alloys are:

$$\text{steels}: \quad @N = 10^6, \quad S_e{}' = \begin{cases} 0.5\,S_{ut}, & \text{for } S_{ut} < 200 \text{ ksi (1400 MPa)} \\ 100 \text{ ksi (700 MPa)}, & \text{for } S_{ut} \geq 200 \text{ ksi (1400 MPa)} \end{cases}$$

$$\text{most irons}: \quad @N = 10^6, \quad S_e{}' = \begin{cases} 0.4\,S_{ut}, & \text{for } S_{ut} < 60 \text{ ksi (400 MPa)} \\ 24 \text{ ksi (160 MPa)}, & \text{for } S_{ut} \geq 60 \text{ ksi (400 MPa)} \end{cases}$$

Nonferrous materials, on the other hand, often exhibit no endurance limit. For some nonferrous materials, approximations for an endurance limit S'_e, using experimental data, has been suggested. Thus, for nonferrous materials we use the fatigue strength S'_N which is the fatigue limit at N cycles. The relationship for the theoretical fatigue strength for common nonferrous alloys are:

$$\text{aluminum}: \quad @N = 5 \times 10^8, \quad S_N{}' = \begin{cases} 0.4\,S_{ut}, & \text{for } S_{ut} < 48 \text{ ksi (330 MPa)} \\ 19 \text{ ksi (130 MPa)}, & \text{for } S_{ut} \geq 48 \text{ ksi (330 MPa)} \end{cases}$$

$$\text{copper}: \quad @N = 10^8, \quad S_N{}' = \begin{cases} 0.4\,S_{ut}, & \text{for } S_{ut} < 60 \text{ ksi (400 MPa)} \\ 14 \text{ ksi (100 MPa)}, & \text{for } S_{ut} \geq 40 \text{ ksi (280 MPa)} \end{cases}$$

Note that aluminum and copper alloys do not have an endurance limit, but a theoretical fatigue strength.

9.6.3 Modified Endurance Stress

Most of the data for S'_e is available for a single specimen test. If we want to use for other parts we need to use the correct value for S_e. Thus in practice we do not use the endurance limit S'_e, but the modified endurance limit:

$$S_e = k_\infty \, S'_e$$

or the fatigue limit at N cycles and is also modified as follows

$$S_N = k_\infty \, S'_N$$

The factor k_∞ accounts for the various influencing factors such as size, surface condition, reliability, loading, temperature, among others. This factor is expressed as the product of

$$k_\infty = k_L \, k_t \, k_{sr} \, k_r \, k_g \, k_e$$

(a) Loading factor

To take into account the low-cycle effects, the loading factor is used:

$$k_L = \begin{cases} 1.00 & \text{bending} \\ 0.85 & \text{axial} \\ 0.59 & \text{torsion} \\ 1.00 & \text{torsion combined with other stresses} \end{cases}$$

(b) Temperature factor

When operating temperatures are below room temperature, brittle fracture is a strong possibility and should be investigated. When operating temperatures are higher than room temperature, yielding should be investigated first because yield strength drops off rapidly with temperature.

$$k_t = \frac{\text{operating temperature}}{\text{room temperature}} = \begin{cases} 1, & \text{for } T \leq 450°\text{C } (840°\text{F}) \\ \\ 1 - 0.0058\,(T - 450), & \text{for } 450°\text{C} < T \leq 550°\text{C} \\ 1 - 0.0032\,(T - 840), & \text{for } 840°\text{C} < T \leq 1020°\text{C} \end{cases}$$

(c) Surface finish factor

Most parts of a machine do not usually have a high-quality surface finish (highly polished). Thus the surface finish factor incorporates the finish effects on the process used to generate the surface. The surface finish factor k_{sr} can be obtained using charts or analytically by

$$k_{sr} = e \, S_{ut}^c$$

where S_{ut} is the ultimate tensile strength of material and the coefficients e and c are defined as

Manufacturing Process	Factor e		Exponent c
	S_{ut} [MPa]	S_{ut} [ksi]	
Grinding	1.58	1.34	-0.085
Machining or cold drawing	4.51	2.70	-0.265
Hot rolling	57.7	14.4	-0.718
As forged	272.0	39.9	-0.995

For mirror-polished surfaces take $k_{sr} = 1.0$.

(d) Reliability factor

Most of the data is empirical however we are often interested in the reliability of the probability of survival, that is the probability of surviving to the life indicated at a particular stress. Thus the reliability factor k_r may be expressed as

$$k_r = 1 - 0.08\,X$$

where X is the transformation variate obtained from any table for a cumulative distribution function. However, the reliability factor for the most common probabilities of survival corresponding to 8% standard deviation of the endurance limit is

Probability of Survival %	Transformation Variate X	Reliability factor k_r
50	0.000	1.000
90	1.288	0.897
95	1.645	0.868
99	2.326	0.814
99.9	3.091	0.753
99.99	3.719	0.702
99.999	4.265	0.659
99.9999	4.753	0.620

(e) Gradient size factor

Choose k_g as follows:

	Bending	Axial	Torsion
$d < 0.4''$ or 10 mm	1	[0.7 0.9]	1
$0.4'' < d < 2''$ or 50 mm	0.9	[0.7 0.9]	0.9
$d > 2''$ or 50 mm	[0.6 0.75]	1	[0.6 0.75]

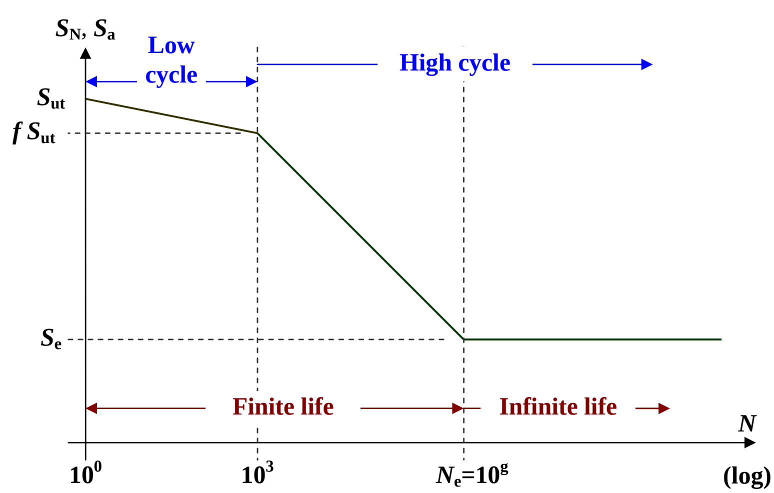
If combined load usually choose 0.9, but in some cases we may choose 1.0.

(f) Miscellaneous Factors

1. Residual stresses
2. Corrosion: There is no fatigue limit
3. Frettage corrosion: $k_e \epsilon [0.24\ 0.90]$
4. Operating speed: $k_e \sim 0.9$

Never use a correction factor greater than one. If any factor gives you a value higher than one, set it to one. Only residual surface stress and operating speed may be higher than one.

9.6.4 Plotting S-N Diagrams



$$N = 10^0 = 1, \qquad S = S_{\rm ut}$$

$$N = 10^3, \qquad S_{\rm f} = f\, S_{\rm ut}$$

$$N = N_{\rm e} = 10^{g_{\rm e}}, \qquad S = S_{\rm e}$$

For the case of pure bending use $f = 0.9$, for the case of pure axial loading use $f = 0.75$, and for the case of pure torsion use $f = 0.72$. For combined loading take f may be approximated as follows:

$$f = 0.93 \quad S_{\rm ut} = 60 \text{ ksi}$$

$$f = 0.86 \quad S_{\rm ut} = 90 \text{ ksi}$$

$$f = 0.82 \quad S_{\rm ut} = 120 \text{ ksi}$$

$$f = 0.77 \quad S_{\rm ut} = 200 \text{ ksi}$$

Stress-Cycle relationship

The common empirical formula relating fatigue strength and number of cycles to failure is

$$S_{\rm N} = a\, N^b \tag{9.8}$$

The constants a and b are derived from (9.8):

$$\log\left(S_{\rm N}\right) = \log\left(a\, N^b\right)$$

$$\log\left(S_{\rm N}\right) = \log\left(a\right) + \log\left(N^b\right)$$

$$\log\left(S_{\rm N}\right) = \log\left(a\right) + b\, \log\left(N\right)$$

Now for high-cycle fatigue use that fact that

$$N = 10^3, \qquad S_{\rm f} = f\, S_{\rm ut}$$

$$N = N_{\rm e} = 10^{g_{\rm e}}, \qquad S = S_{\rm e}$$

to obtain two equations "linear" equations in a and b:

$$\log\left(S_{\rm f}\right) = \log\left(a\right) + b\, \log\left(10^3\right)$$

$$\log\left(S_{\rm e}\right) = \log\left(a\right) + b\, \log\left(10^{g_{\rm e}}\right)$$

Hence we have two equations and two unknown and the constants may be expressed as

$$b = \frac{1}{3 - g_e} \log\left(\frac{S_f}{S_e}\right) \qquad a = \frac{S_f}{10^{3\,b}} = \frac{f\,S_{ut}}{10^{3\,b}} = (f\,S_{ut})^{\frac{g_e}{g_e - 3}}\ (S_e)^{\frac{3}{3 - g_e}}$$

Typically for ferrous materials $g_e = 6$:

$$b = -\frac{1}{3} \log\left(\frac{S_f}{S_e}\right) \qquad a = \frac{S_f}{10^{3\,b}} = \frac{f\,S_{ut}}{10^{3\,b}} = (f\,S_{ut})^2\ (S_e)^{-1}$$

In the S-N diagram, the fatigue stress is also the alternating stress. Thus for a given number of cycles $10^3 < N \leq N_e$, the fatigue stress may be evaluated as follows:

$$S = a\,N^b$$

Also, to obtain the number of cycles for a given alternating stress:

$$S = a\,N^b \qquad \rightarrow \qquad N = \left(\frac{S}{a}\right)^{\frac{1}{b}}$$

Finite and Infinite Life

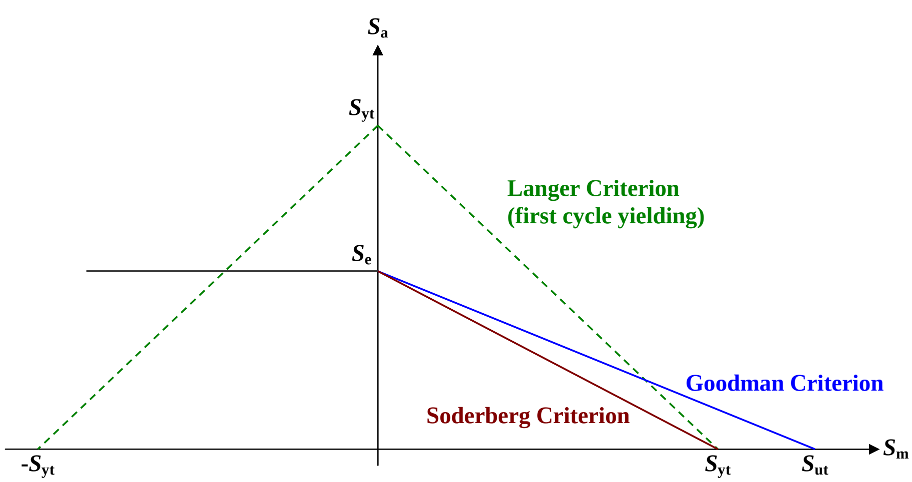
Infinite life begins for stresses bellow the endurance limit, that is

$$S_{eq} < S_e$$

When designing with materials that exhibit no endurance limit, the design will always be for finite life.

Recall, that for nonferrous materials we cannot design for infinite life.

9.6.5 Fatigue Theories of Fatigue Failure



Goodman Criterion

Among the fatigue theories, the Modified Goodman theory is the one widely used. The Modified Goodman criterion, which gives reasonably good results for brittle materials while conservative values for ductile materials is a realistic scheme for most materials. Goodman criterion is widely used because:

1. it is a straight line and the algebra is linear and easy.
2. it is easily graphed, every time for every problem.
3. It reveals subtleties of insight into fatigue problems.
4. Answers can be scaled from the diagrams as a check on the algebra.

For ductile materials, the fatigue equivalent stress is

$$S_{eq} = \frac{S_a}{1 - \dfrac{S_m}{S_{ut}}} \qquad \text{for} \qquad \sigma_m \geq 0 \quad \text{and} \quad \sigma_{max} \leq S_y$$

$$S_{eq} = S_y \qquad \text{for} \qquad \sigma_m \geq 0 \quad \text{and} \quad \sigma_{max} \geq S_y$$

For brittle materials, the fatigue equivalent stress is

$$S_{eq} = \frac{S_a}{1 - \dfrac{S_m}{S_{ut}}}$$

In the above expressions:

$$S_a = n_{SF}\, \sigma_a \qquad S_m = n_{SF}\, \sigma_m$$

where n_{SF} is the safety factor. The maximum and minimum stresses are

$$S_{max} = S_m + S_a \qquad S_{min} = S_m - S_a$$

Soderberg Criterion

Among the fatigue theories, the Soderberg theory may the also be used for ductile materials. It gives conservative values for ductile materials. The criteria states that the fatigue equivalent stress is

$$S_{eq} = \frac{S_a}{1 - \dfrac{S_m}{S_{yt}}}$$

In the above expressions:

$$S_a = n_{SF}\, \sigma_a \qquad S_m = n_{SF}\, \sigma_m$$

where n_{SF} is the safety factor. The maximum and minimum stresses are

$$S_{max} = S_m + S_a \qquad S_{min} = S_m - S_a$$

9.7 Procedure for Multiaxial Fatigue Analysis

If we want to determine the product life:

1. Calculate the mean and alternate loads.
2. Determine the mean and alternate state of stresses.
3. Depending on the type of material (brittle or ductile), determine the equivalent mean and alternate stresses. (Use factor of safety). Calculate the maximum and minimum stresses.
4. Use Goodman or Soderberg theory to determine the equivalent design stress.
5. Use S-N Diagram to determine remaining life.

If we want to design given the product life:

1. Calculate the mean and alternate loads.
2. Determine the mean and alternate state of stresses.
3. Depending on the type of material (brittle or ductile), determine the equivalent mean and alternate stresses. (Use factor of safety). Calculate the maximum and minimum stresses.
4. Use S-N Diagram to determine the equivalent stress.
5. Use Goodman or Soderberg theory to determine the design stress.

Figure 9.5 shows are life cycles and failure criterions are related in fatigue analysis. Always verify for yielding.

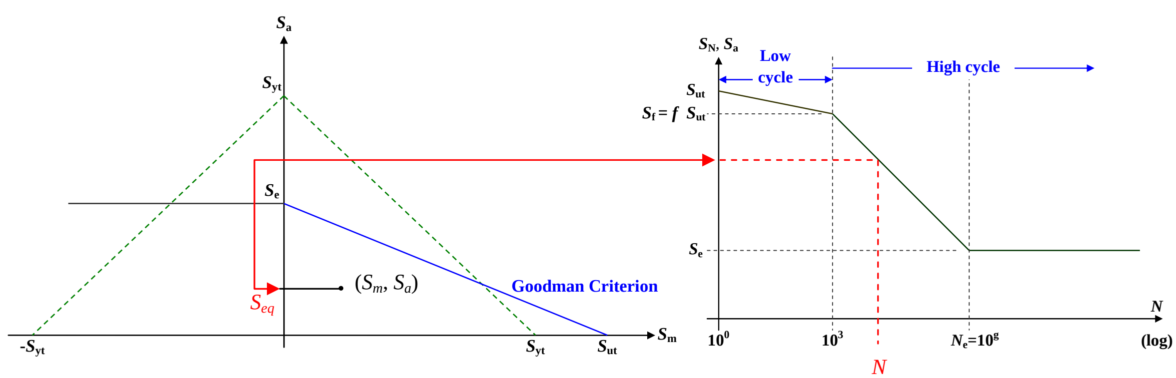
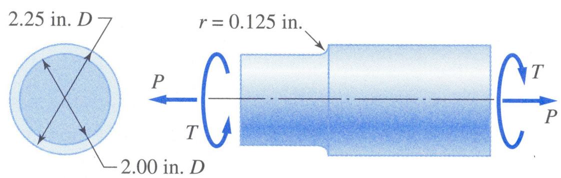


Figure 9.5: Determine life cycle for static fatigue analysis or means/alternate stresses.

Example 9.5.

Life Cycle Example: Brittle Material



The Class 60 gray cast-iron mounting arm is subjected to tension and torsion, as shown in Figure. It takes two minutes for completion of a full cycle and operates for only eight hours a day. The Class 60 gray cast iron has an ultimate strength of 60 ksi in tension, and elongation in 2 inches of less than 0.5%. The design safety factor is 1.5.

(a) The arm is subject to a static axial force of $P = 50000$ lb and a static torsional moment of $T = 18000$ lb–in. For the given dimensions, could the arm support the specified loading without failure?

(b) During a different mode of operation, the axial force P cycles fluctuates from 50000 lb in tension to 10000 lb in compression, and the torsional moment remains zero at all times. What would be the estimated days of life for this mode of operation for a 99% reliability?

Solution:

(a) The arm is subject to a static axial force of $P = 50000$ lb and a static torsional moment of $T = 18000$ lb–in. For the given dimensions, could the arm support the specified loading without failure?

From the problem it is known:

$$S_{\text{ut}} = 60 \text{ ksi}$$

Assuming the state of stress at the most critical location of the shaft's cross-section occurs for an element located at the top:

$$\underline{\boldsymbol{\sigma}}_{\text{A}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{yx}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{zx}} & \tau_{\text{zy}} & \sigma_{\text{zz}} \end{bmatrix}$$

Thus:

$$\sigma_{\text{xx}} = \sigma_{\text{xx}}\big|_{\text{axial}} = \frac{4\,P}{\pi\,d^2} \text{ psi}$$

and

$$\tau_{\rm xz} = \tau\Big|_{\rm torsion} = \frac{16\,T}{\pi\,d^3} \text{ psi}$$

Thus

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\rm xx} & \tau_{\rm xy} & \tau_{\rm xz} \\ \tau_{\rm xy} & \sigma_{\rm yy} & \tau_{\rm yz} \\ \tau_{\rm xz} & \tau_{\rm yz} & \sigma_{\rm zz} \end{bmatrix} = \begin{bmatrix} \frac{4\,P}{\pi\,d^2} & 0 & \frac{16\,T}{\pi\,d^3} \\ 0 & 0 & 0 \\ \frac{16\,T}{\pi\,d^3} & 0 & 0 \end{bmatrix} \quad \text{psi}$$

For the static conditions given, under combined loads of axial tension and torsional shear, the critical point will be at the root of the 0.125″ radius fillet. Stress concentration factors must be separately determined for the tensile load and the torsional load. Thus using figures for stress concentration factor for a shaft with a fillet subject to axial and torsion (from stress concentration charts):

$$d = 2 \text{ in} \qquad r = 0.125 \text{ in} \qquad \rightarrow \qquad \frac{r}{d} = 0.063$$

$$d = 2 \text{ in} \qquad D = 2.25 \text{ in} \qquad \rightarrow \qquad \frac{D}{d} = 1.13$$

Thus

$$K_{\rm t_a} = 1.8 \qquad K_{\rm t_s} = 1.15$$

Thus the state of stress is modified as follows

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} K_{\rm t_a}\frac{4\,P}{\pi\,d^2} & 0 & K_{\rm t_s}\frac{16\,T}{\pi\,d^3} \\ 0 & 0 & 0 \\ K_{\rm t_s}\frac{16\,T}{\pi\,d^3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 28647.9 & 0 & 13178 \\ 0 & 0 & 0 \\ 13178 & 0 & 0 \end{bmatrix} \quad \text{psi}$$

The principal stresses are:

$$\sigma_1 = 33787.7 \text{ psi}$$

$$\sigma_2 = 0$$

$$\sigma_3 = -5139.76 \text{ psi}$$

Now, note that the strain at fracture is less than 0.5 % ($\varepsilon_f < 0.005$), thus the material is brittle, and we can use the Maximum Normal Stress Theory. Since no information is given for $S_{\rm uc}$, let us only use

$$\frac{\sigma_1}{S_{\rm ut}} \leq \frac{1}{n_{\rm SF}} \quad \text{for} \quad \sigma_{\max} \geq 0$$

which leads to

$$\frac{\sigma_1}{S_{\rm ut}} \leq^? \frac{1}{n_{\rm SF}} \qquad \rightarrow \qquad 0.563 < 0.667$$

Thus the arm can support the specified static load without failure for a safety factor of 1.5.

(b) During a different mode of operation, the axial force P cycles fluctuates from 50000 lb in tension to 10000 lb in compression, and the torsional moment remains zero at all times. What would be the estimated days of life for this mode of operation for a 99% reliability?

(a) Calculate the mean and alternate loads.
This is a fluctuating cyclic load problem. Thus,

$$P_{\text{a}} = \frac{P_{\text{max}} - P_{\text{min}}}{2} = \frac{(50000) - (-10000)}{2} = 30000 \text{ lb}$$

$$P_{\text{m}} = \frac{P_{\text{max}} + P_{\text{min}}}{2} = \frac{(50000) + (-10000)}{2} = 20000 \text{ lb}$$

$$\underline{\boldsymbol{\sigma}}_{\text{a}} = \begin{bmatrix} \frac{4\,P_{\text{a}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\text{m}} = \begin{bmatrix} \frac{4\,P_{\text{m}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Determine the mean and alternate state of stresses.
The state of stress is as follows

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \frac{4\,P}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is brittle material, we need to apply the fatigue stress concentration factor to the alternate stresses and static stress concentration factor to the mean stresses. The fatigue stress concentration factor K_{f_a} is calculated as follows,

$$K_{\text{f}_a} = 1 + q\ (K_{\text{t}_a} - 1)$$

For the mean stress

$$K_{\text{fm}_a} = K_{\text{t}_a}$$

From tables we find

$$r = 0.125'' \qquad S_{\text{ut}} = 60 \text{ ksi} \qquad \rightarrow \qquad q \approx 0.75$$

$$K_{\text{f}_a} = 1 + 0.75\ (1.8 - 1) = 1.6$$

Thus

$$\underline{\boldsymbol{\sigma}}_{\mathrm{a}} = \begin{bmatrix} K_{\mathrm{f}_a} \dfrac{4\,P_{\mathrm{a}}}{\pi\, d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 15278.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{m}} = \begin{bmatrix} K_{\mathrm{t}_a} \dfrac{4\,P_{\mathrm{m}}}{\pi\, d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 11459.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Depending on the type of material (brittle or ductile), determine the equivalent mean and alternate stresses. (Use factor of safety). Calculate the maximum and minimum stresses.
The principal stresses are:

$$\sigma_{1,\mathrm{a}} = 15278.9 \quad \sigma_{2,\mathrm{a}} = 0.0 \quad \sigma_{3,\mathrm{a}} = 0.0$$

$$\sigma_{1,\mathrm{m}} = 11459.2 \quad \sigma_{2,\mathrm{m}} = 0.0 \quad \sigma_{3,\mathrm{m}} = 0.0$$

For brittle materials the **mean and alternate stresses** are

$$\sigma_{\mathrm{a}} = \sigma_{1,\mathrm{a}} = 15278.9 \text{ psi} \qquad \sigma_{\mathrm{m}} = \sigma_{1,\mathrm{m}} = 11459.2 \text{ psi}$$

The **maximum and minimum stresses** are

$$\sigma_{\max} = \sigma_{\mathrm{m}} + \sigma_{\mathrm{a}} = 26738 \text{ psi}$$

$$\sigma_{\min} = \sigma_{\mathrm{m}} - \sigma_{\mathrm{a}} = -3819.72 \text{ psi}$$

The **stress range** σ_{r} is the difference range of the maximum and minimum stresses in the cycle:

$$\sigma_{\mathrm{r}} = \Delta\sigma = \sigma_{\max} - \sigma_{\min} = 30557.7 \text{ psi}$$

The **stress ratio** R_{s} is the ratio of minimum to maximum stress amplitudes:

$$R_{\mathrm{s}} = \frac{\sigma_{\min}}{\sigma_{\max}} = -0.142857$$

The **amplitude or load ratio** $\mathbb{A}_{\mathrm{a}}$ is the ratio of the stress amplitude to the mean stress:

$$\mathbb{A}_{\mathrm{a}} = \frac{\sigma_{\mathrm{a}}}{\sigma_{\mathrm{m}}} = 1.333$$

(d) Use Goodman or Soderberg theory to determine the equivalent design stress.

Let us use the modified Goodman criterion:

$$S_{\text{eq}} = \frac{S_{\text{a}}}{1 - \dfrac{S_{\text{m}}}{S_{\text{ut}}}}$$

In the above expressions:

$$S_{\text{a}} = n_{\text{SF}}\,\sigma_{\text{a}} = 22918.3 \qquad S_{\text{m}} = n_{\text{SF}}\,\sigma_{\text{m}} = 17188.7$$

where n_{SF} is the safety factor. Thus

$$S_{\text{eq}} = \frac{S_{\text{a}}}{1 - \dfrac{S_{\text{m}}}{S_{\text{ut}}}} = 32120 \text{ psi}$$

(e) Use S-N Diagram to determine remaining life.
Now we need to find the modified S_e. First of all at $N = 1$ life cycle plot $S = S_{\text{ut}} =$ 60 ksi.
For $N = 10^3$, $S_{\text{f}} = 0.75\,S_{\text{ut}} = 45$ ksi (for pure axial loading $f = 0.75$).
Next for cast irons with $S_{\text{ut}} \leq 88$ ksi:

$$S'_e = 0.4\,S_{\text{ut}} \qquad \text{at } N = 10^6 \text{ cycles}$$

So for the class 60 Gray cast iron alloy used for this mounting arm

$$S'_e = 0.4\,S_{\text{ut}} = 24000 \text{ psi}$$

Since $N_{\text{e}} = 10^6$, $g_{\text{e}} = 6$. For axial loading $k_{\text{L}} = 0.85$ and 99% reliability $k_{\text{r}} = 0.814$. All others are 1.0. Thus

$$k_\infty = 0.6919$$

Thus the modified endurance limit is

$$S_e = k_\infty\, S'_e = 16605.6 \text{ psi}$$

Since $S_{\text{eq}} > S_e$, the part has finite life. In order to find the life cycles, we use the equation

$$N = \left(\frac{S}{a}\right)^{\frac{1}{b}} \quad \rightarrow \quad N = \left(\frac{S_{\text{eq}}}{a}\right)^{\frac{1}{b}}$$

where

$$b = \frac{1}{3 - g_{\text{e}}} \log\left(\frac{S_{\text{f}}}{S_e}\right) = \frac{1}{3 - g_{\text{e}}} \log\left(\frac{f\,S_{\text{ut}}}{S_e}\right) \qquad a = \frac{S_{\text{f}}}{10^{3\,b}} = \frac{f\,S_{\text{ut}}}{10^{3\,b}} = (f\,S_{\text{ut}})^{\frac{g_{\text{e}}}{g_{\text{e}}-3}}\,(S_{\text{e}})^{\frac{3}{3-g_{\text{e}}}}$$

Note that we took $S = S_{\text{eq}}$ because that is the fatigue stress at which we want to calculate the life cycles for failure.
For $g_{\text{e}} = 6$:

$$b = -\frac{1}{3} \log\left(\frac{f\,S_{\text{ut}}}{S_e}\right) = -0.144319 \qquad a = \frac{f\,S_{\text{ut}}}{10^{3\,b}} = 121947.$$

Thus

$$N = \left(\frac{S_{\text{eq}}}{a}\right)^{\frac{1}{b}} = 10343.6 \text{ cycles}$$

The component has 1.03×10^5 life cycles before failure. It is known:

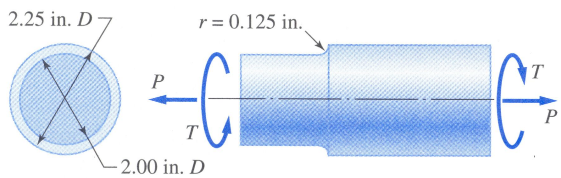
$$N = 1.03 \times 10^5 \text{ cycles} \left(\frac{2 \text{ minutes}}{1 \text{ cycle}}\right) \left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right) \left(\frac{1 \text{ day}}{8 \text{ hours}}\right) = 43.09 \text{ days}$$

Thus the component has about 43 days of life.

End Example □

Example 9.6.

Life Cycle Example: Ductile Material



The wrought ferrous steel arm is subjected to tension and torsion, as shown in Figure. It takes two minutes for completion of a full cycle. It takes two minutes for completion of a full cycle and operates for only eight hours a day. The steel has a yield strength of 70 ksi and ultimate strength of 90 ksi in tension, and elongation in 2 inches of greater than 0.5%. The design safety factor is 1.5.

(a) The arm is subject to a static axial force of $P = 50000$ lb and a static torsional moment of $T = 18000$ lb–in. For the given dimensions, could the arm support the specified loading without failure?

(b) During a different mode of operation, the axial force P cycles fluctuates from 50000 lb in tension to 10000 lb in compression, and the torsional moment remains zero at all times. What would be the estimated life for this cyclic mode of operation for a 99% reliability? Assume that the plastic strain can be avoided at the notch.

Solution:

(a) The arm is subject to a static axial force of $P = 50000$ lb and a static torsional moment of $T = 18000$ lb–in. For the given dimensions, could the arm support the specified loading without failure?

From the problem it is known:

$$S_{\text{ut}} = 90 \text{ ksi} \qquad S_{\text{y}} = 70 \text{ ksi}$$

Assuming the state of stress at the most critical location of the shaft's cross-section occurs for an element located at the top:

$$\underline{\boldsymbol{\sigma}}_{\text{A}} = \begin{bmatrix} \sigma_{\text{xx}} & \tau_{\text{xy}} & \tau_{\text{xz}} \\ \tau_{\text{yx}} & \sigma_{\text{yy}} & \tau_{\text{yz}} \\ \tau_{\text{zx}} & \tau_{\text{zy}} & \sigma_{\text{zz}} \end{bmatrix}$$

Thus:

$$\sigma_{xx} = \sigma_{xx}\big|_{\text{axial}} = \frac{P}{A} = \frac{4\,P}{\pi\,d^2}\ \text{psi}$$

and

$$\tau_{xz} = \tau\big|_{\text{torsion}} = \frac{T}{Q} = \frac{16\,T}{\pi\,d^3}\ \text{psi}$$

Thus

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \frac{4\,P}{\pi\,d^2} & 0 & \frac{16\,T}{\pi\,d^3} \\ 0 & 0 & 0 \\ \frac{16\,T}{\pi\,d^3} & 0 & 0 \end{bmatrix} \qquad \text{psi}$$

For the static conditions given, under combined loads of axial tension and torsional shear, the critical point will be at the root of the 0.125″ radius fillet. For ductile materials, stress concentration may be taken as one:

$$K_{t_a} = 1.0 \qquad K_{t_s} = 1.0$$

Thus the state of stress is modified as follows

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \frac{4\,P}{\pi\,d^2} & 0 & \frac{16\,T}{\pi\,d^3} \\ 0 & 0 & 0 \\ \frac{16\,T}{\pi\,d^3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 15915.5 & 0 & 11459.2 \\ 0 & 0 & 0 \\ 11459.2 & 0 & 0 \end{bmatrix} \qquad \text{psi}$$

Now we determine the principal stresses:

$$\sigma_1 = 21909.\ \text{psi}$$

$$\sigma_2 = 0$$

$$\sigma_3 = -5993.52\ \text{psi}$$

$$\tau_{\max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 13951.3\ \text{psi}$$

Now, note that the strain at fracture is greater than 0.5 % ($\varepsilon_f > 0.005$), thus the material is ductile, and we may use the Distortional Energy Criterion:

$$\frac{\sigma_{eq}}{S_y} \le \frac{1}{n_{SF}}$$

which leads to ($\sigma_{eq} = 25440.9$ psi)

$$\frac{\sigma_{eq}}{S_y} \le^{?} \frac{1}{n_{SF}} \qquad \rightarrow \qquad 0.363442 < 0.667$$

Thus the arm can support the specified static load without failure for a safety factor of 1.5.

(b) During a different mode of operation, the axial force P cycles fluctuates from 50000 lb in tension to 10000 lb in compression, and the torsional moment remains zero at all times. What would be the estimated life for this cyclic mode of operation for a 99% reliability?

(a) Calculate the mean and alternate loads.
This is a fluctuating cyclic load problem. Thus,

$$P_{\text{a}} = \frac{P_{\max} - P_{\min}}{2} = \frac{(50000) - (-10000)}{2} = 30000 \text{ lb}$$

$$P_{\text{m}} = \frac{P_{\max} + P_{\min}}{2} = \frac{(50000) + (-10000)}{2} = 20000 \text{ lb}$$

$$\underline{\boldsymbol{\sigma}}_{\text{a}} = \begin{bmatrix} \dfrac{4\,P_{\text{a}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\text{m}} = \begin{bmatrix} \dfrac{4\,P_{\text{m}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Determine the mean and alternate state of stresses.
The state of stress is as follows

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \dfrac{4\,P}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is ductile material, we need to determine how to apply the fatigue stress concentration factor to mean stresses. We apply the fatigue stress concentration factor to both alternate using K_{f}. The fatigue stress concentration factor K_{f_a} is calculated as follows

$$K_{\text{f}_a} = 1 + q\,(K_{\text{t}_a} - 1)$$

From Tables and from the previous example:

$$r = 0.125'' \qquad S_{\text{ut}} = 60 \text{ ksi} \qquad \rightarrow \qquad q \approx 0.75$$

$$K_{\text{f}_a} = 1 + 0.75\,(1.8 - 1) = 1.6 \qquad (K_{\text{t}_a} = 1.8)$$

Since the plastic strain may be avoided we need to determine what case we will be using:

$$\text{No yielding:} \quad K_{\text{fm}} = K_{\text{f}} \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}}| < S_{\text{y}}$$

$$\text{Initial yielding:} \quad K_{\text{fm}} = \frac{K_{\text{f}} - K_{\text{f}}\,\sigma_{\text{a,nom}}}{|\sigma_{\text{max,nom}}|} \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}}| > S_{\text{y}}$$

$$\text{Reversed yielding:} \quad K_{\text{fm}} = 0 \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}} - \sigma_{\text{min,nom}}| > 2\,S_{\text{y}}$$

Hence, before we proceed we need to determine the maximum nominal stresses:

$$\sigma_{\text{max,nom}} = \frac{4\,P_{\text{max}}}{\pi\,d^2} = 15915.5 \text{ psi}$$

Now checking the first case:

$$\begin{aligned} K_{\text{fa}}\,|\sigma_{\text{max,nom}}| &< S_{\text{y}} \\ (1.6)(15915.5) &<^{?} 70000 \\ 25464.8 &< 70000 \end{aligned}$$

Hence, for the mean stresses we use

$$K_{\text{fm}} = K_{\text{f}_a}$$

Thus

$$\underline{\boldsymbol{\sigma}}_{\text{a}} = \begin{bmatrix} K_{\text{f}_a}\dfrac{4\,P_{\text{a}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 15278.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{\text{m}} = \begin{bmatrix} K_{\text{f}_a}\dfrac{4\,P_{\text{m}}}{\pi\,d^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10185.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Depending on the type of material (brittle or ductile), determine the equivalent mean and alternate stresses. (Use factor of safety). Calculate the maximum and minimum stresses.

For ductile materials the **mean and alternate stresses** are

$$\sigma_{\text{a}} = \sigma_{\text{eq,a}} = \sqrt{I_{\sigma_{1,\text{a}}}^2 - 3\,I_{\sigma_{2,\text{a}}}} = 15278.9 \text{ psi} \qquad \sigma_{\text{m}} = \sigma_{\text{eq,m}} = \sqrt{I_{\sigma_{1,\text{m}}}^2 - 3\,I_{\sigma_{2,\text{m}}}} = 10185.9 \text{ psi}$$

The **maximum and minimum stresses** are

$$\sigma_{\text{max}} = \sigma_{\text{m}} + \sigma_{\text{a}} = 25464.8 \text{ psi}$$

$$\sigma_{\text{min}} = \sigma_{\text{m}} - \sigma_{\text{a}} = -5092.96 \text{ psi}$$

The **stress range** $\sigma_{\rm r}$ is the difference range of the maximum and minimum stresses in the cycle:

$$\sigma_{\rm r} = \Delta\sigma = \sigma_{\rm max} - \sigma_{\rm min} = 30557.7 \text{ psi}$$

The **stress ratio** $R_{\rm s}$ is the ratio of minimum to maximum stress amplitudes:

$$R_{\rm s} = \frac{\sigma_{\rm min}}{\sigma_{\rm max}} = -0.20$$

The **amplitude or load ratio** $\mathbb{A}_{\rm a}$ is the ratio of the stress amplitude to the mean stress:

$$\mathbb{A}_{\rm a} = \frac{\sigma_{\rm a}}{\sigma_{\rm m}} = 1.5$$

(d) Use Goodman or Soderberg theory to determine the equivalent design stress.
Let us use the modified Goodman criterion:

$$S_{\rm a} = n_{\rm SF}\,\sigma_{\rm a} = 22918.3 \qquad S_{\rm m} = n_{\rm SF}\,\sigma_{\rm m} = 15278.9$$

where $n_{\rm SF}$ is the safety factor.

$$S_{\rm m} > 0 \qquad S_{\rm max} = 38.2 \text{ ksi} < S_{\rm y} = 70 \text{ ksi} \quad \rightarrow \quad S_{\rm eq} = \frac{S_{\rm a}}{1 - \dfrac{S_{\rm m}}{S_{\rm ut}}}$$

Thus

$$S_{\rm eq} = \frac{S_{\rm a}}{1 - \dfrac{S_{\rm m}}{S_{\rm ut}}} = 27604.6 \text{ psi}$$

(e) Use S-N Diagram to determine remaining life.
Now we need to find the modified S_e. First of all at $N = 1$ life cycle plot $S = S_{\rm ut} =$ 90 ksi.
For $N = 10^3$, $S_{\rm f} = 0.75\,S_{\rm ut} = 67.5$ ksi (for axial loading $f = 0.75$).
Next for steels with $S_{\rm ut} < 200$ ksi:

$$S'_e = 0.5\,S_{\rm ut} \qquad \text{at } N = 10^6 \text{ cycles}$$

So for the steel used for this mounting arm

$$S'_e = 0.5\,S_{\rm ut} = 40500. \text{ psi}$$

For axial loading $k_{\rm L} = 0.85$ and 99% reliability $k_{\rm r} = 0.814$. All others are 1.0. Thus

$$k_\infty = 0.6919$$

Thus the modified endurance limit is

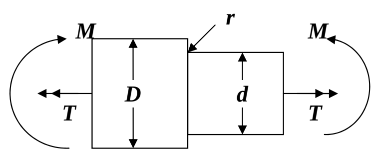
$$S_e = k_\infty\,S'_e = 31135.5 \text{ psi}$$

Since $S_{\rm eq} < S_e$, the part has infinite life.

End Example □

Example 9.7.

Infinite Life Cycle Design Example: Ductile Material



A circular shaft is made of wrought-carbon steel with yield strength of 42 ksi and ultimate strength of 76 ksi in tension, and elongation in 2 inches of greater than 0.5%. The shaft's current mode of operation is such that the loads are as follows: repeated bending moment of 50000 lb–in in tension and the repeated torsional moment of 10000 lb–in in compression. The fillet radius is 0.1 in and $D = 1.5\,d$. Estimate the value of the diameter d, for infinite life. Consider the following operating conditions:

1. Operating speed is 3600 rev/min.
2. The design safety factor is 1.5.
3. A strength reliability level of 99.9%.
4. The part is to be lathe-turned from a bar of the wrought-steel alloy.
5. The plastic strain may be avoided.

Solution:

First let us locate the most critical point in the shaft. It will occur at an element at the top (bending stress in compression) is or at the bottom (bending stress in tension). Let us consider the element at top. Thus,

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} -\dfrac{M}{Z} & 0 & \dfrac{T}{Q} \\ 0 & 0 & 0 \\ \dfrac{T}{Q} & 0 & 0 \end{bmatrix}$$

where

$$Z = \frac{\pi\, d^3}{32} \qquad Q = \frac{\pi\, d^3}{16}$$

1. Calculate the mean and alternate loads.

This is a repeated load in tension for bending moment and repeated load in compression for the torsional load: Thus,

$$M_{\rm a} = \frac{M_{\max} - M_{\min}}{2} = \frac{(50000) - (0)}{2} = 25000 \text{ lb–in}$$

$$M_{\rm m} = \frac{M_{\max} + M_{\min}}{2} = \frac{(50000) + (0)}{2} = 25000 \text{ lb–in}$$

$$T_{\rm a} = \frac{T_{\max} - T_{\min}}{2} = \frac{(0) - (-10000)}{2} = 5000 \text{ lb–in}$$

$$T_{\rm m} = \frac{T_{\max} + T_{\min}}{2} = \frac{(0) + (-10000)}{2} = -5000 \text{ lb–in}$$

2. Determine the mean and alternate state of stresses.

Thus

$$\underline{\boldsymbol{\sigma}}_{\rm a} = \begin{bmatrix} -\dfrac{32\,M_{\rm a}}{\pi\,d^3} & 0 & \dfrac{16\,T_{\rm a}}{\pi\,d^3} \\ 0 & 0 & 0 \\ \dfrac{16\,T_{\rm a}}{\pi\,d^3} & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\rm m} = \begin{bmatrix} -\dfrac{32\,M_{\rm m}}{\pi\,d^3} & 0 & \dfrac{16\,T_{\rm m}}{\pi\,d^3} \\ 0 & 0 & 0 \\ \dfrac{16\,T_{\rm m}}{\pi\,d^3} & 0 & 0 \end{bmatrix}$$

Since it is ductile material, we apply the fatigue stress concentration factor to both alternate and mean stresses according to three yielding criteria. First of all we need to take an initial guess for d, let us say $d = 4''$. Then

$$d = d \text{ in} \qquad r = 0.1 \text{ in} \qquad \rightarrow \qquad \frac{r}{d} = 0.025$$

$$d = d \text{ in} \qquad D = 1.5\,d \text{ in} \qquad \rightarrow \qquad \frac{D}{d} = 1.5$$

Thus

$$K_{{\rm t}_b} = 2.6 \qquad K_{{\rm t}_s} = 2.05$$

The fatigue stress concentration factor $K_{{\rm f}_a}$ is calculated as follows,

$$K_{\rm f} = 1 + q\,(K_{\rm t} - 1)$$

From Tables and from the previous example:

$$r = 0.1'' \qquad S_{\rm ut} = 76 \text{ ksi} \quad \rightarrow \quad q_{\rm b} \approx 0.775 \quad q_{\rm s} \approx 0.82$$

$$K_{{\rm f}_b} = 1 + 0.775\,(2.6 - 1) = 2.24$$

$$K_{{\rm f}_s} = 1 + 0.82\,(2.05 - 1) = 1.861$$

Since the plastic strain may be avoided we need to determine what case we will be using:

$$\text{No yielding:} \quad K_{\text{fm}} = K_{\text{f}} \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}}| < S_{\text{y}}$$

$$\text{Initial yielding:} \quad K_{\text{fm}} = \frac{K_{\text{f}} - K_{\text{f}}\,\sigma_{\text{a,nom}}}{|\sigma_{\text{max,nom}}|} \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}}| > S_{\text{y}}$$

$$\text{Reversed yielding:} \quad K_{\text{fm}} = 0 \qquad K_{\text{f}}\,|\sigma_{\text{max,nom}} - \sigma_{\text{min,nom}}| > 2\,S_{\text{y}}$$

Hence, before we proceed we need to determine the maximum nominal stresses:

$$\sigma_{\text{max,nom}} = \frac{32\,M_{\text{max}}}{\pi\,d^3} = 7957.75 \text{ psi}$$
$$\tau_{\text{max,nom}} = \frac{16\,T_{\text{max}}}{\pi\,d^3} = 0 \text{ psi}$$

Now checking the first case:

$$K_{\text{fb}}\,|\sigma_{\text{max,nom}}| < S_{\text{y}}$$
$$(2.24)(7957.75) <^? 42000$$
$$17825.4 < 42000$$

$$K_{\text{fs}}\,|\tau_{\text{max,nom}}| < S_{\text{y}}$$
$$(1.861)(0) <^? 42000$$
$$(1.861)(0) <^? 42000$$

Hence, for the mean stresses we use

$$K_{\text{fm}_b} = K_{\text{f}_b}, \quad K_{\text{fm}_s} = K_{\text{f}_s}$$

Thus,

$$\underline{\boldsymbol{\sigma}}_{\text{a}} = \begin{bmatrix} -K_{\text{f}_b}\,\frac{32\,M_{\text{a}}}{\pi\,d^3} & 0 & K_{\text{f}_s}\,\frac{16\,T_{\text{a}}}{\pi\,d^3} \\ 0 & 0 & 0 \\ K_{\text{f}_s}\,\frac{16\,T_{\text{a}}}{\pi\,d^3} & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\text{m}} = \begin{bmatrix} -K_{\text{f}_b}\,\frac{32\,M_{\text{m}}}{\pi\,d^3} & 0 & K_{\text{f}_s}\,\frac{16\,T_{\text{m}}}{\pi\,d^3} \\ 0 & 0 & 0 \\ K_{\text{f}_s}\,\frac{16\,T_{\text{m}}}{\pi\,d^3} & 0 & 0 \end{bmatrix}$$

Substituting all values:

$$\underline{\boldsymbol{\sigma}}_{\text{a}} = \begin{bmatrix} -\frac{570411}{d^3} & 0 & \frac{47390}{d^3} \\ 0 & 0 & 0 \\ \frac{47390}{d^3} & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\text{m}} = \begin{bmatrix} -\frac{570411}{d^3} & 0 & -\frac{47390}{d^3} \\ 0 & 0 & 0 \\ -\frac{47390}{d^3} & 0 & 0 \end{bmatrix}$$

3. Depending on the type of material (brittle or ductile), determine the equivalent mean

and alternate stresses. (Use factor of safety). Calculate the maximum and minimum stresses.

For ductile materials the **mean and alternate stresses** are

$$\sigma_{\mathrm{a}} = \sigma_{\mathrm{eq,a}} = \sqrt{\sigma_{\mathrm{xx,a}}^2 + 3\,\tau_{\mathrm{xz,a}}^2} = \frac{258439}{d^3}\ \text{psi} \qquad \sigma_{\mathrm{m}} = \sigma_{\mathrm{eq,m}} = \sqrt{\sigma_{\mathrm{xx,m}}^2 + 3\,\tau_{\mathrm{xz,m}}^2} = \frac{258439}{d^3}\ \text{psi}$$

The **maximum and minimum stresses** are

$$\sigma_{\max} = \sigma_{\mathrm{m}} + \sigma_{\mathrm{a}} = \frac{516879}{d^3}\ \text{psi}$$

$$\sigma_{\min} = \sigma_{\mathrm{m}} - \sigma_{\mathrm{a}} = 0$$

4. Use Goodman or Soderberg theory to determine the design stress.

$$S_{\mathrm{a}} = n_{\mathrm{SF}}\,\sigma_{\mathrm{a}} = \frac{387659}{d^3} \qquad S_{\mathrm{m}} = n_{\mathrm{SF}}\,\sigma_{\mathrm{m}} = \frac{387659}{d^3}$$

where n_{SF} is the safety factor.

Assume

$$S_{\mathrm{m}} > 0 \qquad S_{\max} < S_{\mathrm{y}} \quad \rightarrow \quad S_{\mathrm{eq}} = \frac{S_{\mathrm{a}}}{1 - \dfrac{S_{\mathrm{m}}}{S_{\mathrm{ut}}}}$$

Thus

$$S_{\mathrm{eq}} = \frac{S_{\mathrm{a}}}{1 - \dfrac{S_{\mathrm{m}}}{S_{\mathrm{ut}}}} = \frac{387659}{d^3 - 5.10078}\ \text{psi}$$

5. Use S-N Diagram to determine the equivalent stress.

Now we need to find the modified S_e. First of all at $N = 1$ life cycle plot $S = S_{\mathrm{ut}} = 76$ ksi.

For $N = 10^3$ and combined loading $S_{\mathrm{f}} = 0.9\,S_{\mathrm{ut}} = 68.4$ ksi (for combined loading $f \approx 0.90$).

Next for steels with $S_{\mathrm{ut}} < 200$ ksi:

$$S_e' = 0.5\,S_{\mathrm{ut}} \qquad \text{at } N = 10^6 \text{ cycles}$$

So for the steel used for this circular shaft

$$S_e' = 0.5\,S_{\mathrm{ut}} = 38000.\ \text{psi}$$

The correction factors are:

(a) Loading: $k_{\mathrm{L}} = 1.0$

(b) Temperature: $k_{\mathrm{t}} = 1.0$

(c) Surface finish factor: lathe-turned is a hot rolling treatment:

$$k_{\mathrm{sr}} = e\,S_{\mathrm{ut}}^{c} = 14.4\,(76)^{-0.718} = 0.6426$$

(d) Reliability: 99.9%, $k_{\mathrm{r}} = 0.753$

(e) Gradient Size factor: Bending and torsion combined. Note assumed $d = 4''$ thus $k_{\rm g} = 0.7$

(f) Miscellaneous, Operating speed: $k_{\rm e} = 0.9$

Thus

$$k_\infty = 0.30484$$

Thus the modified endurance limit is

$$S_e = k_\infty \, S'_e = 11584 \text{ psi}$$

6. Determine the diameter:
   For infinite life take

$$S_{\rm eq} < S_e \quad \rightarrow \quad \frac{864430}{d^3 - 5.10078} < 11584$$

Solving for d:

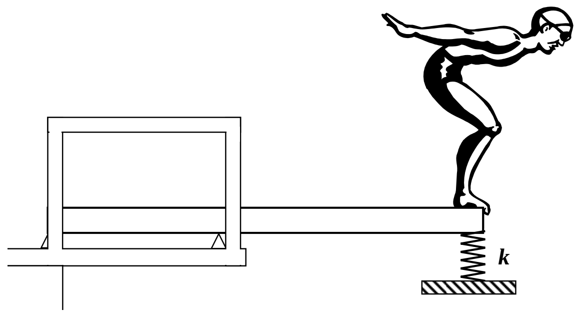
$$d > 3.38''$$

Now, we need to check and ensure that our assumption was correct: $d = 4'' > 3.38''$, which happens to be correct. We would need to iterate in the event the solution did not satisfy the answer.

End Example □

Example 9.8.

Check for Safety



A spring board design is being evaluated by your company. You boss is requesting that you suggest a reasonable warrantee for the product. A typical diver jumps can jump 1 ft on the free end of a diving board before diving into the water. The maximum weight of a typical user is about 225 lbs. The dimensions of the diving board are $L = 4$ m (from the pin to the tip) and is 25.4 mm thick. Has a moment of inertia of $I_{zz} = 5 \times 10^6$ mm^4. The supported end of the diving board is fixed. Assume a 99.999% reliability and that material is a wrought steel with:

$$E = 70 \text{ GPa} \qquad S_y = 400 \text{ MPa} \qquad S_{ut} = 600 \text{ MPa}$$

The spring rate is:

$$k = 16.4 \text{ kN/m}$$

1. First determine, the margin of safety for infinite life using both yielding criterions: Modified Goodman and Soderberg.
2. If the board is to be used during day time (8hr/day) and it is expected that on the average a person uses it every 15 minutes. Suggest a warrantee period for a 400% margin of safety.

Solution: First we need to calculate the maximum load due to impact and consider it as a repeated cyclic load. In order to do so we need to first obtain the static load which can be done using spring analysis. At the tip:

$$\delta_s = \frac{F_s}{k} \quad \delta_b = \frac{F_B\, L^3}{3\, E\, I_{zz}}$$

$$F_s + F_b = F$$

and

$$\delta_s = \delta_b \quad \rightarrow \quad \frac{F_s}{k} = \frac{F_B\, L^3}{3\, E\, I_{zz}} \quad \rightarrow \quad F_B = \frac{3\, E\, I_{zz}\, F_s}{k\, L^3}$$

Thus

$$F_s + F_b = F$$

$$F_s + \frac{3\,E\,I_{zz}\,F_s}{k\,L^3} = W$$

Where $W = 225\,(4.45) = 1001.25$ N. Thus

$$F_s + 1.00038\,F_s = 1001.25 \quad \rightarrow \quad F_s = 500.53 \text{ N}$$

Thus the static deflection is:

$$\delta_{st} = \delta_s = \frac{F_s}{k} = 0.0305201 \text{ m}$$

For a freely falling object the impact factor is:

$$K_{m_1} = 1 + \sqrt{1 + \frac{2\,h}{\delta_{st}}\,\eta_m}$$

For a conservative approximation, take $\eta_m = 1.0$ and $h = 1(12)(25.4)/1000 = 0.3048$ m. Thus impact factor is

$$K_{m_1} = 1 + \sqrt{1 + \frac{2\,h}{\delta_{st}}\,\eta_m} = 5.57971$$

and the impact load will be

$$F_e = W\,K_{m_1} = 5586.68 \text{ N}$$

The most critical point is at the fixed end and at an element at the top. At the top the bending moment is:

$$M_{zz} = -F_e\,L = -22346.7 \text{ N-m}$$

1. First determine, the margin of safety for infinite life using both yielding criterions: Modified Goodman and Soderberg.

   Thus,

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} -\frac{M\,c}{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   where

$$c = \frac{t}{2} = \frac{25.4/1000}{2} = 0.0127 \text{ m}$$

   This is a repeated load in compression for bending moment: Thus,

$$M_a = \frac{M_{max} - M_{min}}{2} = \frac{(0) - (-22346.7)}{2} = 11173.4 \text{ N–m}$$

$$M_m = \frac{M_{max} + M_{min}}{2} = \frac{(0) + (-22346.7)}{2} = -11173.4 \text{ N–m}$$

Thus

$$\underline{\boldsymbol{\sigma}}_{\rm a} = \begin{bmatrix} -2.83803\times10^7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{\rm m} = \begin{bmatrix} 2.83803\times10^7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No stress concentration factors are needed.
The principal stresses are:

$$\sigma_{1,\rm a} = 0 \qquad \sigma_{2,\rm a} = 0 \quad \sigma_{3,\rm a} = -2.83803\times10^7 \text{ Pa}$$

$$\sigma_{1,\rm m} = 2.83803\times10^7 \text{ Pa} \quad \sigma_{2,\rm m} = 0 \quad \sigma_{3,\rm m} = 0$$

For ductile materials the **mean and alternate stresses** are

$$\sigma_{\rm a} = \sigma_{\rm eq,a} = \sqrt{I^2_{\sigma 1,\rm a} - 3\,I_{\sigma 2,\rm a}} = 2.83803\times10^7 \text{ Pa}$$

$$\sigma_{\rm m} = \sigma_{\rm eq,m} = \sqrt{I^2_{\sigma 1,\rm m} - 3\,I_{\sigma 2,\rm m}} = 2.83803\times10^7 \text{ Pa}$$

The **maximum and minimum stresses** are

$$\sigma_{\max} = \sigma_{\rm m} + \sigma_{\rm a} = 5.67607\times10^7 \text{ Pa}$$

$$\sigma_{\min} = \sigma_{\rm m} - \sigma_{\rm a} = 0$$

The **stress range** $\sigma_{\rm r}$ is the difference range of the maximum and minimum stresses in the cycle:

$$\sigma_{\rm r} = \Delta\sigma = \sigma_{\max} - \sigma_{\min} = 5.67607\times10^7 \text{ Pa}$$

The **stress ratio** $R_{\rm s}$ is the ratio of minimum to maximum stress amplitudes:

$$R_{\rm s} = \frac{\sigma_{\min}}{\sigma_{\max}} = 0$$

The **amplitude or load ratio** $\mathbb{A}_{\rm a}$ is the ratio of the stress amplitude to the mean stress:

$$\mathbb{A}_{\rm a} = \frac{\sigma_{\rm a}}{\sigma_{\rm m}} = 1.0$$

Modified Goodman criterion:

$$S_{\rm m} > 0 \qquad S_{\max} < S_{\rm y} \quad \rightarrow \quad S_{\rm eq} = \frac{S_{\rm a}}{1 - \dfrac{S_{\rm m}}{S_{\rm ut}}}$$

$$S_{\rm eq} = \frac{n_{\rm SF}\,\sigma_{\rm a}}{1 - \dfrac{n_{\rm SF}\,\sigma_{\rm m}}{S_{\rm ut}}} \quad \rightarrow \quad \frac{1}{n_{\rm SF}} = \frac{\sigma_{\rm a}}{S_{\rm eq}} + \frac{\sigma_{\rm m}}{S_{\rm ut}}$$

For infinite life take $S_{\rm eq} = S_{\rm e}$, the endurance limit.

Now we need to find the modified S_e. First of all at $N = 1$ life cycle plot $S = S_{\mathrm{ut}} = 600$ MPa.

For $N = 10^3$ and bending $S_{\mathrm{f}} = 0.9\, S_{\mathrm{ut}} = 540$ MPa (for bending $f = 0.90$).

Next wrought steels under pure bending:

$$S'_e = 0.5\, S_{\mathrm{ut}} \qquad \text{at } \ N = 10^6 \text{ cycles}$$

So for the steel used for this spring board

$$S'_e = 0.5\, S_{\mathrm{ut}} = 300. \text{ MPa}$$

The correction factors are:

(a) Loading: $k_{\mathrm{L}} = 1.0$

(b) Temperature: $k_{\mathrm{t}} = 1.0$

(c) Surface finish factor: $k_{\mathrm{sr}} = 1.0$

(d) Reliability: 99.999%, $k_{\mathrm{r}} = 0.659$

(e) Gradient Size factor: For rectangular cross-section:

$$d = 0.8\sqrt{b\, h} = 0.8\sqrt{3.66(25.4/1000)} = 0.243967 \text{ m}$$

$$k_{\mathrm{g}} = 0.7$$

(f) Miscellaneous:$k_{\mathrm{e}} = 1.0$

Thus

$$k_\infty = 0.4613$$

Thus the modified endurance limit is

$$S_e = k_\infty\, S'_e = 138.39 \text{ MPa}$$

Thus

$$\frac{1}{n_{\mathrm{SF}}} = \frac{\sigma_{\mathrm{a}}}{S_{\mathrm{eq}}} + \frac{\sigma_{\mathrm{m}}}{S_{\mathrm{ut}}} \quad \rightarrow \quad n_{\mathrm{SF}} = 3.96$$

and the margin of safety is 296%.

Soderberg Criterion

$$S_{\mathrm{eq}} = \frac{n_{\mathrm{SF}}\, \sigma_{\mathrm{a}}}{1 - \dfrac{n_{\mathrm{SF}}\, \sigma_{\mathrm{m}}}{S_{\mathrm{y}}}} \quad \rightarrow \quad \frac{1}{n_{\mathrm{SF}}} = \frac{\sigma_{\mathrm{a}}}{S_{\mathrm{eq}}} + \frac{\sigma_{\mathrm{m}}}{S_{\mathrm{y}}}$$

For infinite life take $S_{\mathrm{eq}} = S_{\mathrm{e}}$, the endurance limit. From above the modified endurance limit is

$$S_e = 138.39 \text{ MPa}$$

Thus

$$\frac{1}{n_{\mathrm{SF}}} = \frac{\sigma_{\mathrm{a}}}{S_{\mathrm{eq}}} + \frac{\sigma_{\mathrm{m}}}{S_{\mathrm{y}}} \quad \rightarrow \quad n_{\mathrm{SF}} = 3.623$$

and the margin of safety is 262%.

2. If the board is to be used during day time (8hr/day, 5 days) and it is expected that on the average a person uses it every 15 minutes. Suggest a warrantee period for a 400% margin of safety.

   The safety factor is $n_{\text{SF}} = 5.0$:

$$S_{\text{a}} = n_{\text{SF}}\,\sigma_{\text{a}} = 141.902 \text{ MPa} \qquad S_{\text{m}} = n_{\text{SF}}\,\sigma_{\text{m}} = 141.902 \text{ MPa}$$

   Using the modified Goodman criterion:

$$\sigma_{\text{m}} > 0 \qquad \sigma_{\max} < S_{\text{y}} \quad \rightarrow \quad S_{\text{eq}} = \frac{S_{\text{a}}}{1 - \dfrac{S_{\text{m}}}{S_{\text{ut}}}} = 185.858 \text{ MPa}$$

   The modified endurance limit is

$$S_e = 138.39 \text{ MPa}$$

   Since $S_{\text{eq}} > S_e$, the part has finite life.

   In order to find the life cycles, we use the equation

$$N = \left(\frac{S}{a}\right)^{\frac{1}{b}} \quad \rightarrow \quad N = \left(\frac{S_{\text{eq}}}{a}\right)^{\frac{1}{b}}$$

   where

$$b = \frac{1}{3 - g_{\text{e}}}\,\log\left(\frac{S_{\text{f}}}{S_e}\right) \qquad a = \frac{S_{\text{f}}}{10^{3\,b}} = \frac{f\,S_{\text{ut}}}{10^{3\,b}} = (f\,S_{\text{ut}})^{\frac{g_{\text{e}}}{g_{\text{e}}-3}}\,(S_{\text{e}})^{\frac{3}{3-g_{\text{e}}}}$$

   Note that we took $S = S_{\text{eq}}$ because that is the fatigue stress at which we want to calculate the life cycles for failure.

   For $g_{\text{e}} = 6$:

$$b = -\frac{1}{3}\,\log\left(\frac{f\,S_{\text{ut}}}{S_e}\right) = -0.197096 \qquad a = \frac{f\,S_{\text{ut}}}{10^{3\,b}} = 2.10709 \times 10^{9}$$

   Thus

$$N = \left(\frac{S_{\text{eq}}}{a}\right)^{\frac{1}{b}} = 223969. \text{ cycles}$$

   The component has 2.24×10^5 life cycles before failure.

$$N = 2.24 \times 10^5 \text{ cycles}\left(\frac{1 \text{ hour}}{4 \text{ cycle}}\right)\left(\frac{1 \text{ day}}{8 \text{ hours}}\right)\left(\frac{1 \text{ week}}{5 \text{ days}}\right)\left(\frac{1 \text{ year}}{52 \text{ weeks}}\right) = 26.91 \text{ years}$$

   The part has 26.91 years of life. Thus warrantee could be for 25 years.

   Using the Soderberg criterion:

$$S_{\text{eq}} = \frac{S_{\text{a}}}{1 - \dfrac{S_{\text{m}}}{S_{\text{y}}}} = 220.00 \text{ MPa}$$

The modified endurance limit is

$$S_e = 138.39 \text{ MPa}$$

Since $S_{\text{eq}} > S_e$, the part has finite life.
For $g_{\text{e}} = 6$:

$$b = -\frac{1}{3} \log\left(\frac{f\,S_{\text{ut}}}{S_e}\right) = -0.197096 \qquad a = \frac{f\,S_{\text{ut}}}{10^{3\,b}} = 2.10709 \times 10^9$$

Thus

$$N = \left(\frac{S_{\text{eq}}}{a}\right)^{\frac{1}{b}} = 95365.9 \text{ cycles}$$

The component has 9.54×10^5 life cycles before failure.

$$N = 9.54 \times 10^5 \text{ cycles} \left(\frac{1 \text{ hour}}{4 \text{ cycle}}\right)\left(\frac{1 \text{ day}}{8 \text{ hours}}\right)\left(\frac{1 \text{ week}}{5 \text{ days}}\right)\left(\frac{1 \text{ year}}{52 \text{ weeks}}\right) = 11.46 \text{ years}$$

Thus warrantee should be for ten years.

End Example □

9.8 Cumulative fatigue damage

Aerospace structural components are not always subjected to the constant stress cycles. Many parts may be under different severe levels of reversed stress cycles or randomly varying stress levels. Examples include aircraft structural components operating at stress levels between the fracture strength and endurance limit. We must examine the cumulative damage when a structural component is to operate for a finite time at higher stress. It is important to note that predicting the cumulative damage of parts stressed above the endurance limit is at best a rough procedure. Clearly, for parts subjected to randomly varying loads, the damage prognosis is further complicated. The simplest and most widely accepted criterion used to explain cumulative damage is known as the Miner's rule. In 1945, M. A. Miner popularized a rule that had first been proposed by A. Palmgren in 1924.

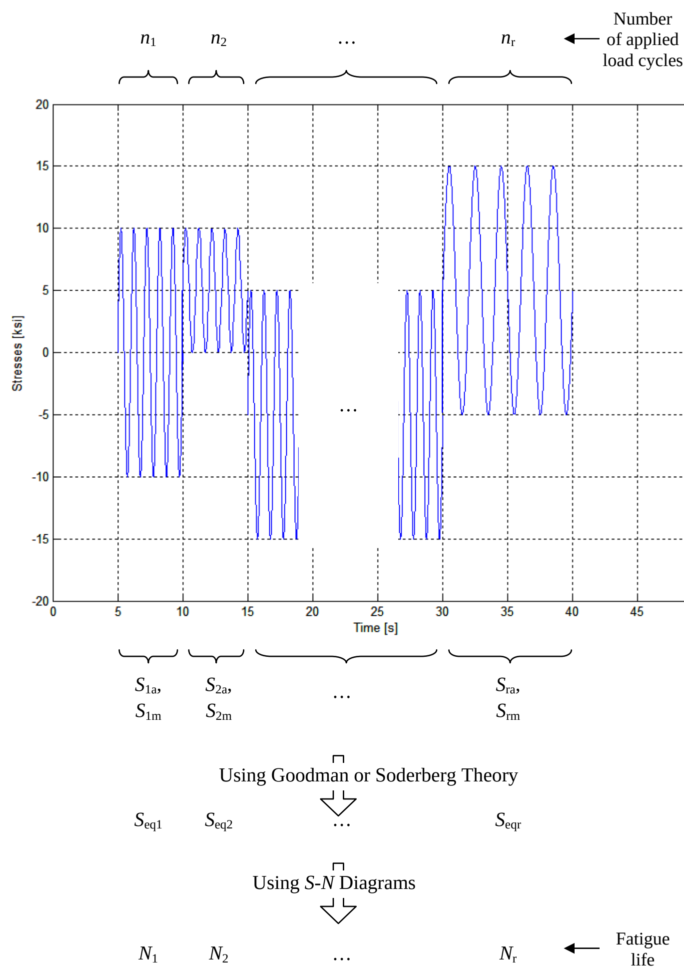


Figure 9.6: Stress spectrum.

The Linear Damage Rule (also known as Palmgren-Miner cycle-ratio summation rule or simply Miner's rule) is based on the concept of fatigue damage. Miner's rule states that a damage fraction, D, is defined as the fraction of life used up by an event or a series of events. In this context, we may predict failure for cumulative fatigue damage using the Miner's rule which is defined as follows

$$\sum_{i=1}^{r} \frac{n_i}{N_i} = c$$

$$D = B_{\rm f} \sum_{i=1}^{r} \frac{n_i}{N_i} = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \cdots + \frac{n_r}{N_r} \tag{9.9}$$

where $B_{\rm f}$ is the number of block or duty cycles, n_i the number of cycles at stress levels σ_i, and N_i the number of cycles to fail at stress level σ_i. These life cycles are taken from the appropriate S–N diagram. Figure 9.6 shows a schematic on how to evaluate each of these terms.

The Miner's equation assumes that the damage to the material is directly proportional to the number of cycles at a given stress. The rule also presupposes that the stress sequence does not matter and the rate of damage accumulation at a stress level is independent of the stress history. These have not been completely verified in tests. Sometimes specifications are used in which the right side of Eq. (9.9) is taken as

$$0.7 \leq c \leq 2.2$$

Failure is then determined by using

$D = c$ onset of failure due to fatigue is predicted

$D > c$ failure due to fatigue has occurred

$D < c$ safety due to fatigue failure

Usually for design purposes, we take $c = 1$. Though Miner's rule is a useful approximation in many circumstances, it has two major limitations:

1. It fails to recognize the probabilistic nature of fatigue and there is no simple way to relate life predicted by the rule with the characteristics of a probability distribution.
2. There is sometimes an effect in the order in which the reversals occur. In some circumstances, cycles of low stress followed by high stress cause more damage than would be predicted by the rule. It does not consider the effect of overload or high stress which may result in a compressive residual stress. High stress followed by low stress may have less damage due to the presence of compressive residual stress.

Despite these limitations, the Linear Damage Rule (Miner's rule) is still widely used. This is due both to its simplicity and the fact that more sophisticated methods do not always result in better predictions.

Example 9.9.

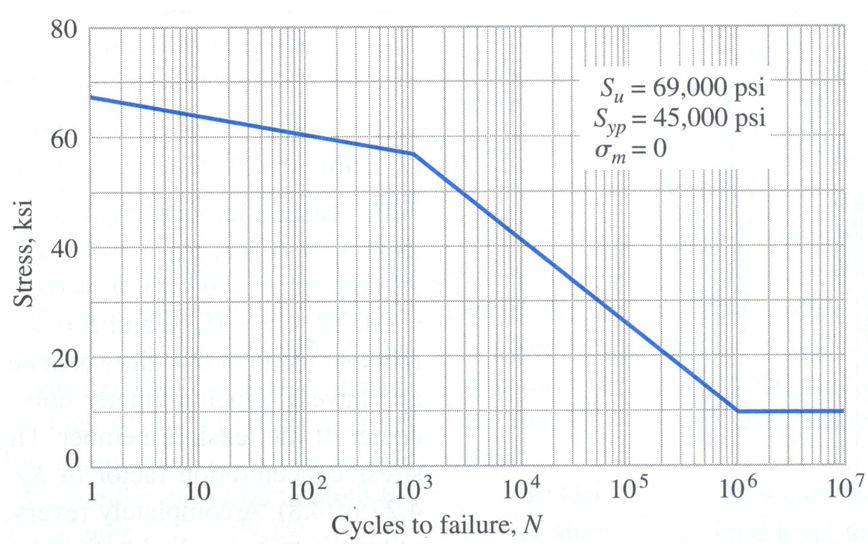
Number of cycles

A hollow square tube with outside dimension of 1.75 inches and wall thickness of 0.125 inch with a S–N curve for the material shown in Figure.

1. This hollow square tube is to be subject to the following sequence of completely reversed axial force amplitudes: First 20000 lb for 5200 cycles; next, 5500 lb for 948000 cycles; then, 8450 lb for 11100 cycles. After this loading sequence has been imposed, it is desired to change the force amplitude to 19000 lb, still in the axial direction. Assuming only one duty cycle, how many remaining cycles of life would you predict for the tube at this final level of loading?
2. It is desired to design a hollow circular tube with an outside diameter of 1.75 inches with the same material and manufacturing conditions mentioned above and with a margin of safety of 1.5. Let the ratio of the outer diameter to be inner diameter:

$$\beta = \frac{d_o}{d_i}$$

Determine the value of β such that the remaining life for the 19000 lb force amplitude is $n_4 = 1.3615 \times 10^5$ cycles.



Solution:

1. This hollow square tube is to be subject to the following sequence of completely reversed axial force amplitudes: First 20000 lb for 5200 cycles; next, 5500 lb for 948000 cycles;

then, 8450 lb for 11100 cycles. After this loading sequence has been imposed, it is desired to change the force amplitude to 19000 lb, still in the axial direction. Assuming only one duty cycle, how many remaining cycles of life would you predict for the tube at this final level of loading?

From the problem it is known:

$$S_{\text{ut}} = 69000 \text{ psi} \qquad S_{\text{y}} = 45000 \text{ psi}$$

The Miner's rule for cumulative fatigue damage for four different stress cycles is defined as

$$D = B_{\text{f}} \sum_{i=1}^{4} \frac{n_i}{N_i} = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4}$$

For one duty cycle

$$B_{\text{f}} = 1$$

and when $D = c = 1$ failure is predicted by fatigue and the Miner's rule is written as

$$(1)\left(\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4}\right) = 1$$

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4} = 1$$

For our problem,

$$P_1 = 20000 \text{ lb} \qquad n_1 = 5200$$

$$P_2 = 5500 \text{ lb} \qquad n_2 = 948000$$

$$P_3 = 8450 \text{ lb} \qquad n_3 = 11100$$

$$P_4 = 19000 \text{ lb} \qquad n_4 = ?$$

The problems reduces to find number of life cycles.

Since the loads are completely reversed:

$$P_{1\text{max}} = 20000 \text{ lb} \qquad P_{1\text{min}} = -20000 \text{ lb}$$

$$P_{2\text{max}} = 5500 \text{ lb} \qquad P_{2\text{min}} = -5500 \text{ lb}$$

$$P_{3\text{max}} = 8450 \text{ lb} \qquad P_{3\text{min}} = -8450 \text{ lb}$$

$$P_{4\text{max}} = 19000 \text{ lb} \qquad P_{4\text{min}} = -19000 \text{ lb}$$

Thus the alternate and mean loads are

$$P_{1\mathrm{a}} = 20000 \text{ lb} \qquad P_{1\mathrm{m}} = 0 \text{ lb}$$

$$P_{2\mathrm{a}} = 5500 \text{ lb} \qquad P_{2\mathrm{m}} = 0 \text{ lb}$$

$$P_{3\mathrm{a}} = 8450 \text{ lb} \qquad P_{3\mathrm{m}} = 0 \text{ lb}$$

$$P_{4\mathrm{a}} = 19000 \text{ lb} \qquad P_{4\mathrm{m}} = 0 \text{ lb}$$

Only axial loads are applied,

$$\sigma_{\mathrm{xx}} = \frac{P}{A}$$

Thus the state of stress is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \frac{P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$A = a_o^2 - a_i^2 = a_o^2 - (a_o - 2\,t)^2 = 0.8125 \text{ in}^2$$

Hence the state of stress for each load will be

$$\underline{\boldsymbol{\sigma}}_1 = \begin{bmatrix} \frac{P_1}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_2 = \begin{bmatrix} \frac{P_2}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_3 = \begin{bmatrix} \frac{P_3}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_4 = \begin{bmatrix} \frac{P_4}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The alternate state of stress is

$$\underline{\boldsymbol{\sigma}}_{1a} = \begin{bmatrix} \frac{P_{1a}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{2a} = \begin{bmatrix} \frac{P_{2a}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{3a} = \begin{bmatrix} \frac{P_{3a}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{4a} = \begin{bmatrix} \frac{P_{4a}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The mean state of stress is

$$\underline{\boldsymbol{\sigma}}_{1m} = \begin{bmatrix} \frac{P_{1m}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{2m} = \begin{bmatrix} \frac{P_{2m}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{3m} = \begin{bmatrix} \frac{P_{3m}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{4m} = \begin{bmatrix} \frac{P_{4m}}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we are working with ductile materials the fully reversed equivalent mean stresses are

$$\sigma_{1m} = 0 \quad \sigma_{2m} = 0 \quad \sigma_{3m} = 0 \quad \sigma_{4m} = 0$$

and the fully reversed equivalent alternate stresses are

$$\sigma_{1a} = \frac{P_{1a}}{A} = 24615.4 \text{ psi}$$

$$\sigma_{2a} = \frac{P_{2a}}{A} = 6769.23 \text{ psi}$$

$$\sigma_{3a} = \frac{P_{3a}}{A} = 10400 \text{ psi}$$

$$\sigma_{4a} = \frac{P_{4a}}{A} = 23384.6 \text{ psi}$$

Assuming a safety factor of one:

$$S_{1\mathrm{m}} = n_{\mathrm{SF}}\, \sigma_{1\mathrm{m}} = 0$$

$$S_{2\mathrm{m}} = n_{\mathrm{SF}}\, \sigma_{2\mathrm{m}} = 0$$

$$S_{3\mathrm{m}} = n_{\mathrm{SF}}\, \sigma_{3\mathrm{m}} = 0$$

$$S_{4\mathrm{m}} = n_{\mathrm{SF}}\, \sigma_{4\mathrm{m}} = 0$$

$$S_{1\mathrm{a}} = n_{\mathrm{SF}}\, \sigma_{1\mathrm{a}} = 24615.4 \text{ psi}$$

$$S_{2\mathrm{a}} = n_{\mathrm{SF}}\, \sigma_{2\mathrm{a}} = 6769.23 \text{ psi}$$

$$S_{3\mathrm{a}} = n_{\mathrm{SF}}\, \sigma_{3\mathrm{a}} = 10400 \text{ psi}$$

$$S_{4\mathrm{a}} = n_{\mathrm{SF}}\, \sigma_{4\mathrm{a}} = 23384.6 \text{ psi}$$

The fully reversed fatigue equivalent stresses, using Goodman criterion, are:

$$S_{1\mathrm{eq}} = \frac{S_{1\mathrm{a}}}{1 - \dfrac{S_{1\mathrm{m}}}{S_{\mathrm{ut}}}} = S_{1\mathrm{a}} = 24615.4 \text{ psi}$$

$$S_{2\mathrm{eq}} = \frac{S_{2\mathrm{a}}}{1 - \dfrac{S_{2\mathrm{m}}}{S_{\mathrm{ut}}}} = S_{2\mathrm{a}} = 6769.23 \text{ psi}$$

$$S_{3\mathrm{eq}} = \frac{S_{3\mathrm{a}}}{1 - \dfrac{S_{3\mathrm{m}}}{S_{\mathrm{ut}}}} = S_{3\mathrm{a}} = 10400 \text{ psi}$$

$$S_{4\mathrm{eq}} = \frac{S_{4\mathrm{a}}}{1 - \dfrac{S_{4\mathrm{m}}}{S_{\mathrm{ut}}}} = S_{4\mathrm{a}} = 23384.6 \text{ psi}$$

First verify failure due to yielding has not occurred:

$$S_{1\max} = 24615.4 \text{ psi} \quad < \quad S_{\mathrm{y}} = 45000 \text{ psi}$$

$$S_{2\max} = 6769.23 \text{ psi} \quad < \quad S_{\mathrm{y}} = 45000 \text{ psi}$$

$$S_{3\max} = 10400 \text{ psi} \quad < \quad S_{\mathrm{y}} = 45000 \text{ psi}$$

$$S_{4\max} = 23384.6 \text{ psi} \quad < \quad S_{\mathrm{y}} = 45000 \text{ psi}$$

Hence we continue. Next for each completely reversed stress spectrum obtain the number of life cycles (or cycles to fail). For this we use the S–N diagram (or the analytical equations) to obtain the number of life cycles:

$$S_{1_{\rm eq}} = 24.6 \text{ ksi} \quad \rightarrow \quad N_1 = 1.0 \times 10^5$$

$$S_{2_{\rm eq}} = 6.77 \text{ ksi} \quad \rightarrow \quad N_2 = \infty$$

$$S_{3_{\rm eq}} = 10.4 \text{ ksi} \quad \rightarrow \quad N_3 = 1.0 \times 10^6$$

$$S_{4_{\rm eq}} = 23.4 \text{ ksi} \quad \rightarrow \quad N_4 = 1.5 \times 10^5$$

Thus using the Miner's rule,

$$\frac{5200}{1.0 \times 10^5} + \frac{948000}{\infty} + \frac{11100}{1.0 \times 10^6} + \frac{n_4}{1.5 \times 10^5} = 1 \qquad \rightarrow \qquad n_4 = 1.41 \times 10^5$$

Thus 1.41×10^5 cycles remain for fatigue failure.

2. It is desired to design a hollow circular tube with an outside diameter of 1.75 inches with the same material and manufacturing conditions mentioned above and with a margin of safety of 1.5. Let the ratio of the outer diameter to be inner diameter:

$$\beta = \frac{d_o}{d_i}$$

Determine the value of β such that the remaining life for the 19000 lb force amplitude is $n_4 = 1.3615 \times 10^5$ cycles.

Here

$$P_1 = 20000 \text{ lb} \qquad n_1 = 5200$$

$$P_2 = 5500 \text{ lb} \qquad n_2 = 948000$$

$$P_3 = 8450 \text{ lb} \qquad n_3 = 11100$$

$$P_4 = 19000 \text{ lb} \qquad n_4 = 1.3615 \times 10^5$$

The problems reduces to find number of life cycles for the last load.

Thus the state of stress is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} \frac{P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) = \frac{\pi}{4} d_o^2 \left(1 - \left(\frac{d_i}{d_o}\right)^2\right) = \frac{\pi}{4} d_o^2 \left(1 - \frac{1}{\beta^2}\right)$$

Assume that the life cycles of the first three sequences of fully reversed axial forces does not change. Since the material specification are the same we may use the same S–N diagram. The problem requires a factor of safety of:

$$n_{\text{SF}} = MS + 1 = 2.5$$

Using the Miner's rule we need to determine the number of cycles to fail if $n_4 = 1.3615 \times 10^5$:

$$\frac{5200}{1.0 \times 10^5} + \frac{948000}{\infty} + \frac{11100}{1.0 \times 10^6} + \frac{1.3615 \times 10^5}{N_4} = 1 \qquad \rightarrow \qquad N_4 = 1.45 \times 10^4$$

Thus there are 1.45×10^4 life cycles for fatigue failure. From the S–N diagram we proceed to find the equivalent fatigue stress

$$S_{4\text{eq}} = 29 \text{ ksi}$$

Note that the above is the maximum allowable stress, thus

$$n_{\text{SF}} = 2.5 = \frac{S_{4\text{eq}}}{\sigma_{4\text{eq}}} \qquad \rightarrow \qquad \sigma_{4\text{eq}} = \frac{S_{4\text{eq}}}{n_{\text{SF}}} = 11.6 \text{ ksi}$$

And the area can be calculated using P_4:

$$\sigma_{4\text{eq}} = \frac{P_4}{A} \qquad \rightarrow \qquad 11600 = \frac{19000}{A} \qquad \rightarrow \qquad A = 1.638 \text{ in}^2$$

The area for a hollow tubular tube is

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) \qquad \rightarrow \qquad d_i = \sqrt{d_o^2 - \frac{4\,A}{\pi}} = 0.988 \text{ in}$$

Thus

$$\beta = \frac{d_o}{d_i} = 1.77$$

If one assume that the life cycles of the first three sequences of fully reversed axial forces does not change, the process becomes an iterative one. One should check the stresses and the associated life cycles and check if it is safe, if not then continue to increase or decrease the diameter until Miner's rule is satisfied.

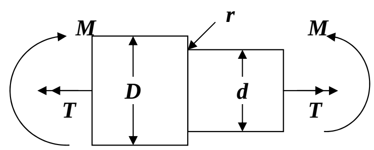
End Example □

Example 9.10.

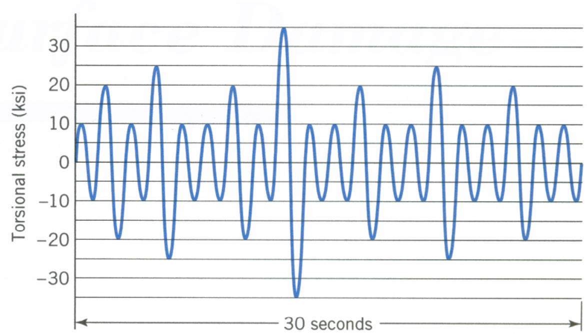
Duty Life Cycles

A stepped shaft, as shown in Figure, has dimensions of

$$D = 2'' \quad d = 1'' \quad r = 0.1''$$



It is machined from Titanium Ti-6Al-4V. The loading is one of completely reversed torsion and constant bending stress of 3.08642 ksi. During a typical 30 seconds of operation under overload conditions the nominal stress in the l-inch diameter section was measured to be as shown in Figure. Estimate the life of the shaft for a 95% survivability when operating continuously under these conditions. Take a safety factor of 1.2 and be conservative.



The 30 second test shows that the stepped tube is subject to the following sequence of completely reversed torsional stress amplitudes: one 35 ksi torsional stress amplitude; two 25 ksi torsional stress amplitudes; four 20 ksi torsional stress amplitudes; thirteen 10 ksi torsional stress amplitudes. The goal is to find the number of duty cycles in 30 second for failure.

For Titanium Ti-6Al-4V it is known:

$$S_{ut} = 150 \text{ ksi} \qquad S_y = 128 \text{ ksi}$$

The Miner's rule for cumulative fatigue damage for four different stress cycles is defined as

$$D = B_\text{f} \sum_{i=1}^{4} \frac{n_i}{N_i} = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4}$$

For B_f duty cycle and taking $D = c = 1$ for failure prediction by fatigue, the Miner's rule is written as

$$B_\text{f} \left(\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4} \right) = 1$$

For our problem,

$$\tau_1 = 35 \text{ ksi} \qquad n_1 = 1$$

$$\tau_2 = 25 \text{ ksi} \qquad n_2 = 2$$

$$\tau_3 = 20 \text{ ksi} \qquad n_3 = 4$$

$$\tau_4 = 10 \text{ ksi} \qquad n_4 = 13$$

For our problem we have the nominal stresses and the loads are a constant load in bending and completely reversed load in torsional. Considering an element at the top we get

$$\underline{\boldsymbol{\sigma}}_\text{a} = \begin{bmatrix} \sigma_\text{xxa} & 0 & \tau_\text{xza} \\ 0 & 0 & 0 \\ \tau_\text{xza} & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_\text{m} = \begin{bmatrix} \sigma_\text{xxm} & 0 & \tau_\text{xzm} \\ 0 & 0 & 0 \\ \tau_\text{xzm} & 0 & 0 \end{bmatrix}$$

For each bending load cycle

$$\sigma_\text{max} = \sigma_\text{min} = 3.08642$$

$$\sigma_\text{xxa} = \frac{\sigma_\text{max} - \sigma_\text{min}}{2} = \frac{(3.08642) - (3.08642)}{2} = 0$$

$$\sigma_\text{xxm} = \frac{\sigma_\text{max} + \sigma_\text{min}}{2} = \frac{(3.08642) + (3.08642)}{2} = 3.08642 \text{ ksi}$$

For each torsional load cycle

$$\tau_{xza_1} = \frac{\tau_{xz_{max1}} - \tau_{xz_{min1}}}{2} = \frac{(35) - (-35)}{2} = 35 \text{ ksi}$$

$$\tau_{xzm_1} = \frac{\tau_{xz_{max1}} + \tau_{xz_{min1}}}{2} = \frac{(35) + (-35)}{2} = 0$$

$$\tau_{xza_2} = \frac{\tau_{xz_{max2}} - \tau_{xz_{min2}}}{2} = \frac{(25) - (-25)}{2} = 25 \text{ ksi}$$

$$\tau_{xzm_2} = \frac{\tau_{xz_{max2}} + \tau_{xz_{min2}}}{2} = \frac{(25) + (-25)}{2} = 0$$

$$\tau_{xza_3} = \frac{\tau_{xz_{max3}} - \tau_{xz_{min3}}}{2} = \frac{(20) - (-20)}{2} = 20 \text{ ksi}$$

$$\tau_{xzm_3} = \frac{\tau_{xz_{max3}} + \tau_{xz_{min3}}}{2} = \frac{(20) + (-20)}{2} = 0$$

$$\tau_{xza_4} = \frac{\tau_{xz_{max4}} - \tau_{xz_{min4}}}{2} = \frac{(10) - (-10)}{2} = 10 \text{ ksi}$$

$$\tau_{xzm_4} = \frac{\tau_{xz_{max4}} + \tau_{xz_{min4}}}{2} = \frac{(10) + (-10)}{2} = 0$$

Thus

$$\underline{\boldsymbol{\sigma}}_{a1} = \begin{bmatrix} 0 & 0 & 35 \\ 0 & 0 & 0 \\ 35 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{m1} = \begin{bmatrix} 3.08642 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{a2} = \begin{bmatrix} 0 & 0 & 25 \\ 0 & 0 & 0 \\ 25 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{m2} = \begin{bmatrix} 3.08642 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{a3} = \begin{bmatrix} 0 & 0 & 20 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{m3} = \begin{bmatrix} 3.08642 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{a4} = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 0 \end{bmatrix} \qquad \underline{\boldsymbol{\sigma}}_{m4} = \begin{bmatrix} 3.08642 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is ductile material, we apply the fatigue stress concentration factor to both alternate and mean stresses.

$$d = 1'' \qquad r = 0.1'' \qquad \rightarrow \qquad \frac{r}{d} = 0.1$$

$$d = 1'' \qquad D = 2'' \qquad \rightarrow \qquad \frac{D}{d} = 2.0$$

Thus

$$K_{\mathrm{t}_b} = 1.75 \qquad K_{\mathrm{t}_s} = 1.46$$

The fatigue stress concentration factor K_{f_a} is calculated as follows,

$$K_{\mathrm{f}} = 1 + q\ (K_{\mathrm{t}} - 1)$$

From Tables and from the previous example:

$$r = 0.1'' \qquad S_{\mathrm{ut}} = 100\ \mathrm{ksi} \quad \rightarrow \quad q_{\mathrm{b}} \approx 0.83 \quad q_{\mathrm{s}} \approx 0.86$$

$$K_{\mathrm{f}_b} = 1 + 0.83\ (1.75 - 1) = 1.62$$

$$K_{\mathrm{f}_b} = 1 + 0.86\ (1.46 - 1) = 1.40$$

Thus,

$$\underline{\boldsymbol{\sigma}}_{\mathrm{a}1} = \begin{bmatrix} (1.62)0 & 0 & (1.40)35 \\ 0 & 0 & 0 \\ (1.40)35 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 48.846 \\ 0 & 0 & 0 \\ 48.846 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{m}1} = \begin{bmatrix} (1.62)(3.08642) & 0 & (1.40)0 \\ 0 & 0 & 0 \\ (1.40)0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 8.1125 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{a}2} = \begin{bmatrix} (1.62)0 & 0 & (1.40)25 \\ 0 & 0 & 0 \\ (1.40)25 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 34.89 \\ 0 & 0 & 0 \\ 34.89 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{m}2} = \begin{bmatrix} (1.62)(3.08642) & 0 & (1.40)0 \\ 0 & 0 & 0 \\ (1.40)0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5.0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{a3} = \begin{bmatrix} (1.62)0 & 0 & (1.40)20 \\ 0 & 0 & 0 \\ (1.40)20 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 27.912 \\ 0 & 0 & 0 \\ 27.912 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{m3} = \begin{bmatrix} (1.62)(3.08642) & 0 & (1.40)0 \\ 0 & 0 & 0 \\ (1.40)0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5.0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{a4} = \begin{bmatrix} (1.62)0 & 0 & (1.40)10 \\ 0 & 0 & 0 \\ (1.40)10 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 13.956 \\ 0 & 0 & 0 \\ 13.956 & 0 & 0 \end{bmatrix}$$

$$\underline{\boldsymbol{\sigma}}_{m4} = \begin{bmatrix} (1.62)(3.08642) & 0 & (1.40)0 \\ 0 & 0 & 0 \\ (1.40)0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5.0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal stresses for loading cycle 1 are:

$$\sigma_{1,a1} = 48.846 \text{ ksi} \quad \sigma_{2,a1} = 0 \quad \sigma_{3,a1} = -48.846 \text{ ksi}$$

$$\sigma_{1,m1} = 5 \text{ ksi} \quad \sigma_{2,m1} = 0 \quad \sigma_{3,m1} = 0$$

The principal stresses for loading cycle 2 are:

$$\sigma_{1,a2} = 34.89 \text{ ksi} \quad \sigma_{2,a2} = 0 \quad \sigma_{3,a2} = -34.89 \text{ ksi}$$

$$\sigma_{1,m2} = 5 \text{ ksi} \quad \sigma_{2,m2} = 0 \quad \sigma_{3,m2} = 0$$

The principal stresses for loading cycle 3are:

$$\sigma_{1,a3} = 27.912 \text{ ksi} \quad \sigma_{2,a3} = 0 \quad \sigma_{3,a3} = -27.912 \text{ ksi}$$

$$\sigma_{1,m3} = 5 \text{ ksi} \quad \sigma_{2,m3} = 0 \quad \sigma_{3,m3} = 0$$

The principal stresses for loading cycle 4 are:

$$\sigma_{1,a4} = 13.956 \text{ ksi} \quad \sigma_{2,a4} = 0 \quad \sigma_{3,a4} = -13.956 \text{ ksi}$$

$$\sigma_{1,m4} = 5 \text{ ksi} \quad \sigma_{2,m4} = 0 \quad \sigma_{3,m4} = 0$$

For ductile materials the **mean and alternate stresses** for each cycle are

$$\sigma_{a1} = \sigma_{eq,a1} = \sqrt{I_{\sigma_{1,a1}}^2 - 3\,I_{\sigma_{2,a1}}} = 84.60 \text{ ksi} \qquad \sigma_{m1} = \sigma_{eq,m1} = \sqrt{I_{\sigma_{1,m1}}^2 - 3\,I_{\sigma_{2,m1}}} = 5 \text{ ksi}$$

$$\sigma_{a2} = \sigma_{eq,a2} = \sqrt{I_{\sigma_{1,a2}}^2 - 3\,I_{\sigma_{2,a2}}} = 60.43 \text{ ksi} \qquad \sigma_{m2} = \sigma_{eq,m2} = \sqrt{I_{\sigma_{1,m2}}^2 - 3\,I_{\sigma_{2,m2}}} = 5 \text{ ksi}$$

$$\sigma_{a3} = \sigma_{eq,a3} = \sqrt{I_{\sigma_{1,a3}}^2 - 3\,I_{\sigma_{2,a3}}} = 48.35 \text{ ksi} \qquad \sigma_{m3} = \sigma_{eq,m3} = \sqrt{I_{\sigma_{1,m3}}^2 - 3\,I_{\sigma_{2,m3}}} = 5 \text{ ksi}$$

$$\sigma_{a4} = \sigma_{eq,a4} = \sqrt{I_{\sigma_{1,a4}}^2 - 3\,I_{\sigma_{2,a4}}} = 24.1725 \text{ ksi} \qquad \sigma_{m4} = \sigma_{eq,m4} = \sqrt{I_{\sigma_{1,m4}}^2 - 3\,I_{\sigma_{2,m4}}} = 5 \text{ ksi}$$

For a safety factor of 1.2:

$$S_{1m} = n_{SF}\,\sigma_{m1} = 6.0 \text{ ksi}$$

$$S_{2m} = n_{SF}\,\sigma_{m2} = 6.0 \text{ ksi}$$

$$S_{3m} = n_{SF}\,\sigma_{m3} = 6.0 \text{ ksi}$$

$$S_{4m} = n_{SF}\,\sigma_{m4} = 6.0 \text{ ksi}$$

$$S_{1a} = n_{SF}\,\sigma_{a1} = 101.525 \text{ ksi}$$

$$S_{2a} = n_{SF}\,\sigma_{a2} = 72.5175 \text{ ksi}$$

$$S_{3a} = n_{SF}\,\sigma_{a3} = 58.014 \text{ ksi}$$

$$S_{4a} = n_{SF}\,\sigma_{a4} = 29.007 \text{ ksi}$$

Verify safety due to yielding:

$$S_{1max} = 107.525 \text{ ksi} \quad < \quad S_y = 128 \text{ ksi}$$

$$S_{2max} = 78.5175 \text{ ksi} \quad < \quad S_y = 128 \text{ ksi}$$

$$S_{3max} = 64.014 \text{ ksi} \quad < \quad S_y = 128 \text{ ksi}$$

$$S_{4max} = 35.007 \text{ ksi} \quad < \quad S_y = 128 \text{ ksi}$$

The conservative fatigue criterion for ductile materials, we use the Soderberg criterion:

$$S_{1\text{eq}} = \frac{S_{1\text{a}}}{1 - \dfrac{S_{1\text{m}}}{S_{\text{yt}}}} = 106.518 \text{ ksi}$$

$$S_{2\text{eq}} = \frac{S_{2\text{a}}}{1 - \dfrac{S_{2\text{m}}}{S_{\text{yt}}}} = 76.0839 \text{ ksi}$$

$$S_{3\text{eq}} = \frac{S_{3\text{a}}}{1 - \dfrac{S_{3\text{m}}}{S_{\text{yt}}}} = 60.8672 \text{ ksi}$$

$$S_{4\text{eq}} = \frac{S_{4\text{a}}}{1 - \dfrac{S_{4\text{m}}}{S_{\text{yt}}}} = 30.4336 \text{ ksi}$$

Hence we continue. Next for each cyclic stress spectrum obtain the number of life cycles (or cycles to fail). For this we use the S–N diagram (or the analytical equations) to obtain the number of life cycles. Thus we need to find the constants a and b in

$$S_{\text{N}} = a\, N^b$$

In order to do so, we need to find the modified S_e. First of all at $N = 1$ life cycle plot $S = S_{\text{ut}} = 150$ ksi. For $N = 10^3$ and fatigue in torsional loading only $S_{\text{f}} = 0.72\, S_{\text{ut}} = 108$ ksi ($f = 0.72$). Next for Titanium alloys:

$$S'_e = 0.45\, S_{\text{ut}} \qquad \text{at } N = 10^6 \text{ cycles}$$

So for the steel used for this circular shaft

$$S'_e = 0.65\, S_{\text{ut}} = 97.5 \text{ ksi}$$

The correction factors are:

(a) Loading (fatigue in torsion only): $k_{\text{L}} = 0.59$

(b) Temperature: $k_{\text{t}} = 1.0$

(c) Surface finish factor (machined or cold-drawn):

$$k_{\text{sr}} = e\, S_{\text{ut}}^c = 2.70\,(150)^{-0.265} = 0.715648$$

(d) Reliability: 95%, $k_{\text{r}} = 0.868$

(e) Gradient Size factor: Bending and torsion combined. For $d = 1''$, $k_{\text{g}} = 0.9$

(f) Miscellaneous: $k_{\text{e}} = 1.0$

Thus

$$k_\infty =$$

Thus the modified endurance limit is

$$S_e = k_\infty \, S'_e = 32.1602 \text{ ksi}$$

For $g_{\rm e} = 6$:

$$b = -\frac{1}{3}\,\log\left(\frac{f\,S_{\rm ut}}{S_e}\right) = -0.175368 \qquad a = \frac{f\,S_{\rm ut}}{10^{3\,b}} = 362.685$$

Thus

$$N = \left(\frac{S_{\rm eq}}{a}\right)^{\frac{1}{b}}$$

Thus

$$S_{1\rm eq} = 106.518 \text{ ksi} > S_{\rm e} \quad \rightarrow \quad N_1 = 1.082 \times 10^3$$

$$S_{2\rm eq} = 76.0839 \text{ ksi} > S_{\rm e} \quad \rightarrow \quad N_2 = 7.370 \times 10^3$$

$$S_{3\rm eq} = 60.8672 \text{ ksi} > S_{\rm e} \quad \rightarrow \quad N_3 = 2.631 \times 10^4$$

$$S_{4\rm eq} = 30.4336 \text{ ksi} < S_{\rm e} \quad \rightarrow \quad N_4 = \infty$$

Thus

$$B_{\rm f}\left(\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4}\right) = 1$$

$$B_{\rm f} = \frac{1}{\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4}}$$

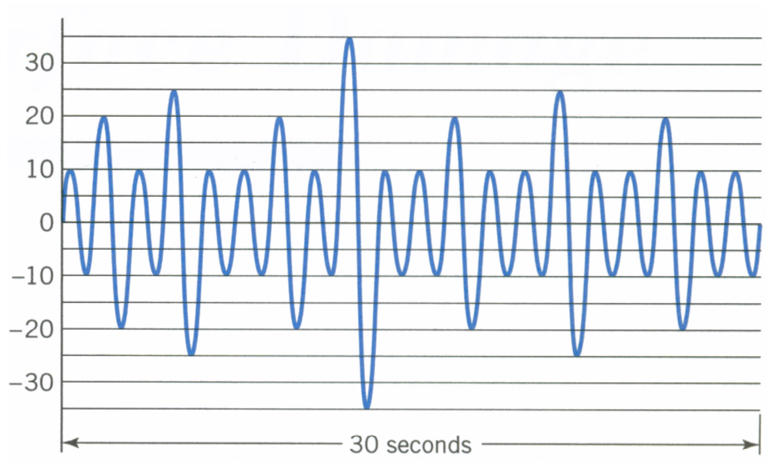
$$= \frac{1}{0.00134761} = 742.057 \text{ duty cycles}\left(\frac{30 \text{ seconds}}{1 \text{ duty}}\right)\left(\frac{1 \text{ minute}}{60 \text{ seconds}}\right)\left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right) = 6.18$$

Thus there are approximately 6 hours of life.

End Example □

Example 9.11.

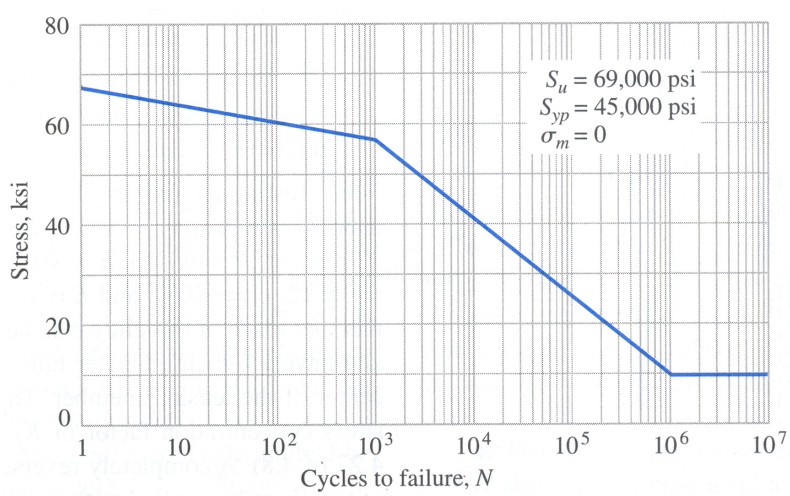
$n \times 10^{-1}$



During a repeating flight time recorded data (shown in the figure above), the load factor fluctuated steadily between n^+ and n^-. The maximum and minimum stress values are found in terms of the load factors and the 1.0 g stress by means of the relations:

$$\sigma_{\text{max}} = n^+ \, \sigma_{\text{o}} \qquad \sigma_{\text{min}} = n^- \, \sigma_{\text{o}}$$

where it σ_{o} is the von Mises stress and its calculated value is 5 ksi. The S–N diagram is given:



If the margin of safety is 10%, determine the hours of operation before failure. Useful equations:

$$B_f\left(\frac{n_1}{N_1}+\frac{n_2}{N_2}+\frac{n_3}{N_3}+\frac{n_4}{N_4}\right)=1 \qquad \sigma_a=\frac{\sigma_{max}-\sigma_{min}}{2} \qquad \sigma_m=\frac{\sigma_{max}+\sigma_{min}}{2}$$

$$S_{eq}=\frac{S_a}{1-\dfrac{S_m}{S_{yt}}} \qquad S_m=n_{SF}\,\sigma_m \qquad S_m=n_{SF}\,\sigma_a$$

Solution:

1. **The first load factor is read as:**

$$n^+=1 \qquad n^-=-1$$
$$\sigma_{max1}=n^+\,\sigma_o=5\text{ ksi} \qquad \sigma_{min1}=n^-\,\sigma_o=-5\text{ ksi}$$

Note that this is a case of fully reversed load, thus

$$\sigma_{a1}=5\text{ ksi} \quad \sigma_{m1}\quad=0$$

Using the safety factor ($n_{SF}=1.10$):

$$S_{a1}=n_{SF}\,\sigma_{a1}=5.5\text{ ksi} \quad S_{m1}\quad=n_{SF}\,\sigma_{m1}=0$$

Using the Goodman criteria

$$S_{eq1}=\frac{S_{a1}}{1-\dfrac{S_{m1}}{S_{yt}}}=S_{a1}=5.5\text{ ksi}<S_y$$

Since it is less than the yield strength, we use the S–N Diagram to determine the life cycles for this stress:

$$N_1=\infty$$

and from the gust load-time diagram:

$$n_1=13$$

2. **The second load factor is read as:**

$$n^+=2 \qquad n^-=-2$$
$$\sigma_{max2}=n^+\,\sigma_o=10\text{ ksi} \qquad \sigma_{min2}=n^-\,\sigma_o=-10\text{ ksi}$$

Note that this is a case of fully reversed load, thus

$$\sigma_{a2}=10\text{ ksi} \quad \sigma_{m2}\quad=0$$

Using the safety factor ($n_{SF}=1.10$):

$$S_{a2}=n_{SF}\,\sigma_{a2}=11.0\text{ ksi} \quad S_{m2}\quad=n_{SF}\,\sigma_{m2}=0$$

Using the Goodman criteria

$$S_{\text{eq2}} = \frac{S_{\text{a2}}}{1 - \dfrac{S_{\text{m2}}}{S_{\text{yt}}}} = S_{\text{a2}} = 11.0 \text{ ksi} < S_{\text{y}}$$

Since it is less than the yield strength, we use the S–N Diagram to determine the life cycles for this stress:

$$N_2 \approx 8 \times 10^5$$

and from the gust load-time diagram:

$$n_2 = 4$$

3. **The third load factor is read as:**

$$n^+ = 2.5 \qquad n^- = -2.5$$

$$\sigma_{\text{max3}} = n^+ \sigma_{\text{o}} = 12.5 \text{ ksi} \qquad \sigma_{\text{min3}} = n^- \sigma_{\text{o}} = -12.5 \text{ ksi}$$

Note that this is a case of fully reversed load, thus

$$\sigma_{\text{a3}} = 12.5 \text{ ksi} \quad \sigma_{\text{m3}} = 0$$

Using the safety factor ($n_{\text{SF}} = 1.10$):

$$S_{\text{a3}} = n_{\text{SF}}\, \sigma_{\text{a3}} = 13.75 \text{ ksi} \quad S_{\text{m3}} = n_{\text{SF}}\, \sigma_{\text{m3}} = 0$$

Using the Goodman criteria

$$S_{\text{eq3}} = \frac{S_{\text{a3}}}{1 - \dfrac{S_{\text{m3}}}{S_{\text{yt}}}} = S_{\text{a3}} = 13.75 \text{ ksi} < S_{\text{y}}$$

Since it is less than the yield strength, we use the S–N Diagram to determine the life cycles for this stress:

$$N_3 \approx 6 \times 10^5$$

and from the gust load-time diagram:

$$n_3 = 2$$

4. **The fourth load factor is read as:**

$$n^+ = 3.5 \qquad n^- = -3.5$$

$$\sigma_{\text{max4}} = n^+ \sigma_{\text{o}} = 17.5 \text{ ksi} \qquad \sigma_{\text{min4}} = n^- \sigma_{\text{o}} = -17.5 \text{ ksi}$$

Note that this is a case of fully reversed load, thus

$$\sigma_{\text{a4}} = 17.5 \text{ ksi} \quad \sigma_{\text{m4}} = 0$$

Using the safety factor ($n_{SF} = 1.10$):

$$S_{a4} = n_{SF}\,\sigma_{a4} = 19.25 \text{ ksi} \quad S_{m4} \quad = n_{SF}\,\sigma_{m4} = 0$$

Using the Goodman criteria

$$S_{eq4} = \frac{S_{a4}}{1 - \dfrac{S_{m4}}{S_{yt}}} = S_{a4} = 19.25 \text{ ksi} < S_y$$

Since it is less than the yield strength, we use the S–N Diagram to determine the life cycles for this stress:

$$N_4 \approx 2.5 \times 10^5$$

and from the gust load-time diagram:

$$n_4 = 1$$

5. **Now applying the Miner's rule:**

$$B_f \left(\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \frac{n_4}{N_4} \right) = 1$$

$$B_f \left(\frac{13}{\infty} + \frac{4}{8 \times 10^5} + \frac{2}{6 \times 10^5} + \frac{1}{2.5 \times 10^5} \right) = 1$$

$$B_f\,(0.0000123333) = 1$$

Thus

$$B_f = 81081.1 \text{ cycles} \left(\frac{30 \text{ seconds}}{1 \text{ cycle}} \right) \left(\frac{1 \text{ minute}}{60 \text{ seconds}} \right) \left(\frac{1 \text{ hour}}{60 \text{ minutes}} \right) = 675.676 \text{ h}$$

Thus, assuming the same pattern is repeated each time and no other external loads affect the system, there are approximately 675 hours of life remaining before failure.

End Example □

9.9 References

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Juvinall, R. C., and Marsheck, K. A., *Fundamentals of Machine Component Design*, 2000, John Wiley and Sons, New York, NY.

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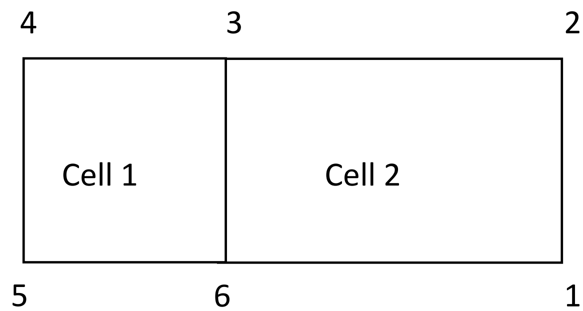
Thomas, G. B., Finney R. L., Weir, M. D., and Giordano F. R., *Thomas Calculus, Early Transcendental Update*, 2003, Tenth Edition, Addison-Wesley, Massachusetts. Entire book.

9.10 Suggested Problems

Problem 9.1.

Consider a two rectangular squared cell with different isotropic materials subject to a torque T.



The mechanical properties are:

$$E_{12} = E_{45} = E_{36} = 3.5 \times 10^6 \text{ psi} \qquad \nu_{12} = \nu_{45} = \nu_{36} = 0.30$$
$$E_{34} = E_{23} = 2.5 \times 10^6 \text{ psi} \qquad \nu_{34} = \nu_{23} = 0.25$$
$$E_{56} = E_{61} = 1.5 \times 10^6 \text{ psi} \qquad \nu_{56} = \nu_{61} = 0.20$$

The geometric properties are:

$$a_{12} = a_{45} = a_{36} = a \qquad a_{34} = a_{56} = a \qquad a_{23} = a_{61} = 2\,a$$
$$t_{12} = t_{45} = t_{36} = t \qquad t_{34} = t_{23} = 2\,t$$

where a's are the branch lengths and t's the branch thicknesses.

If the reference modulus E_0 is made of Titanium Ti-6Al-4V and $a = 10\,t$, determine the values of a and t for 30 years of life. Consider a 95% survivability. The wing box will be used during 10 hours a day and 5 days a week. During one year four week is used for maintenance; hence during that period no loads are applied. Take a margin of safety of 50%. A 30 second test shows that the wing box is subject to the following sequence of completely reversed torsional load amplitudes: five 35 kips-in torsional load amplitudes; four 25 kips-in torsional load amplitudes; one 20 kips-in torsional load amplitude; ten 10 kips-in torsional load amplitudes.

□

Appendix A
Math Review Using MATLAB

Instructional Objectives of Appendix A

After completing this appendix, the student should be able to:

1. Get familiar with MATLAB®: basic commands, functions and scripts, plotting.
2. Understand basic concepts in linear algebra: *hand* and MATLAB® solutions.
3. Solve linear system of equations: *hand* and MATLAB® solutions.
4. Obtain interpolation polynomials: *hand* and MATLAB® solutions.
5. Solve the eigenvalue problem: *hand* and MATLAB® solutions.
6. Approximate one-, two-, and three- dimensional integrals using numerical approximations: *hand* and MATLAB® solutions.

We often encounter a great deal of problems that cannot be solved without the help of various programs that work for high-performance numerical computation. MATLAB[1] is a high-performance language for technical computing that integrates computation, visualization, and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation. It has become very popular for its power and ease of use. Hence, all concepts discussed in this chapter are explained using MATLAB®.

The finite element analysis heavily uses arrays to obtain the solution. In recent years, MATLAB® is becoming a popular tool for "homemade" FEA for its versatility and robustness using matrices and vectors.

[1]Trademark of The MathWorks, Inc.

A.1 What is MATLAB®

The name **MATLAB®** stands for **Mat**rix **Lab**oratory, because the basic data element of this software is a matrix. MATLAB® has been developed assuming its users know basic languages in *C* and *FORTRAN*. Lately, university students have been introduced to programming with little or no knowledge of other programming languages. Thus, this tutorial has the purpose to help students learn the basics of MATLAB® programming and its capabilities.

MATLAB® provides an interactive environment with numerous built-in functions for technical computation, graphics, and animation. It also provides easy extensibility with its own programming language. MATLAB®'s built-in functions provide excellent tools for linear algebra computations, signal processing, data analysis, optimization, numerical solution of ordinary differential equations, and many other types of scientific computations. It also provides an external interface to run programs from other softwares, such as Fortran or C, from within MATLAB®.

The basic building block of MATLAB® is the matrix, and the fundamental data-type is the array. Vectors, scalars, real and complex matrices are all handled as special cases of the basic data-type. The dimensions of a matrix almost never have to be specified. The built-in functions are optimized for vector operations, therefore, vectorized commands or codes run much faster in MATLAB®.

MATLAB® contains hundreds of commands used in mathematics. You can use it to graph functions, solve equations, perform statistical tests, and do much more. You can produce sound and animate graphics. You can do simulations and modeling. You can prepare materials for export to the World Wide Web, and you can use it in conjunction with a word processing program to combine mathematical computations with text and graphics to produce a polished, integrated, and interactive document.

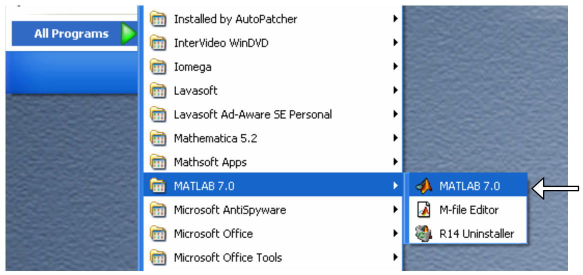
MATLAB® can be a very useful tool for a number of CAD/CAE problems. A Computer-Aided Engineering Design covers a variety of topics that require the knowledge of a programming language; MATLAB® being the one that suits it best because of its programming and graphical power. Topics like working with matrices, design optimization and functionals require the use of programming, making MATLAB® essential for CAD/CAE problems.

Through this appendix we will learn MATLAB® programming and function creation will be introduced and explained. Please remember that MATLAB® is so vast of a program, that it can hardly be covered in these few pages, in fact entire books are dedicated to specific applications in MATLAB®. The scope of this tutorial is to get a new user started with the basics, and hopefully this tutorial will make the beginners experience more enjoyable.

A.1.1 Getting Familiar with MATLAB®

No matter what software you use for the very first time, it is a challenge to learn and master it. Mastering a software only comes with experience in solving real engineering problems. Here, we enclose some *fast-learning* steps on how to start using MATLAB®.

In order to get started, find the MATLAB® icon in the All Programs Bar and click on it, as shown here[2]



Once it is opened, the main MATLAB® Window will show up, as shown in Fig. A.1. This window is composed of four *sub-windows*: the *command window*, *workspace*, *current directory* and *command history*. The ***command window*** is the input area, where you type your work (i.e., define variables, solve equations, call functions, etc.). The ***workspace window*** shows all defined variables, vectors and matrices. The ***current directory*** window allows you to see your M-Files (we will talk about these files shortly) while working in the command window so you can recall them if you need. The ***current directory*** and the ***workspace*** windows are found in the same window, and you can see them one at a time by using the switch tab. The ***command history*** encloses a record of all operations performed for future reference.

A.1.2 Basic commands and syntax

The first thing you will see in the command window is the MATLAB® prompt

```
>>
```

This indicates that the program is ready to receive instructions. Here, you write the desired arguments (i.e., variables, equations, functions, etc.). If you type in a valid expression and press Enter, MATLAB® will immediately execute it and return the result, just like a calculator.

```
>> 2+2

ans = 4

>> 4^2

ans = 16

>> sin(pi/2)

ans = 1

>> 1/0
```

[2]This is shown for Windows XP and MATLAB® 2007, other versions and platforms might have the icon differently located.

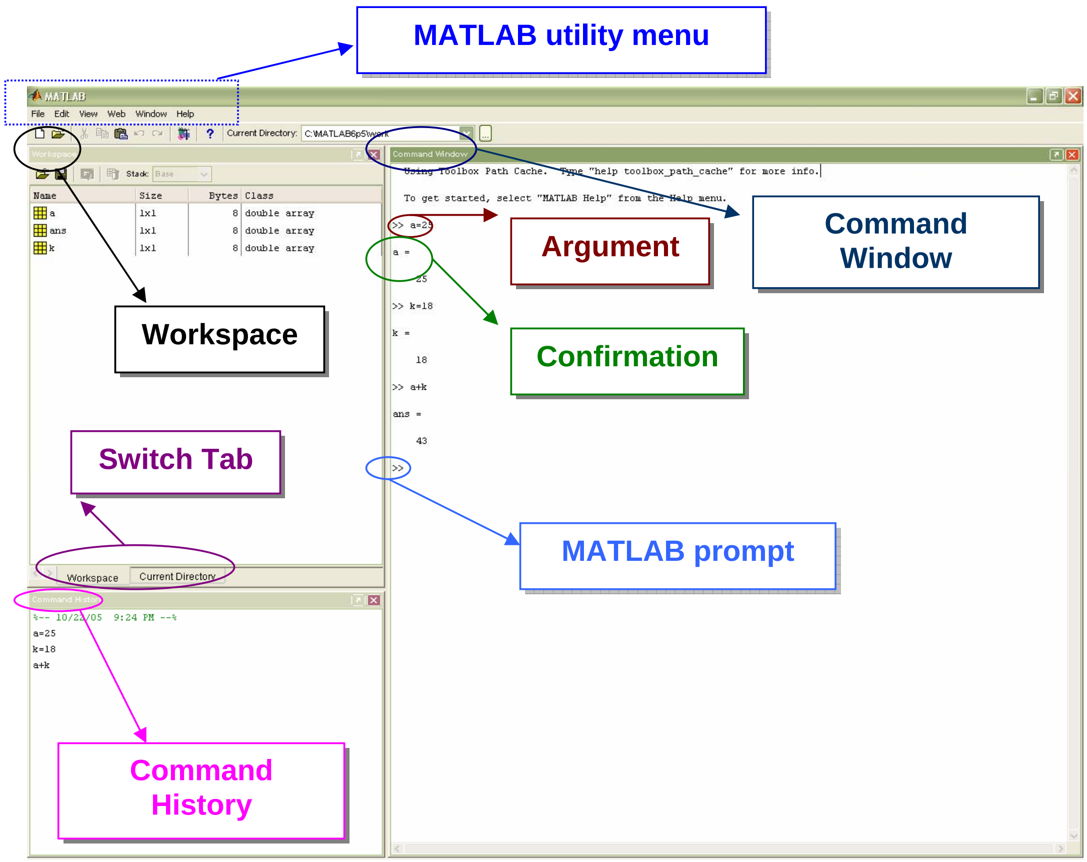


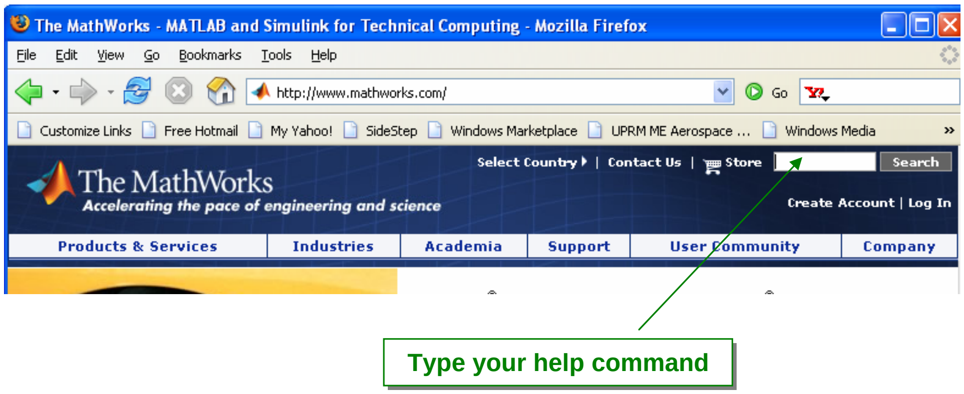
Figure A.1: Basic MATLAB® working environment.

```
Warning: Divide by zero. ans = Inf

>> exp(i*pi)

ans = -1.0000 + 0.0000i
```

Notice some of the special expressions here: `pi` for π, `Inf` for ∞, and `i` or `j` for $\sqrt{-1}$. Another special value is `NaN`, which stands for not-a-number. `NaN` is used to express an undefined value.

A.1.3 MATLAB® Help Command

MATLAB® is huge. You can't learn everything about it, or even always remember things you have done before. It is essential that you become familiar with the online help. MATLAB® has an excellent embedded help and an excellent online tutorial (http://www.mathworks.com/).



The help command written just alone provides a list of fields in which the software can help. The command is written as follows:

```
>> help
```

The main help list gives the main command categories in which all the commands can be grouped. You can either click in the link given after prompting for help or you can after knowing in what category you are interested write the command

```
>> help category
```

For example MATLAB®/general, gives the list of all the commands that are related to the general capabilities of the software. The general category help command gives help for commands like save, quit, and exit. You can either click the link MATLAB®/general or write the command

```
>> help general
```

The help command can provide help for every command possible in the MATLAB® language. Just write the command

```
>> help command
```

For an example, if you want to learn how to use the roots command you would type:

```
>> help zeros
```

the display will be

```
>> help zeros

ZEROS  Zeros array.
ZEROS(N) is an N-by-N matrix of zeros.

ZEROS(M,N) or ZEROS([M,N]) is an M-by-N matrix of zeros.

ZEROS(M,N,P,...) or ZEROS([M N P ...]) is an M-by-N-by-P-by-... array of
zeros.

ZEROS(SIZE(A)) is the same size as A and all zeros.

ZEROS with no arguments is the scalar 0.

ZEROS(M,N,...,CLASSNAME) or ZEROS([M,N,...],CLASSNAME) is an
```

```
M-by-N-by-... array of zeros of class CLASSNAME.

Example:
   x = zeros(2,3,'int8');

See also eye, ones.

Reference page in Help browser
   doc zeros
```

It can be seen here that several ways of using the command zero are displayed with the explanation of how to use all the ways using all the necessary variables. It also gives a simple example of the command used with real numerical values, and gives also a reference to click. In this way, we can access a more detailed information.

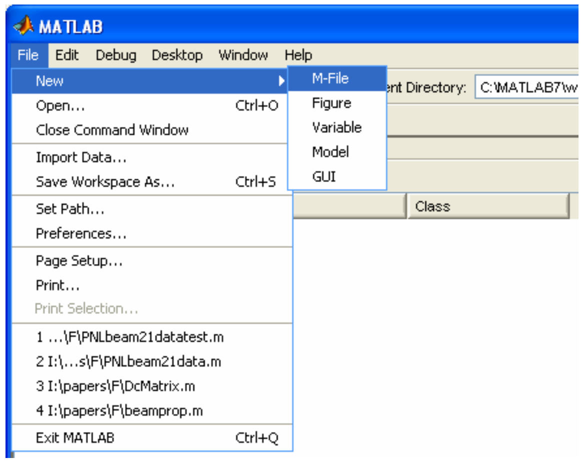
A.1.4 M-Files

More than a complete algebra system, MATLAB® can be used as a very powerful programming tool. The logic behind programming in MATLAB® is very similar to that employed when programming in C or C++, but in here we will limit to only explain what are M-File, which are the script files used when programming in MATLAB®, and how they are created in MATLAB®.

An M-File, or script file, is a simple text file where you can place MATLAB® commands. When you run the script file, MATLAB® reads the commands and executes them exactly as it would if you had typed each command sequentially at the MATLAB® prompt or command window. All M-File names must end with the extension "*.m". If you create a new m-file with the same name as an existing m-file, MATLAB® will choose the one which appears first in the path order. To make life easier, choose a name for your m-file which doesn't already exist. To see if a `filename.m` exists, type help *filename* at the MATLAB® command window.

For simple problems, entering your requests at the MATLAB® command window is fast and efficient. However, as the number of commands increases iterations are needed, typing the commands over and over at the MATLAB® command window becomes tedious, thus writing simple programs in the form of M-files will be helpful and almost necessary in these cases.

In order to open MATLAB®'s text editor to create an M-File we must follow the following procedure: From the MATLAB® utility menu choose the File menu → New → select M-File



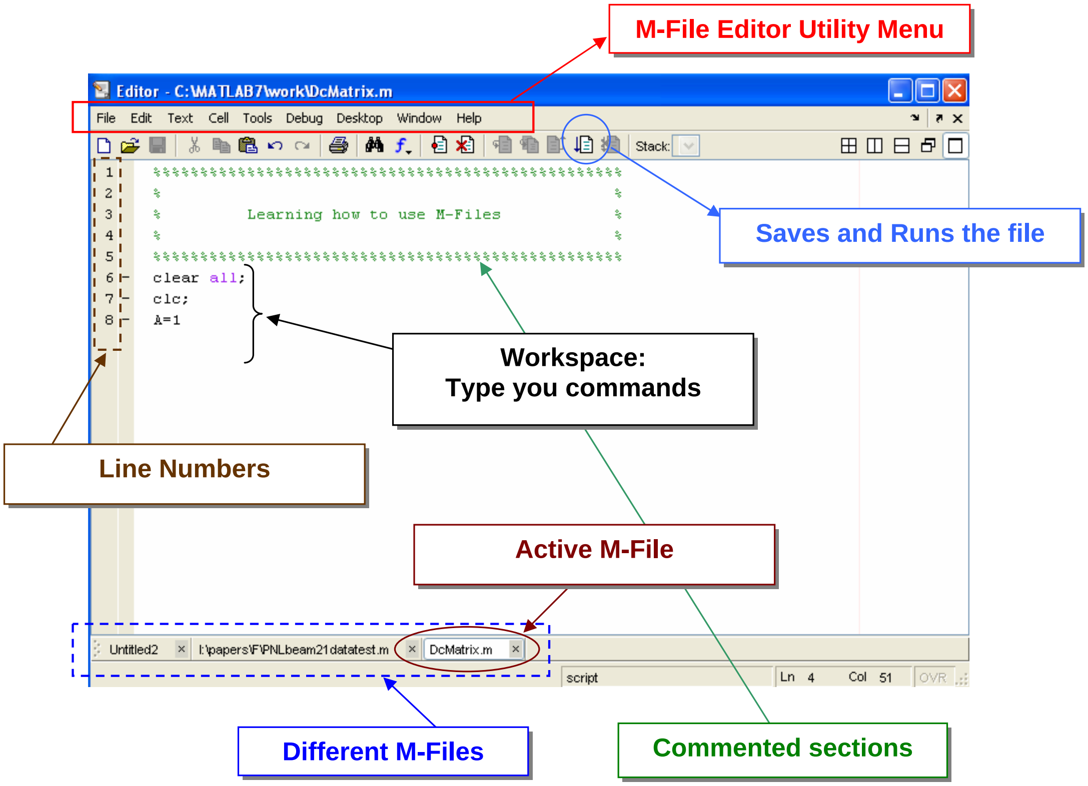
This procedure brings up a text editor window in which you can enter MATLAB® commands.

Now, in order to execute the M-File the user must make sure that he is working in the correct directory where the file is stored and then he can simply type the name of the M-File in the command window and press enter and MATLAB® will automatically execute the whole code. If there are errors within it, MATLAB® will display an error message telling the number of the line where the program thinks there is an error.

We run the M-Files directly from the prompt line: type the file's name and this will execute the commands included in the M-File. As for an example suppose we create an M-File with the name `filename.m` then the file is called as follows:

```
>> filename
```

and this will execute whatever the files tells MATLAB® to do. The following figure shows how the M-File editor works:



A.1.5 Programming in MATLAB®

Defining variables

A variable is a name made of a letter or several letters that are assigned to a numerical value that can be used in statements, functions, or any MATLAB® command. It is a name of a memory location. In MATLAB®, the = sign is the assignment operator. In other words,

`Variable = Numerical_Value or computable_expression.`

Notice that the left hand side of the assignment operator can include only one variable name. There are basically two types of variables: *Local* and *Global*.

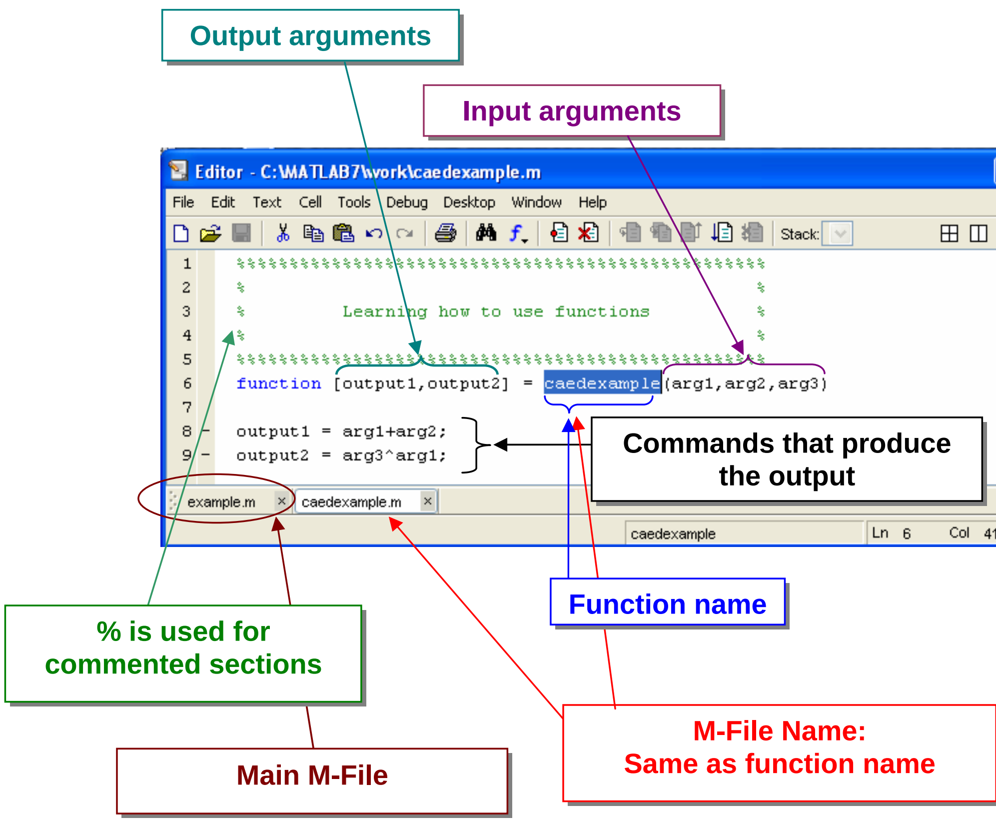
Local Variables are those that are called within a function or script, and are only valid for that application. If several functions, and possibly the base workspace, all declare a particular name as *global*, they all share a single copy of that variable. Any assignment to that variable, in any function, is available to all the functions declaring it global. Students are encouraged to further investigate the use of these definitions.

Scripts and Functions

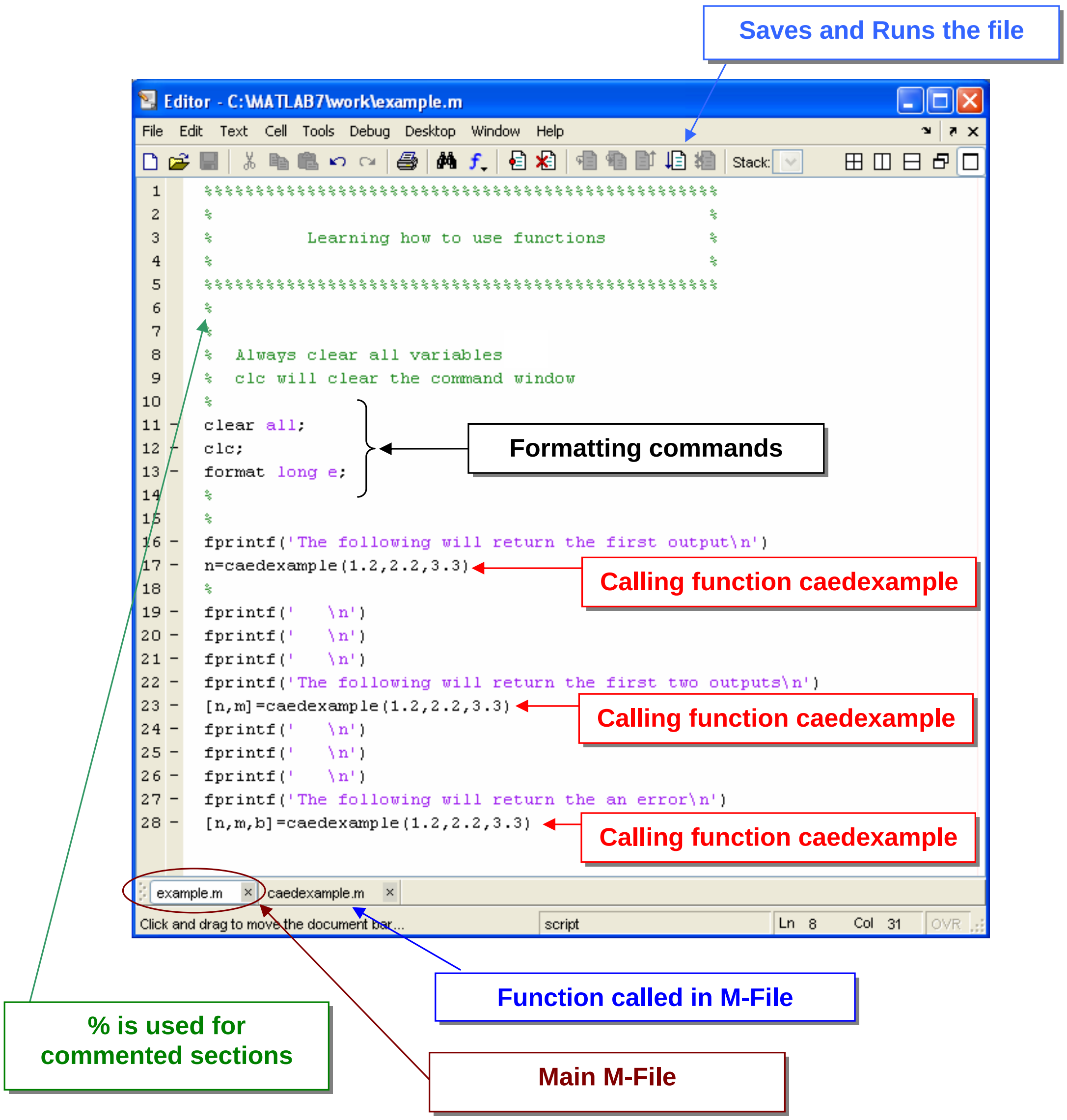
A function is capable of taking particular variables (called arguments) and doing something specific to "return" some particular type of result. A function needs to start with the following line:

```
Function [return-values] = function_name(arguments)
```

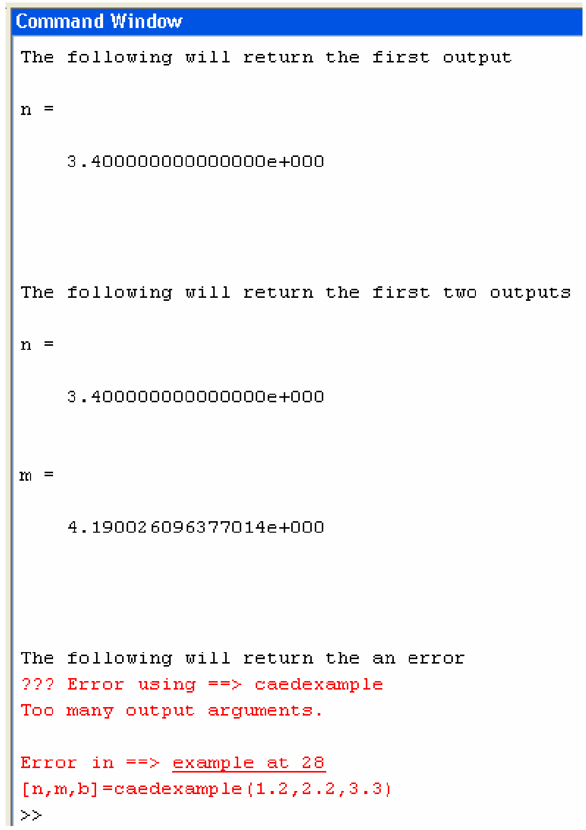
and must save this file as a M-File with the function's name `function_name.m`. MATLAB® will recognize this as a function. For an example:



A script is just a list of commands to be ran in some order. Placing these commands in a file that ends in `.m` allows you to "run" the script by typing its name at the command line of the command window. As an example let us create a script that calls the above function:



This produces the following output in the MATLAB® Command Window:



```
Command Window
The following will return the first output

n =

    3.400000000000000e+000



The following will return the first two outputs

n =

    3.400000000000000e+000


m =

    4.190026096377014e+000



The following will return the an error
??? Error using ==> caedexample
Too many output arguments.

Error in ==> example at 28
[n,m,b]=caedexample(1.2,2.2,3.3)
>>
```

As we can see, by calling the function and asking for one output argument it will give the first output-argument. The structure consists in providing the output from left to right as defined in the function and whenever the arguments requested as an output are exceeded an error is returned.

Sub Functions

A subfunction, visible only to the other functions in the same file, is created by defining a new function with the **function** keyword after the body of the preceding function or subfunction. For example, `avg` is a subfunction within the file `stat.m`:

```
function [mean,stdev] = stat(x)

n = length(x);
mean = avg(x,n);
stdev =sqrt(sum((x-avg(x,n)).^2)/n);

function mean = avg(x,n)
mean = sum(x)/n;
```

Subfunctions are not visible outside the file where they are defined. Functions normally return when the end of the function is reached. Use a return statement to force an early return.

A.1.6 Diary on and diary off

Often we are interested in printing the output to an external file rather then seeing it on the command window. MATLAB® has the command **diary** for this purpose. `DIARY` saves all the text of the MATLAB® session to an output file. It causes a copy of all subsequent command window input and most of the resulting command window output to be appended to the named file.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                               %
%        How diary on and diary off works       %
%                                               %
%      All output will be saved in the file:    %
%      ex3.txt                                  %
%                                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                        %
%  Initial Commands                      %
%                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
clc;
delete('ex3.txt')   % ensure that the file does not exist
% otherwise it keeps writing below
% the previous data
diary('ex3.txt')    % create the file
% can have any text extension
% including *.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                        %
%  Body of the script file               %
%                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
diary on            % begins to print all output to file
.
.
.
diary off           % stops printing to file
.
.                   % will not print to file after diary off
.
diary on            % begins to print to file again
.
.
.
```

```
diary off           % stops printing to file
.
.
.
return;
```

A.1.7 Graphical Display of Functions

Plots are graphical representation of the data or function(s). A graphical display usually better communicates the behavior described by the function(s) or data provided. This substitutes huge tables with the output and input data. MATLAB®'s ability to plot is quite versatile and powerful. By no means, it is intended that you will become an expert in plotting using MATLAB® through this section, but that you will be able to get started and with experience keeping expanding that knowledge.

2-D Plots

The most fundamental plotting command in MATLAB® is **plot**. Normally, it uses line segments to connect points given by two vectors of $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ coordinates. As an example:

```
>> t = pi*(0:0.02:2);

>> plot(t,sin(t))
```

The line may seem to be a smooth, continuous curve although it is connection of points. For example,

```
>> plot(t,sin(t),'o-')
```

In the above example a circle is drawn at each of the points being connected. Just as $\underline{\mathbf{t}}$ and $\sin(\underline{\mathbf{t}})$ are vectors, and not functions, curves in MATLAB® are joined line segments. We can change line colors as we plot, i.e., you now say

```
>> plot(t,cos(t),'r')
```

which outputs a red curve representing $\cos(\underline{\mathbf{t}})$.

Now, each time we run a MATLAB® script to plot, it will overwrite the previous plot. If we do not want to overwrite but plot two or more curves in MATLAB®, we use the command **hold on**, i.e.,

```
>> plot(t,sin(t),'b')
>> hold on
>> plot(t,cos(t),'r')
```

We could also do multiple curves at once as shown in the following example (use column vectors):

```
>> t = (0:0.01:1)';
>> plot(t,[t t.^2 t.^3])
```

We could give the plot a figure number, i.e.,

```
>> figure(1)
>> plot(t,sin(t),'b')
>> figure(2)
>> plot(t,cos(t),'r')
```

and each plot will be plotted in different figures opposed to overwriting the plots or having multiple plots in one figure.

Other useful 2D plotting commands can be found in MATLAB®'s help by typing:

```
>> help graph2d
```

3-D Plots

We often, graph 3D functions, for instance, plots of surfaces for functions such as $f(x, y)$. These surface plots also follow the connecting the points principle, although the details may be more complicated. The first step for a surface plot is to create a grid of points in the x–y plane. This set of grid points are points where f is evaluated to get the "dots" in 3D space. As an example:

```
>> x = pi*(0:0.02:1);
>> y = 2*x;
>> [X,Y] = meshgrid(x,y);
>> surf(X,Y,sin(X.^2+Y))
```

Once a 3D plot has been made, you can use the rotation button in the figure window to manipulate the 3D viewpoint. There are additional menus that give you much more control of the view, too. The *surf* plot begins by using *meshgrid* to make an x–y grid that is stored in the arrays $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$. To see the grid graphically, use

```
>> plot(X(:),Y(:),'k.')
```

With the grid so defined, the expression

```
>> sin(X.^2+Y)
```

is actually an array of values of $\underline{\mathbf{z}} = \sin(\underline{\mathbf{X}}^2 + \underline{\mathbf{Y}})$ on the grid. (This array could be assigned to another variable if you wish.) Finally, **surf** makes a solid-looking surface in which color and apparent height describe the given values of f. An alternative command mesh is similar to surf but makes a "wireframe" surface.

We can also have 3D multi-plots, as follows:

```
>> x = pi*(0:0.02:1);
>> y = 2*x;
>> [X,Y] = meshgrid(x,y);
>> contour(X,Y,sin(X.^2+Y))
>> hold on
>> t = (0:0.01:3)';
>> plot(t,[t t.^3])
```

Annotation

All graphs and figure should be always carefully labeled and explained to avoid any confusion or misinterpretation from the student. Hence, once the plots have been created, we proceed to label the axes and give a title (even add a legend, if necessary). For example,

```
>> t = 2*pi*(0:0.01:1);
>> plot(t,sin(t))
>> xlabel('time')
>> ylabel('amplitude')
>> title('Simple Harmonic Oscillator')
```

You can also add legends, text, arrows, or text/arrow combinations to help label different data. If we want to add Greek letters to the labels, titles, or legends we would do the following:

```
>> xlabel('\gamma')
>> ylabel('\beta')
>> title('f(\alpha)')
```

This will produce a x-label γ, y-label β and a title $f(\alpha)$.

Auto function plots

When plotting a mathematical expression, you must pick the evaluation points of the plot before calling **plot** or **surf**. This extra step can be skipped by using special alternative MATLAB® plotting commands, i.e.,

```
>> ezplot( @(x) exp(3*sin(x)-cos(2*x)), [0 4] )
```

```
>> ezsurf( '1/(1+x^2+2*y^2)', [-3 3], [-3 3] )
>> ezcontour( @(x,y) x.^2-y.^2, [-1 1], [-1 1] )
```

Color

The coloring of lines and text is easy to understand. Each object has a Color property that can be assigned an RGB (red, green, blue) vector whose entries are between zero and one. In addition many one-letter string abbreviations are understood.

Surfaces are different. To begin with, the edges and faces of a surface may have different color schemes, accessed by EdgeColor and FaceColor properties. You specify color data at all the points of your surface. In between the points the color is determined by shading. In flat shading, each face or mesh line has constant color determined by one boundary point. In interpolated shading, the color is determined by interpolation of the boundary values. While interpolated shading makes much smoother and prettier pictures, it can be very slow to render, particularly on printers. Finally, there is faceted shading which uses flat shading for the faces and black for the edges. You select the shading of a surface by calling shading after the surface is created.

Furthermore, there are two models for setting color data:

1. **Indirect:** Also called indexed. The colors are not assigned directly, but instead by indexes in a lookup table called a colormap. This is how things work by default.
2. **Direct:** Also called truecolor. You specify RGB values at each point of the data.

Direct color is more straightforward, but it produces bigger files and is most suitable for photos and similar images. Here's how indirect mapping works. Just as a surface has XData, YData, and ZData properties, with axes limits in each dimension, it also has a CData property and "color axis" limits. The color axis is mapped linearly to the colormap, which is a 64×3 list of RGB values stored in the figure. A point's CData value is located relative to the color axis limits in order to look up its color in the colormap. By changing the figure's colormap, you can change all the surface colors instantly. Consider these examples:

```
>> [X,Y,Z] = peaks; % some built-in data
>> surf(X,Y,Z), colorbar
>> caxis % current color axis limits
ans = -6.5466 8.0752
>> caxis([-8 8]), colorbar % a symmetric scheme
>> shading interp
>> colormap pink
>> colormap gray
>> colormap(flipud(gray)) % reverse order
```

By default, the CData of a surface is equal to its ZData. But you can make it different and thereby display more information. One use of this is for functions of a complex variable.

```
>> [T,R] = meshgrid(2*pi*(0:0.02:1),0:0.05:1);
>> [X,Y] = pol2cart(T,R);
>> Z = X + 1i*Y;
>> W = Z.^2;
>> surf(X,Y,abs(W),angle(W)/pi)
>> axis equal, colorbar
>> colormap hsv                                % ideal for this situation
```

Saving figures

In MATLAB® we have the versatility to make changes to an existing plot by avoiding the pain of having to re-run the MATLAB® script files. We can save a figure in a format that allows it to be recreated by typing the following

```
>> saveas(gcf,'goyal.fig')
```

If you save the current figure in a file `goyal.fig`. Later you can enter `openfig goyal` to recreate it.

There are three major issues that come up when you want to include some MATLAB® graphics in a document: file format, size and position, and color. The big difference in graphics file formats is between vector and bitmap graphics. Bitmaps, including `GIF`, `JPEG`, `PNG`, and `TIFF`, are fine for photographs. These formats fix the resolution of your image forever, whereas the resolution of your screen, your printer, and a journal's printer may be different. Vector formats are usually a much better choice and they include `EPS` and `WMF`. The choice here depends somewhat on your platform and word processor. For example, to export the current MATLAB® figure to file `goyal.eps`, use

```
>> saveas(gcf,'myfig.eps')
or
>> print -deps myfig
```

For color output use `-depsc` in the second position. EPS also works with `MS Word`, if you print on a postscript printer. In this case it's handy to use

```
>> print -deps -tiff myfig
```

in which case `Word` will be able to show a crude version of the graph in the document on screen.

To scale a figure before saving you need to enter

```
>> set(gcf,'paperpos',[0 0 3 2.25])
```

where the units are in inches. Unfortunately, sometimes the axes or other elements need to be repositioned. To make the display match the paper output, you should enter

```
>> set(gcf,'unit','inch','pos',[0 0 3 2.25])
```

The colored plots can be converted to grayscale. To do so, you should consider using colormap(gray) or colormap(flipud(gray)), whichever gives less total black, before exporting the figure. Finally, the edges of wireframe surfaces created by mesh are also converted to gray, often with poor results. Make them all the lines black by entering

```
>> set(findobj(gcf,'type','surface'),'edgecolor','k')
```

or by pointing and clicking.

A.1.8 Final Remarks on MATLAB®

When it comes to programming, the basic tools for MATLAB® are outlined in these past sections. Best programming practices for MATLAB®, as well as solutions to common problems can usually be found within the MATLAB® user groups throughout the internet. These groups are filled with experts and new-comers of MATLAB® who are most probably dealing with issues relevant to any question that you as a new user might come across. Herein lays one of the most crucial benefits this software package has to offer: its universality. It is so widely used in almost every science and research based industry, that the knowledge and support base it has to offer is unequaled. Remember, there is no "right" way to possibly teach a person how to program or develop an algorithm, since these tasks depend so much on the application and the user's taste and skills. These are skills a programmer acquires through practice, experience, and of course, consulting with others who have had previous experience with many of the same problems you might encounter.

MATLAB® is a very powerful programming and simulation tool which has grown throughout the years to better meet the demands of each industry it strives to serve. Hopefully these tools provided may serve as a stepping stone for further programming knowledge and skill development as you become acquainted with MATLAB®.

MATLAB® is a very practical, convenient tool to have when solving many computational problems we are faced with, not only as engineering students, but as engineers and researches as well.

A.2 Linear Algebra

Let us begin by defining some algebraic terms used throughout this book. A linear system of equations can often be expressed in terms of a matrix and vectors, which contains all the information about the system that is necessary to determine its solution, but in a compact form. Thus, let us review basic concepts regarding matrices and vectors.

A.2.1 Matrices

An $\mathbf{n} \times \mathbf{m}$ matrix is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array. The matrices will be denoted by a capital letter in a bold font along with a bar under the letter, e.g., $\underline{\mathbf{A}}$. For an example, an $n \times m$ rectangular matrix, $\underline{\mathbf{A}}$, is:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{bmatrix} \tag{A.1}$$

In MATLAB®, matrices are defined as follows:

```
>> A=[1 2 3; 2 3 4]

A =

 1      2      3
 2      3      4
```

Note that each rows is separated by a semicolon and column are separated by spaces. The square brackets represent the beginning of the matrix ("[") and the ending of the matrix ("]"). Columns may also be separated by commas:

```
>> A=[1, 2, 3; 2, 3, 4]

A =

 1      2      3
 2      3      4
```

Another alternative is to *directly* type the matrix:

```
>> A = [1 2 3        % press "return" after the number "3" is typed
2 3 4]

A =

 1     2     3
 2     3     4

>> A = [1, 2, 3        % press "return" after the number "3" is typed
2, 3, 4]       % with commas (no comma at the end of "3" because
   % a new row will begin)

A =

 1     2     3
 2     3     4
```

The above is a 2×3 matrix: a matrix with two rows and three columns. In MATLAB®, we can determine the size of a matrix as follows:

```
>> [n,m]=size(A)

n =

 2


m =

 3
```

Each entry of the matrix is called *element of matrix*. The lowercase letters with double subscripts, such as a_{ij} are used to refer to the entry at the intersection of the i^{th} row and j^{th} column. In the double-subscript notation for each element, the first script always denotes the row and the second the column in which the given entry stands. In MATLAB®, the element in the second row and third column is extracted as follows:

```
>> A(2,3)

ans =

 4
```

The matrices in MATLAB® can also be defined element by element:

```
>> A(1,1) = 1;
A(1,2) = 2;
```

```
A(1,3) = 3;
A(2,1) = 2;
A(2,2) = 3;
A(2,3) = 4;
A

A =

 1     2     3
 2     3     4
```

In extracting values from a matrix, the use of subscripts outside the current matrix dimensions results in error:

```
>> A(1,5)
??? Index exceeds matrix dimensions.
```

Squared Matrix

A square matrix has the same number of rows as columns, e.g., $n = m$. For an example, an $n \times n$ square matrix, $\underline{\mathbf{A}}$, is:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A.2}$$

An $n \times n$ matrix is known as a squared matrix of order n. In MATLAB®, a squared matrix is represented as follows:

```
>> B=[1 2 3; 2 3 4; -2 3 0]

B =

 1     2     3
 2     3     4
-2     3     0

>> size(B)

ans =

 3     3
```

As we can see the above matrix has the same number of rows ($n = 3$) and columns ($m = 3$).

Symmetric Matrix

A symmetric matrix is a squared matrix where $a_{ij} = a_{ji}$. For an example, an $n \times n$ symmetric matrix, $\underline{\mathbf{A}}$, is:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \tag{A.3}$$

If we have only the upper half of the matrix in MATLAB®,

```
>> A=[1 2 3; 0 7 4; 0 0 -10]

A =

 1     2     3
 0     7     4
 0     0   -10
```

(note that the matrix was initialized to zero), then the symmetric matrix can be found as follows:

```
>> A=A'+A-diag(diag(A))

A =

 1     2     3
 2     7     4
 3     4   -10
```

Diagonal Matrix

A diagonal matrix is a square matrix that has nonzero entries along the principal diagonal and any entry above or below the principal diagonal must be zero. For an example, an $n \times n$ diagonal matrix, $\underline{\mathbf{D}}$, is:

$$\underline{\mathbf{D}} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix} \tag{A.4}$$

A diagonal matrix in MATLAB® can be easily defined as follows:

```
>> d=[1 2 3 4]

d =
```

```
 1     2     3     4

>> diag(d)

ans =

 1     0     0     0
 0     2     0     0
 0     0     3     0
 0     0     0     4
```

We can also extract the diagonal of a matrix as follows:

```
>> A=[1 2 3; 2 7 4; 3 4 -10]

A =

 1     2     3
 2     7     4
 3     4   -10

>> diag(A)

ans =

1
7
-10

>> diag(diag(A))

ans =

 1     0     0
 0     7     0
 0     0   -10
```

Identity Matrix

A special case of diagonal matrices is the identity matrix. The identity matrix of order n is defined by a capital bold letter I along with a bar under the letter. For an example,

$$\underline{\mathbf{I}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{A.5}$$

In MATLAB®, the identity matrix can be defined in two different ways:

```
>> I=eye(3)

I =

 1     0     0
 0     1     0
 0     0     1

>> I=eye(3,3)

I =

 1     0     0
 0     1     0
 0     0     1
```

Suppose we have defined the matrix $\underline{\mathbf{A}}$

```
>> A=[1 2 3 5; 3 4 4 6; 0 1 -10 9; 10 -2 0 73]

A =

 1     2     3     5
 3     4     4     6
 0     1   -10     9
10    -2     0    73
```

In order to generate an identity matrix of the same size in MATLAB®, we do the following:

```
>> eye(size(A))

ans =

 1     0     0     0
 0     1     0     0
 0     0     1     0
 0     0     0     1
```

Zero Matrix

A special type of matrices is the zero matrix. The zero matrix of order n is defined by a capital bold letter 0 along with a bar under the letter. For an example,

$$\underline{\mathbf{0}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tag{A.6}$$

In MATLAB®, we can define the zero matrix as follows:

```
>> T=zeros(4)

T =

 0     0     0     0
 0     0     0     0
 0     0     0     0
 0     0     0     0

>> T=zeros(4,4)

T =

 0     0     0     0
 0     0     0     0
 0     0     0     0
 0     0     0     0

>> T=zeros(4,7)

T =

 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0

>> T=zeros(4,7,2)

T(:,:,1) =

 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
```

```
T(:,:,2) =

 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0
 0     0     0     0     0     0     0

>> T=zeros(4,3,2,2)

T(:,:,1,1) =

 0     0     0
 0     0     0
 0     0     0
 0     0     0


T(:,:,2,1) =

 0     0     0
 0     0     0
 0     0     0
 0     0     0


T(:,:,1,2) =

 0     0     0
 0     0     0
 0     0     0
 0     0     0


T(:,:,2,2) =

 0     0     0
 0     0     0
 0     0     0
 0     0     0
```

As we can see, we can have zero matrices in two-dimensions, three-dimensions, and so-and-so forth. As in any computer programming, all matrices must be initialized to zero before we start to use them within the program.

A.2.2 Vectors

Column Vectors

An n-dimensional column vector is an $n \times 1$ matrix and it is defined with a lowercase bold letter along with a bar under the letter. For an example, for an n-dimensional column vector, $\underline{\mathbf{x}}$, is:

$$\underline{\mathbf{x}} = \left\{ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{matrix} \right\} \tag{A.7}$$

In MATLAB®, we define each element separated by a semicolon:

```
>> x=[1;2;3;4;5]

x =

 1
 2
 3
 4
 5
```

or

```
>> x = [1         % press "return" after the number "1" is typed
2         % press "return" after the number "2" is typed
3         % press "return" after the number "3" is typed
4         % press "return" after the number "4" is typed
5]

x =

 1
 2
 3
 4
 5
```

or we can define it element by element:

```
>> x(1,1) = 1;
```

```
x(2,1) = 2;
x(3,1) = 3;
x(4,1) = 4;
x(5,1)=5;
x

x =

 1
 2
 3
 4
 5
```

When defining a vector element by element in MATLAB®, the default is a column vector.

Row Vectors

An n-dimensional row vector is a $1 \times n$ matrix and it is defined with a lowercase bold letter along with a bar under the letter. For an example, for an n-dimensional row vector, $\underline{\mathbf{y}}$, is:

$$\underline{\mathbf{y}} = \left\{ \begin{array}{ccccccc} y_1 & y_2 & \cdots & y_j & \cdots & y_n \end{array} \right\} \tag{A.8}$$

In MATLAB®, we define each element by separating them by a comma or a space:

```
>> y=[1 2 3 4 5]

y =

 1     2     3     4     5


>> y=[1, 2, 3, 4, 5]

y =

 1     2     3     4     5
```

or we can define it element by element:

```
>> y(1,1) = 1;
y(1,2) = 2;
y(1,3) = 3;
y(1,4) = 4;
y(1,5) = 5;
y
```

```
y =

 1     2     3     4     5
```

Note that we have to specify the two dimensions (row number as one and the column numbers) in order to obtain a row vector.

Zero Column Vectors

An n-dimensional zero column vector is a vector where all elements have a zero value. For an example, for an n-dimensional column vector, $\underline{\mathbf{0}}$, is:

$$\underline{\mathbf{0}} = \left\{ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\} \tag{A.9}$$

A zero column vector in MATLAB® is defined as follows:

```
>> w=zeros(3,1)

w =

 0
 0
 0
```

Zero Row Vectors

An n-dimensional zero row vector is a vector where all elements have a zero value. For an example, for an n-dimensional row vector, $\underline{\mathbf{0}}$, is:

$$\underline{\mathbf{0}} = \left\{ \begin{array}{cccc} 0 & 0 & \cdots & 0 \end{array} \right\} \tag{A.10}$$

A zero row vector in MATLAB® is defined as follows:

```
>> q=zeros(1,3)

q =

 0     0     0
```

Equally spaced vector

In MATLAB®, there is no need to use a for loop to create a vector with $\underline{\mathbf{q}}$ equally spaced elements between s and t. We can easily create a row vector using the command

```
linspace(startValue,endValue,numberofElements)=linspace(s,t,q)
```

Example A.1.

Suppose we want to create a vector with 7 equally-spaced elements between 0 and 20:

```
>> x=linspace(0,20,7)

x =

 0    3.3333    6.6667   10.0000   13.3333   16.6667   20.0000
```

Suppose we want to create a vector with 7 equally-spaced elements between -0.3 and 0.2:

```
>> x=linspace(-0.3,0.2,7)

x =

   -0.3000   -0.2167   -0.1333   -0.0500    0.0333    0.1167    0.2000
```

Note that only row vectors are created. To obtain column vectors, take the transpose:

```
>> x=linspace(-0.5,0.5,11)'

x =

   -0.5000
   -0.4000
   -0.3000
   -0.2000
   -0.1000
 0
0.1000
0.2000
0.3000
0.4000
0.5000
```

End Example □

Also, a simple vector with numbers can be created using

```
>> x=-1:5

x =

-1     0     1     2     3     4     5

>> x=1:5

x =

 1     2     3     4     5
```

MATLAB® has a very powerful and highly compact syntaxis referred to as colon notation. This notation can be used either to create vectors or, combined with subscript notation, to extract ranges of matrix elements. We can use two forms of colon notation:

```
vector = startValue:endValue

vector = startValue:increment:endValue
```

Example A.2.

For an example,

```
>> y=3:5

y =

 3     4     5


>> y=0:0.1:0.5

y =

 0    0.1000    0.2000    0.3000    0.4000    0.5000
```

End Example □

A.2.3 Matrix and Vector Operations

Extracting Rows or Columns from a Matrix

A colon can be used as a wild card to refer to an entire row or column of a matrix. For an example,

```
>> A=[1 2 3 4
5 6 7 8
6 1 0 3]

A =

 1     2     3     4
 5     6     7     8
 6     1     0     3

>> A(:,1)

ans =

 1
 5
 6

>> A(1,:)

ans =

 1     2     3     4
```

Transpose of a Matrix

The transpose of a matrix is obtained by *flipping* the rows and columns. For an example, the transpose of the matrix $\underline{\mathbf{A}}$ would be:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \underline{\mathbf{A}}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix} \tag{A.11}$$

where $\underline{\mathbf{A}}^{\mathrm{T}}$ is called the transpose of $\underline{\mathbf{A}}$. Likewise we can take the transpose of a vector:

$$\underline{\mathbf{c}} = \left\{ \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \right\} \qquad \underline{\mathbf{c}}^{\mathrm{T}} = \left\{ \begin{array}{cccc} c_1 & c_2 & \dots & c_n \end{array} \right\} \tag{A.12}$$

In MATLAB® the transpose operator is the single quote appended to a matrix or vector.

Example A.3.

Determine the transpose of the following vectors and matrices:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad \underline{\mathbf{y}} = \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \end{Bmatrix} \quad \underline{\mathbf{x}} = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix}$$

```
>> A = [1 2 3
2 3 4]

A =

 1     2     3
 2     3     4

>> A'

ans =

 1     2
 2     3
 3     4

>> y=[1 2 3 4 5]

y =

 1     2     3     4     5

>> y'

ans =

 1
 2
 3
 4
 5

>> x=[1;2;3;4;5]

x =
```

```
 1
 2
 3
 4
 5

>> x'

ans =

 1     2     3     4     5
```

We should be very careful in using transposes in MATLAB®, because a single quote is a small character that may be easy to miss. As a result, errors caused by extraneous or missing transpose operators can be literally hard to see.

End Example □

As with the *linspace* function, colon expressions create row vectors by default. To create a column vector instead, enclose the expression in parentheses and append the transpose operator:

```
>> y=(0:0.1:0.5)'

y =

 0
0.1000
0.2000
0.3000
0.4000
0.5000
```

The parentheses are necessary because the transpose operator has higher precedence than the colon:

```
>> y=0:0.1:0.5'

y =

 0    0.1000    0.2000    0.3000    0.4000    0.5000
```

Tabular Column Vectors

The MATLAB® statement that follows create a matrix using column vectors

```
>> x=[1 2 3 4 5]

x =

 1     2     3     4     5

>> y=[1;2;3;4;5]

y =

 1
 2
 3
 4
 5

>> D=[x' y]

D =

 1     1
 2     2
 3     3
 4     4
 5     5
```

Trigonometric Function Values

Apply the trigonometric functions to vectors, creates vectors containing the values of trigonometric functions.

```
>> x=linspace(0,2*pi,6)

x =

 0    1.2566    2.5133    3.7699    5.0265    6.2832

>> s=sin(x)

s =

 0    0.9511    0.5878   -0.5878   -0.9511   -0.0000

>> c=cos(x)

c =

1.0000    0.3090   -0.8090   -0.8090    0.3090    1.0000
```

```
>> t=tan(x)

t =

 0    3.0777   -0.7265    0.7265   -3.0777   -0.0000

>> ee=exp(x)

ee =

1.0000    3.5136   12.3453   43.3762  152.4060  535.4917
```

If the vector $\underline{\mathbf{x}}$ is a column vector then the result will be a column vector; if the vector $\underline{\mathbf{x}}$ is a row vector then the result will be a row vector.

Matrix Addition or substraction

Two matrices can be added together or subtracted from each other provided that they are of the same size (each matrix, has the same number of rows and columns). For an example, we can add or subtract the matrix $\underline{\mathbf{A}}$ of dimension $m \times n$ to matrix $\underline{\mathbf{B}}$ of dimension $m \times n$ by adding or subtracting the like elements:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \underline{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$\begin{aligned} \underline{\mathbf{A}} \pm \underline{\mathbf{B}} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} (a_{11} \pm b_{11}) & (a_{12} \pm b_{12}) & \cdots & (a_{1n} \pm b_{1n}) \\ (a_{21} \pm b_{21}) & (a_{22} \pm b_{22}) & \cdots & (a_{2n} \pm b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1} \pm b_{m1}) & (a_{m2} \pm b_{m2}) & \cdots & (a_{mn} \pm b_{mn}) \end{bmatrix} \qquad \text{(A.13)} \\ &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \qquad \text{(A.14)} \end{aligned}$$

In compact form,

$$c_{ij} = a_{ij} \pm b_{ij} \quad \text{for} \quad i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n$$

Example A.4.

Consider the following rectangular matrices:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \end{bmatrix} \quad \underline{\mathbf{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Determine

a) Size of $\underline{\mathbf{A}}$
b) Size of $\underline{\mathbf{B}}$
c) $\underline{\mathbf{A}} + \underline{\mathbf{B}}$
d) $\underline{\mathbf{A}} + \underline{\mathbf{B}}^{\mathrm{T}}$
e) $\underline{\mathbf{A}}^{\mathrm{T}} + \underline{\mathbf{B}}$

```
>> A=[1 2 3 4; 5 6 7 8; 4 5 6 7]

A =

 1     2     3     4
 5     6     7     8
 4     5     6     7

>> [nA,mA]=size(A)

nA =

 3


mA =

 4

>> B=[1 2 3; 4 5 6; 7 8 4; 5 6 7]

B =

 1     2     3
 4     5     6
 7     8     4
 5     6     7

>> [nB,mB]=size(B)
```

```
nB =

 4


mB =

 3

>> A+B          % note [3x4]+[4x3] is not the same size
??? Error using ==> plus Matrix dimensions must agree.

>> A'+B         % note [4x3]+[4x3] is the same size

ans =

 2     7     7
 6    11    11
10    15    10
 9    14    14

>> A+B'         % note [3x4]+[3x4] is the same size

ans =

 2     6    10     9
 7    11    15    14
 7    11    10    14
```

End Example □

Inner (dot) Product of Vectors

The inner product or dot product is obtained by the multiplication of a row vector, say $\underline{\mathbf{a}}$, by the column vector, say $\underline{\mathbf{b}}$. The length of the vectors must be the same (the number of rows of $\underline{\mathbf{a}}$ must equal the number of columns of $\underline{\mathbf{b}}$):

$$\underline{\mathbf{a}} = \underbrace{\left\{ \begin{matrix} a_1 & a_2 & \dots & a_n \end{matrix} \right\}}_{1\times n} \qquad \underline{\mathbf{b}} = \underbrace{\left\{ \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \right\}}_{n\times 1} \tag{A.15}$$

Thus, the inner product is then defined as:

$$\begin{aligned}\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} &= \underbrace{\left\{ \begin{matrix} a_1 & a_2 & \ldots & a_n \end{matrix} \right\}}_{1\times n} \cdot \underbrace{\left\{ \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \right\}}_{n\times 1} \\ &= \sum_{i=1}^{n} a_i\, b_i = a_1\, b_1 + a_2\, b_2 + \ldots + a_n\, b_n\end{aligned} \tag{A.16}$$

Note it produces a scalar. If the number of rows of the first vector is not the same as the number of columns of the second vector then the dot product cannot be performed.

One should not confuse the dot product with:

$$\underline{\mathbf{b}} \cdot \underline{\mathbf{a}} = \underbrace{\left\{ \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} \right\}}_{n\times 1} \cdot \underbrace{\left\{ \begin{matrix} a_1 & a_2 & \ldots & a_n \end{matrix} \right\}}_{1\times n} = \text{produces a } n \times n \text{ matrix} \tag{A.17}$$

A dot product will produce a scalar and not a matrix.

Example A.5.

Let $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ be both row vectors of size 1×4 and defined as:

$$\underline{\mathbf{x}} = \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \right\} \quad \underline{\mathbf{y}} = \left\{ \begin{array}{cccc} -1 & -2 & -3 & -4 \end{array} \right\}$$

Determine the inner (dot) product.

```
>> x=[1 2 3 4]

x =

 1     2     3     4

>> y= [-1 -2 -3 -4]

y =

-1    -2    -3    -4
```

Note that the inner dimensions must be the same:

```
>> x*y              % [1x4] x [1x4]
??? Error using ==> mtimes Inner matrix dimensions must agree.
```

The following produces a dot product:

```
>> dot(x,y)

ans =

   -30
```

End Example □

Scalar-Matrix Multiplication

When a matrix $\underline{\mathbf{A}}$ of size $m \times n$ is multiplied by a scalar quantity such as β, the operation results in a matrix of the same size $m \times n$, whose elements are the product of elements in the original matrix and the scalar quantity. For an example, consider a matrix $\underline{\mathbf{A}}$ of size $m \times n$. Then, the product $\underline{\mathbf{C}} = \beta\,\underline{\mathbf{A}}$ will produce an $m \times n$ matrix $\underline{\mathbf{C}}$:

$$\underline{\mathbf{C}} = \beta\,\underline{\mathbf{A}} = \beta \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} = \underbrace{\begin{bmatrix} \beta\,a_{11} & \beta\,a_{12} & \cdots & \beta\,a_{1n} \\ \beta\,a_{21} & \beta\,a_{22} & \cdots & \beta\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta\,a_{m1} & \beta\,a_{m2} & \cdots & \beta\,a_{mn} \end{bmatrix}}_{m\times n}$$
$$= \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}}_{m\times n} \tag{A.18}$$

Example A.6.

Consider the following 3×4 matrix $\underline{\mathbf{A}}$:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 3 & -4 & 5 \\ 10 & 3 & 5 & -20 \end{bmatrix}$$

Determine:

a) $\beta\,\underline{\mathbf{A}}$ where $\beta = -1$
b) $\beta\,\underline{\mathbf{A}}$ where $\beta = 0$
c) $\beta\,\underline{\mathbf{A}}$ where $\beta = 10$

```
>> A=[1 -2 0 1
  2 3 -4 5
  10 3 5 -20]

A =

 1     -2      0      1
 2      3     -4      5
10      3      5    -20
```

Let us take $\beta = -1$:

```
>> (-1)*A

ans =

-1     2     0    -1
-2    -3     4    -5
   -10    -3    -5    20

>> -A                  % same as multiplying a minus one to A

ans =

-1     2     0    -1
-2    -3     4    -5
   -10    -3    -5    20
```

Let us take $\beta = 0$:

```
>> 0*A

ans =

 0     0     0     0
 0     0     0     0
 0     0     0     0
```

It produces the zero matrix instead of a scalar zero! Finally, let us take $\beta = 10$:

```
>> 10*A

ans =

10   -20     0    10
20    30   -40    50
   100    30    50  -200
```

As we can see, each matrix element is multiplied by the scalar.

End Example □

Matrix-Matrix Multiplication

Whereas any size matrix can be multiplied by a scalar quantity, matrix multiplication cannot be performed for any size of matrices. Consider the following two matrices

$$\underline{\mathbf{A}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} \qquad \underline{\mathbf{B}} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{bmatrix}}_{n\times q} \qquad (\text{Assume } m \neq q)$$

The product of the two matrices can be performed if and only if the number of columns of the premultiplier matrix is equal to the number of rows of the postmultiplier matrix. For an example, consider the above matrices. Then, the product $\underline{\mathbf{C}} = \underline{\mathbf{A}}\,\underline{\mathbf{B}}$ (in this order) of an $m \times n$ matrix $\underline{\mathbf{A}}$ and an $n \times q$ matrix $\underline{\mathbf{B}}$ will produce an $m \times q$ matrix $\underline{\mathbf{C}}$:

$$\begin{aligned} \underline{\mathbf{A}}\,\underline{\mathbf{B}} &= \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{bmatrix}}_{n\times q} \\ &= \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mq} \end{bmatrix}}_{m\times q} \end{aligned} \tag{A.19}$$

In order to obtain the elements of matrix $\underline{\mathbf{C}}$, recall that the first subscript in c_{mq} represents the row and the second one represents the column. Then,

$$\begin{aligned} c_{mq} &= \left(m^{\text{th}} \text{ row of } \underline{\mathbf{A}}\right) \cdot \left(q^{\text{th}} \text{ column of } \underline{\mathbf{B}}\right) \\ &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{1q} \\ b_{2q} \\ \vdots \\ b_{nq} \end{bmatrix} = \sum_{k=1}^{n} a_{mk}\, b_{kq} \end{aligned} \tag{A.20}$$

where the "·" represents the inner or dot product.

However,

$$\underline{\mathbf{B}}\,\underline{\mathbf{A}} = \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{bmatrix}}_{n\times q} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} \tag{A.21}$$

$$= \text{can not be performed when } q \neq m$$

Example A.7.

Consider the following matrices

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \end{bmatrix} \quad \underline{\mathbf{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

Determine:

a) $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$

b) $\underline{\mathbf{B}}\,\underline{\mathbf{A}}$

c) $\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{B}}$

d) $\underline{\mathbf{A}}\,\underline{\mathbf{B}}^{\mathrm{T}}$

```
>> A=[1 2 3 4; 5 6 7 8; 4 5 6 7]

A =

 1      2      3      4
 5      6      7      8
 4      5      6      7

>> B=[1 2 3; 4 5 6; 7 8 4; 5 6 7]

B =

 1      2      3
 4      5      6
 7      8      4
 5      6      7

>> A*B                  % [3x4] x [4x3] = [3x3]

ans =
```

```
50    60    55
118   144   135
101   123   115

>> B*A               % [4x3] x [3x4] = [4x4]

ans =

23    29    35    41
53    68    83    98
63    82   101   120
63    81    99   117

>> A'*B               % [4x3] x [4x3]:  m not equal to n
??? Error using ==> mtimes Inner matrix dimensions must agree.

>> A*B'              % [3x4] x [3x4]:  m not equal to n
??? Error using ==> mtimes Inner matrix dimensions must agree.
```

End Example □

Example A.8.

Consider the following vectors

$$\underline{\mathbf{x}} = \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \right\} \quad \underline{\mathbf{y}} = \left\{ \begin{array}{cccc} -1 & -2 & -3 & -4 \end{array} \right\}$$

Determine:

a) $\underline{\mathbf{y}}\,\underline{\mathbf{x}}^{\mathrm{T}}$
b) $\underline{\mathbf{x}}\,\underline{\mathbf{y}}^{\mathrm{T}}$
c) $\underline{\mathbf{y}}^{\mathrm{T}}\,\underline{\mathbf{x}}$
d) $\underline{\mathbf{x}}^{\mathrm{T}}\,\underline{\mathbf{y}}$

Note we can obtain the dot product by using matrix multiplication rules (note the transpose on $\underline{\mathbf{x}}$)

```
>> x=[1 2 3 4]
```

```
x =

 1     2     3     4

>> y=[-1 -2 -3 -4]

y =

-1    -2    -3    -4

>> y*x'               % [1x4] x [4x1] = [1x1] =scalar
  % (note the transpose on x)

ans =

   -30

>> x*y'               % [1x4] x [4x1] = [1x1] =scalar
  % (note the transpose on y)

ans =

   -30
```

The following is not a dot product but a matrix

```
>> x'*y               % [4x1] x [1x4] = [4x4]

ans =

-1    -2    -3    -4
-2    -4    -6    -8
-3    -6    -9   -12
-4    -8   -12   -16

>> y'*x               % [4x1] x [1x4] = [4x4]

ans =

-1    -2    -3    -4
-2    -4    -6    -8
-3    -6    -9   -12
-4    -8   -12   -16
```

End Example □

Element-by-Element Multiplication

The array operator symbols are a combination of a period and one of the conventional operators. Element-by-element multiplication is obtained with the " .* " operator, element-by-element division is obtained with the " . / " operator:

```
>> x=[1     2     3     4]

x =

 1     2     3     4

>> y=[9; 10; 11; 12]

y =

 9
10
11
12

>> x.*y
??? Error using ==> times Matrix dimensions must agree.

>> x.*y'

ans =

 9    20    33    48

>> x./y'

ans =

0.1111    0.2000    0.2727    0.3333

>> x.^2

ans =

 1     4     9    16
```

Array operations apply to matrices as well as vectors (note that this is an element-by-element multiplication, thus the matrices must be of the same size):

```
>> A=[1 2 3 4; 5 6 7 8; 4 5 6 7]
```

```
A =

 1     2     3     4
 5     6     7     8
 4     5     6     7

>> B=[1 2 3; 4 5 6; 7 8 4; 5 6 7]

B =

 1     2     3
 4     5     6
 7     8     4
 5     6     7

>> A.*B               % [3x4] .x [4x3] = not possible

??? Error using ==> times Matrix dimensions must agree.

>> B.*A               % [4x3] x [3x4] = not possible

??? Error using ==> times Matrix dimensions must agree.

>> A'.*B               % [4x3] x [4x3]

ans =

 1    10    12
 8    30    30
21    56    24
20    48    49

>> A.*B'              % [3x4] x [3x4]
??? Error using ==> mtimes Inner matrix dimensions must agree.

ans =

 1     8    21    20
10    30    56    48
12    30    24    49
```

The array exponentiation operator raises the individual elements of a matrix to a power:

```
>> A=[1 2 3 4; 5 6 7 8; 4 5 6 7]

A =

 1     2     3     4
```

```
 5     6     7     8
 4     5     6     7

>> A.^2

ans =

 1     4     9    16
25    36    49    64
16    25    36    49

>> A.^(1/2)

ans =

1.0000    1.4142    1.7321    2.0000
2.2361    2.4495    2.6458    2.8284
2.0000    2.2361    2.4495    2.6458
```

Also, trigonometric functions may be evaluated:

```
>> x=linspace(0,2*pi,5)

x =

 0    1.5708    3.1416    4.7124    6.2832

>> y=sin(x)./cos(x)

y =

  1.0e+016 *

 0    1.6331   -0.0000    0.5444   -0.0000

>> y=tan(x)

y =

  1.0e+016 *

 0    1.6331   -0.0000    0.5444   -0.0000
```

In the above example, we see that both methods produce the same result.

Determinant of a matrix

Obtaining the determinant of a matrix can be of great importance in engineering problems such as solving eigenvalue problems. Thus, here only a brief review on how to obtain the determinant of a square matrix is given. It should be clear that we can only obtain the determinant of square matrices. A determinant is represented by long bars (|.|) and it is an expression associated with a squared matrix.

Various methods exist in finding the determinant of a matrix. However, here we discuss only one of them. Let us begin with a 2 × 2 matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{A.22}$$

Then, the determinant of matrix $\underline{\mathbf{A}}$ is

$$\det[\underline{\mathbf{A}}] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$$

Now, consider a 3 × 3 matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{A.23}$$

Let us eliminate the first row. However, we can eliminate any row or even columns, just follow the following checkerboard pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Then, the determinant of matrix $\underline{\mathbf{A}}$ is

$$\det[\underline{\mathbf{A}}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Only the determinant of a squared matrix is defined. In general, it is a single number although for some problems, such as dynamics problems, it may lead to a polynomial expression.

In MATLAB® the determinants are found as follows:

```
>> A=[1 2 3 4; 5 6 7 8; 4 5 6 7; 3 -4 -5 6]

A =

 1     2     3     4
 5     6     7     8
 4     5     6     7
 3    -4    -5     6

>> det(A)
```

```
ans =

 0

>> B=[10 2 3 4; 5 6 7 8; 4 5 6 7; 3 -4 -5 6]

B =

10     2     3     4
 5     6     7     8
 4     5     6     7
 3    -4    -5     6

>> det(B)

ans =

   108
```

Singular matrix

A squared matrix is a singular matrix if its determinant is zero. An alternative approach could be to find the eigenvalues of the matrix and show that at least one eigenvalue is zero. Solution to the eigenvalue problem will be discussed later in this chapter.

In the above example, matrix $\underline{\mathbf{A}}$ is singular, whereas matrix $\underline{\mathbf{B}}$ is not.

Inverse of a matrix

In the set of real numbers, we know that for each real number $a \neq 0$ there exists a real number a^{-1} such that

$$a\,a^{-1} = 1$$

The number a^{-1} is called the inverse of the number a relative to multiplication, or multiplicative inverse of a. We use this idea to define the inverse of a squared matrix.

If $\underline{\mathbf{A}}$ is a nonsingular squared matrix of order n and if there exists a matrix $\underline{\mathbf{A}}^{-1}$ such that

$$\underline{\mathbf{A}}\,\underline{\mathbf{A}}^{-1} = \underline{\mathbf{I}}$$

then $\underline{\mathbf{A}}^{-1}$ is called the inverse of $\underline{\mathbf{A}}$. To better understand the definition, let us consider the following example.

Review of Matrix Row Operations:

Before we proceed let us review basic row operations. There are three basic operations used on the rows of a matrix when we are trying to find either the inverse of a matrix or the solve to a system of linear equations. The goal is usually to get the left part of the matrix to look like the identity matrix. To illustrate three operations let us consider the following matrix:

$$\left[\begin{array}{rrr|rrr} 2 & -2 & 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

1. **Switching rows:** We can switch any row and obtain a new matrix. For instance, let us start by choosing the leftmost nonzero column and get a 1 at the top row. This is done by switching the first and third rows:

$$\left[\begin{array}{rrr|rrr} 2 & -2 & 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \quad \mathcal{R}_1 \leftrightarrow \mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{rrr|rrr} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

   We can repeat this step anytime during the procedure. Note that, it is a good idea to use some form of notation (such as the arrows and subscripts above) in order to keep track of the work. Matrices operations can be very messy and this may help to check the work in case of errors.

2. **Multiplying a Row by a Number:** We can multiply any row by a number. (This means multiplying every entry in the row by the same number.) For instance, we may multiply the first row by three:

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad \mathcal{R}_1 \rightarrow 3\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{rrr|rrr} 3 & -9 & 6 & 0 & 0 & 3 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

   Note all others remained untouched. We may multiply two rows at once:

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{c} \mathcal{R}_1 \rightarrow 3\,\mathcal{R}_1 \\ \mathcal{R}_3 \rightarrow 0.5\,\mathcal{R}_3 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|rrr} 3 & -9 & 6 & 0 & 0 & 3 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0.5 & 0.5 & 0 & 0 \end{array}\right]$$

3. **Adding or Subtracting Rows:** We can add two rows together, and replace a row with the result. By doing so, we can "reduce" (get more leading zeroes in) the second or third row by adding the first row to it (the general goal with matrices at this stage being to get a "1" or "0's" and then a "1" at the beginning of each matrix row). For instance, use multiples of the first row to get zeros below the one obtained in the previous step:

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad \mathcal{R}_2 \rightarrow \mathcal{R}_2 - 3\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{rrr|rrr} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 10 & -7 & 0 & 1 & -3 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

   As we can see we get a zero in the first column of the second row.

4. **Important Note:** If the equations represented by the original matrix represent identical or parallel

lines, we will not be able to get the identity matrix using these row operations. In this case, the solution either does not exist or there are infinitely many solutions to the system.

Example A.9.

Determine the inverse of the following squared matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

We are looking for

$$\underline{\mathbf{A}}^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

such that

$$\underline{\mathbf{A}}\,\underline{\mathbf{A}}^{-1} = \underline{\mathbf{I}}$$
$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We are trying to find a, b, c, and d so that the product of $\underline{\mathbf{A}}$ and $\underline{\mathbf{A}}^{-1}$ is the indentity matrix $\underline{\mathbf{I}}$. Hence, multiplying the left side we get

$$\underline{\mathbf{A}}\,\underline{\mathbf{A}}^{-1} = \underline{\mathbf{I}}$$
$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} (2\,a + 3\,b) & (2\,c + 3\,d) \\ (a + 2\,b) & (c + 2\,d) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This produces a system of equations with four unknowns:

$$\begin{aligned} (2\,a + 3\,b) &= 1 \\ (2\,c + 3\,d) &= 0 \\ (a + 2\,b) &= 0 \\ (c + 2\,d) &= 1 \end{aligned}$$

The solution to this system of equations is $a = 2, b = -1, c = -3, d = 2$. Thus, the inverse is

$$\underline{\mathbf{A}}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

However, the method outlined above for finding the inverse, if it exists, gets very involved for matrices of order larger than 2. Now, that we know what we are looking for let us use the concept of augmented matrices to simplify the process. Basically, the idea is the following. Write the matrix as follows:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] = \left[\ \underline{\mathbf{A}} \mid \underline{\mathbf{I}}\ \right]$$

Now, we try to perform operations on matrix 1 until we obtain a row-equivalent matrix that looks like matrix 2, i.e.,

$$\left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right] = [\ \underline{\mathbf{I}} \mid \underline{\mathbf{B}}\ ]$$

If the above can be done then the new matrix to the right of the vertical bar is the inverse of $\underline{\mathbf{A}}$,

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^{-1}$$

Let us do this step-by-step.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] = [\ \underline{\mathbf{A}} \mid \underline{\mathbf{I}}\ ]$$

Let us start by multiplying the second row by 2 and subtract it from the first row

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \quad \mathcal{R}_2 \rightarrow 2\,\mathcal{R}_2 - \mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Now, let us multiply the second row by 3 and subtract first row from the second row

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right] \quad \mathcal{R}_1 \rightarrow \mathcal{R}_1 - 3\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & 0 & 4 & -6 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Lastly, let us divide the first row by 2:

$$\left[\begin{array}{cc|cc} 2 & 0 & 4 & -6 \\ 0 & 1 & -1 & 2 \end{array}\right] \quad \mathcal{R}_1 \rightarrow \frac{\mathcal{R}_1}{2} \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

End Example □

The inverse of a matrix may be easily evaluated in MATLAB® using the following command

```
inv(matrix)
```

The command **inv(A)** gives the inverse of the square matrix $\underline{\mathbf{A}}$. A warning message is printed if $\underline{\mathbf{A}}$ is badly scaled or nearly singular.

Example A.10.

Determine the inverse of the following matrices

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \underline{\mathbf{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -9 & -8 & -7 \end{bmatrix} \quad \underline{\mathbf{C}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

```
>> A=[1:3; 4:6; 7:9]

A =

 1     2     3
 4     5     6
 7     8     9

>> inv(A)
Warning: Matrix is close to singular or badly scaled.
 Results may be inaccurate. RCOND = 1.541976e-018.

ans =

  1.0e+016 *

   -0.4504    0.9007   -0.4504
0.9007   -1.8014    0.9007
   -0.4504    0.9007   -0.4504

>> B=[1:3; 4:6; -9:-7]

B =

 1     2     3
 4     5     6
-9    -8    -7

>> inv(B)
Warning: Matrix is singular to working precision.

ans =

   Inf   Inf   Inf
   Inf   Inf   Inf
   Inf   Inf   Inf

>> C=[2 -1 0 0; -1 2 -1 0; 0 -1 2 -1; 0 0 -1 2]

C =

 2    -1     0     0
-1     2    -1     0
 0    -1     2    -1
```

```
 0     0    -1     2

>> inv(C)

ans =

0.8000    0.6000    0.4000    0.2000
0.6000    1.2000    0.8000    0.4000
0.4000    0.8000    1.2000    0.6000
0.2000    0.4000    0.6000    0.8000
```

End Example □

Example A.11.

Determine the inverse *by hand* and using MATLAB of the following matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

The solution is obtained by:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 & 1 \end{array}\right] = \left[\ \underline{\mathbf{A}} \mid \underline{\mathbf{I}}\ \right]$$

If the inverse exists then the above can be expressed as follows:

$$\left[\ \underline{\mathbf{I}} \mid \underline{\mathbf{B}}\ \right]$$

Let's apply matrix row operations. First, let us switch the first and third rows:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \quad \mathcal{R}_1 \leftrightarrow \mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

Now, continue to apply the matrix operations.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad \mathcal{R}_2 \rightarrow \mathcal{R}_2 - 3\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 10 & -7 & 0 & 1 & -3 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 10 & -7 & 0 & 1 & -3 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad \mathcal{R}_3 \to \mathcal{R}_3 - 2\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 10 & -7 & 0 & 1 & -3 \\ 0 & 4 & -3 & 1 & 0 & -2 \end{array}\right]$$

Now, we repeat the previous step but to make sure that the element in the second row and second column is one:

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 10 & -7 & 0 & 1 & -3 \\ 0 & 4 & -3 & 1 & 0 & -2 \end{array}\right] \quad \mathcal{R}_2 \to \frac{1}{10}\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 4 & -3 & 1 & 0 & -2 \end{array}\right]$$

Now, we need a zero in the third row and second column:

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 4 & -3 & 1 & 0 & -2 \end{array}\right] \quad \mathcal{R}_3 \to \mathcal{R}_3 - 4\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 0 & -0.2 & 1 & -0.4 & -0.8 \end{array}\right]$$

Now, we need a one at B_{33},

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 0 & -0.2 & 1 & -0.4 & -0.8 \end{array}\right] \quad \mathcal{R}_3 \to -5\,\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right]$$

Now, we need to have zeros at B_{12}, B_{13} and B_{23}:

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & -0.7 & 0 & 0.1 & -0.3 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right] \quad \mathcal{R}_2 \to \mathcal{R}_2 + \frac{7}{10}\,\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3.5 & 1.5 & 2.5 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3.5 & 1.5 & 2.5 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right] \quad \mathcal{R}_1 \to \mathcal{R}_1 - 2\,\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 10 & -4 & -7 \\ 0 & 1 & 0 & -3.5 & 1.5 & 2.5 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 10 & -4 & -7 \\ 0 & 1 & 0 & -3.5 & 1.5 & 2.5 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right] \quad \mathcal{R}_1 \to \mathcal{R}_1 + 3\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & 0.5 & 0.5 \\ 0 & 1 & 0 & -3.5 & 1.5 & 2.5 \\ 0 & 0 & 1 & -5 & 2 & 4 \end{array}\right]$$

Hence, the inverse is:

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us verify this solution using MATLAB:

```
>> A=[2 -2 1; 3 1 -1; 1 -3 2]

A =

  2    -2     1
  3     1    -1
  1    -3     2

>> B=inv(A)
```

```
B =

   -0.5000    0.5000    0.5000
   -3.5000    1.5000    2.5000
   -5.0000    2.0000    4.0000
```

End Example □

A.2.4 General Rules for Matrix Operations

When performing matrix operations, we should keep in mind couple of rules. MATLAB® has these rules integrated, so the output will always satisfy the following rules of operation:

(a) Matrix operation is not commutative except for very special cases, e.g.,

$$\underline{\mathbf{A}}\,\underline{\mathbf{B}} \neq \underline{\mathbf{B}}\,\underline{\mathbf{A}}$$

(b) Matrix multiplication is associative, e.g,

$$\underline{\mathbf{A}}\,(\underline{\mathbf{B}}\,\underline{\mathbf{C}}) = (\underline{\mathbf{A}}\,\underline{\mathbf{B}})\,\underline{\mathbf{C}}$$

(c) The distributive law holds true for matrix multiplication, e.g.,

$$\underline{\mathbf{A}}\,(\underline{\mathbf{B}} + \underline{\mathbf{C}}) = \underline{\mathbf{A}}\,\underline{\mathbf{B}} + \underline{\mathbf{A}}\,\underline{\mathbf{C}}$$

or

$$(\underline{\mathbf{A}} + \underline{\mathbf{B}})\,\underline{\mathbf{C}} = \underline{\mathbf{A}}\,\underline{\mathbf{C}} + \underline{\mathbf{B}}\,\underline{\mathbf{C}}$$

(d) For a squared matrix, the matrix may be raised to an integer power n in the following manner:

$$\underline{\mathbf{A}}^n = \underbrace{\underline{\mathbf{A}}\,\underline{\mathbf{A}}\cdots\underline{\mathbf{A}}}_{n \text{ times}}$$

(e) The identity matrix is a squared matrix matching size of $\underline{\mathbf{A}}$. Then, the following can be shown:

$$\underline{\mathbf{A}}\,\underline{\mathbf{I}} = \underline{\mathbf{I}}\,\underline{\mathbf{A}} = \underline{\mathbf{A}}$$

(f) If the transpose of a matrix produces the same matrix then the matrix is symmetric:

$$\underline{\mathbf{A}}^{\mathrm{T}} = \underline{\mathbf{A}} = \text{symmetric}$$

(g) The transpose of a matrix times the same matrix, always produces a symmetric matrix:

$$\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{A}} = \underline{\mathbf{B}} = \text{symmetric} \qquad \text{or} \qquad \underline{\mathbf{A}}\,\underline{\mathbf{A}}^{\mathrm{T}} = \underline{\mathbf{C}} = \text{symmetric}$$

(h) When we take the transpose of the transpose of the matrix, we get the same matrix:

$$\left[\underline{\mathbf{A}}^{\mathrm{T}}\right]^{\mathrm{T}} = \underline{\mathbf{A}}$$

(i) When performing matrix operations dealing with transpose of matrices, the following identities are true:

$$\left[\underline{\mathbf{A}} + \underline{\mathbf{B}} + \cdots + \underline{\mathbf{Z}}\right]^{\mathrm{T}} = \underline{\mathbf{A}}^{\mathrm{T}} + \underline{\mathbf{B}}^{\mathrm{T}} + \cdots + \underline{\mathbf{Z}}^{\mathrm{T}}$$

$$\left[\underline{\mathbf{A}}\,\underline{\mathbf{B}}\cdots\underline{\mathbf{Z}}\right]^{\mathrm{T}} = \underline{\mathbf{Z}}^{\mathrm{T}}\cdots\underline{\mathbf{B}}^{\mathrm{T}}\,\underline{\mathbf{A}}^{\mathrm{T}}$$

Note the change in the order of multiplication.

(j) When performing matrix operations dealing with inverse of matrices, the following identities are true:

$$\left[\underline{\mathbf{A}} + \underline{\mathbf{B}} + \cdots + \underline{\mathbf{Z}}\right]^{-1} = \underline{\mathbf{A}}^{-1} + \underline{\mathbf{B}}^{-1} + \cdots + \underline{\mathbf{Z}}^{-1}$$

$$\left[\underline{\mathbf{A}}\,\underline{\mathbf{B}}\cdots\underline{\mathbf{Z}}\right]^{-1} = \underline{\mathbf{Z}}^{-1}\cdots\underline{\mathbf{B}}^{-1}\,\underline{\mathbf{A}}^{-1}$$

Note the change in the order of multiplication.

A.2.5 Norm of a Vector

Suppose we have an $n \times 1$ column vector $\underline{\mathbf{x}}$:

$$\underline{\mathbf{x}} = \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{array} \right\}$$

The Euclidean norm of the vector $\underline{\mathbf{x}}$ is defined as

$$\|\underline{\mathbf{x}}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{A.24}$$

The above norm is also known as the L_2 norm and it represents the usual notion of the distance from the origin. If x is a column vector then the norm can also be expressed as

$$\|\underline{\mathbf{x}}\|_2 = \sqrt{\underline{\mathbf{x}}^{\mathrm{T}} \cdot \underline{\mathbf{x}}} \tag{A.25}$$

where $\underline{\mathbf{x}}^{\mathrm{T}}$ is the transpose of $\underline{\mathbf{x}}$ and "$\cdot$" represents the inner product. If $\underline{\mathbf{y}}$ is a $1 \times n$ row vector then the norm is

$$\|\underline{\mathbf{y}}\|_2 = \sqrt{\underline{\mathbf{y}} \cdot \underline{\mathbf{y}}^{\mathrm{T}}} \tag{A.26}$$

where $\underline{\mathbf{y}}^{\mathrm{T}}$ is the transpose of $\underline{\mathbf{y}}$.

The built-in *norm* function in MATLAB® computes p-norms of vectors:

$$\|\underline{\mathbf{x}}\|_p = \left\{ |x_1|^p + |x_2|^p + \cdots + |x_n|^p \right\}^{1/p}$$

$$\|\underline{\mathbf{x}}\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$$

Example A.12.

Consider the row vector:

$$\underline{\mathbf{x}} = \left\{ \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \right\}$$

Determine

a) L_1 norm.
b) L_2 norm.
c) L_3 norm.
d) L_∞ norm.

Consider the following row vector

```
>> x=-2:2

x =

-2    -1     0     1     2
```

If only one argument is passed to norm, then the Euclidean (L_2) norm is returned:

```
>> norm(x)

ans =

3.1623
```

A second argument is used to specify the value of p:

```
>> norm(x,1)

ans =

 6

>> norm(x,2)

ans =

3.1623

>> norm(x,3)
```

```
ans =

2.6207

>> norm(x,Inf)

ans =

 2
```

End Example □

A.3 Solution to Linear System of Equations

Consider the following linear system of equations:

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{n1}\,x_1 & + & a_{n2}\,x_2 & + & \cdots & + & a_{nn}\,x_n & = & b_n \end{array}$$

The above system of equations can be expressed in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

or in compact form as follows:

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

where $\underline{\mathbf{A}}$ is an $n \times n$ known square matrix, $\underline{\mathbf{x}}$ an $n \times 1$ unknown column vector, and $\underline{\mathbf{b}}$ an $n \times 1$ known column vector. To obtain the unknown vector $\underline{\mathbf{x}}$, we can do the following:

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}} \quad \rightarrow \quad \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{b}} \quad \rightarrow \quad \underline{\mathbf{I}}\,\underline{\mathbf{x}} = \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{b}} \quad \rightarrow \quad \underline{\mathbf{x}} = \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{b}}$$

In MATLAB,

```
x=inv(A)*b
```

The inverse of a matrix is very costly, and in some cases cumbersome to obtain, especially for very large matrices. In fact, it is seldom necessary to form the explicit inverse of a matrix. A frequent misuse of *inv* arises when solving the system of linear equations of the form

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

A more efficient way to solve this in MATLAB®, from both an execution time and numerical accuracy standpoint, is to use the matrix division operator "\":

```
x=A\b
```

This produces the solution using Gaussian elimination, without forming the inverse. The MATLAB® command

```
MLDIVIDE(A,b)   % should be all lowercase
```

is called for the syntax "$\underline{\mathbf{A}}\backslash\underline{\mathbf{b}}$" when $\underline{\mathbf{A}}$ or $\underline{\mathbf{b}}$ is an object.

Before we proceed with an example, let us review some important statements, which are equivalent for any $n \times n$ matrix $\underline{\mathbf{A}}$, in solving linear system of equations:

a) The equation $\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has the unique solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

b) The linear system $\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has a unique solution for any n-dimensional column vector $\underline{\mathbf{b}}$.

c) The matrix $\underline{\mathbf{A}}$ is nonsingular; that is, $\underline{\mathbf{A}}^{-1}$ exists, and $\det[\underline{\mathbf{A}}] \neq 0$.

Example A.13.

Solve the following system of equations:

$$\begin{array}{ccccccc} 2\,x_1 & - & 2\,x_2 & + & x_3 & = & 3 \\ 3\,x_1 & + & x_2 & - & x_3 & = & 7 \\ x_1 & - & 3\,x_2 & + & 2\,x_3 & = & 0 \end{array}$$

The above can be written in matrix form as follows:

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 7 \\ 0 \end{Bmatrix}$$

The solution is obtained by:

$$\begin{aligned} \underline{\mathbf{A}}\,\underline{\mathbf{x}} &= \underline{\mathbf{b}} \\ \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{A}}\,\underline{\mathbf{x}} &= \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{b}} \\ \underline{\mathbf{I}}\,\underline{\mathbf{x}} &= \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{b}} = \underline{\mathbf{c}} \end{aligned}$$

Using Gauss-Jordan elimination,

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] = \left[\ \underline{\mathbf{A}} \mid \underline{\mathbf{b}}\ \right]$$

If the above can be done then the new vector to the right of the vertical bar will be the solution,

$$\left[\ \underline{\mathbf{I}} \mid \underline{\mathbf{c}}\ \right]$$

Let us start by choosing the leftmost nonzero column and get a 1 at the top row:

$$\left[\begin{array}{rrr|r} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \quad \mathcal{R}_1 \leftrightarrow \mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right]$$

Now, we use multiples of the first row to get zeros below the one obtained in the previous step:

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \mathcal{R}_2 \rightarrow \mathcal{R}_2 - 3\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \mathcal{R}_3 \rightarrow \mathcal{R}_3 - 2\,\mathcal{R}_1 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right]$$

Now, we repeat the previous step but to make sure that the element in the second row and second column is one:

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad \mathcal{R}_2 \rightarrow \frac{1}{10}\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right]$$

Now, we need a zero in the third row and second column:

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad \mathcal{R}_3 \rightarrow \mathcal{R}_3 - 4\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right]$$

Now, we need a one at B_{33},

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \quad \mathcal{R}_3 \rightarrow -5\,\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now, we need to have zeros at B_{12}, B_{13} and B_{23}:

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \mathcal{R}_2 \rightarrow \mathcal{R}_2 + \frac{7}{10}\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \mathcal{R}_1 \rightarrow \mathcal{R}_1 - 2\,\mathcal{R}_3 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \mathcal{R}_1 \rightarrow \mathcal{R}_1 + 3\,\mathcal{R}_2 \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Hence, the solution is:

$$\left\{\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\} = \left\{\begin{array}{c} 2 \\ 0 \\ -1 \end{array}\right\}$$

Let us verify this solution using MATLAB:

```
>> A=[2 -2 1; 3 1 -1; 1 -3 2]

A =

 2    -2     1
 3     1    -1
 1    -3     2

>> b=[3; 7; 0]

b =

 3
 7
 0

>> x=inv(A)*b

x =

2.0000
0.0000
   -1.0000

>> x=mldivide(A,b)

x =

2.0000
0.0000
   -1.0000

>> x=A\b

x =

2.0000
0.0000
   -1.0000
```

End Example □

A.4 Polynomial Approximation

For many type of problems, we do not have the actual polynomial but a set of data points. For such cases, we may built an approximate polynomial that fits the data using interpolation functions.

A.4.1 Lagrange Interpolation Functions

For a set of data points:

$$\{(x_1, y_1), (x_2, y_2), \dots, (x_i, y_i), \dots, (x_n, y_n)\} \quad \text{for} \quad i = 1, 2, \dots, n \quad (n > 0)$$

the elementary Lagrange interpolation formula is

$$L_i^n(x) = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \qquad i = 1, 2, \dots, n \tag{A.27}$$

where $L_i^n(x)$ is a polynomial with degree no greater than $n - 1$. Its value at any data point x_k within the data set is either 1 or 0:

$$L_i^n(x_k) = \prod_{j=1, j\neq i}^{n} \frac{x_k - x_j}{x_i - x_j} = \delta_{ik} = \begin{cases} 0, & \text{for} i \neq k; \\ 1, & \text{for} i = k \end{cases} \tag{A.28}$$

Thus, the Lagrange interpolation polynomial of degree $n - 1$ is

$$p_{n-1}(x) = \sum_{i=1}^{n} y_i \, L_i^n(x) \tag{A.29}$$

For the simplest case where $n = 1$, there are only two data points:

$$\{(x_1, y_1), (x_2, y_2)\}$$

and is a linear function which passes through the two data points. Thus, is just a straight line with its two end points being the two data points:

$$p_1(x) = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}$$

For $n = 3$, there are only three data points:

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$$

and it is a quadratic polynomial that passes through three data points:

$$p_1(x) = y_1 \frac{(x - x_2)\,(x - x_3)}{(x_1 - x_2)\,(x_1 - x_3)} + y_2 \frac{(x - x_1)\,(x - x_3)}{(x_2 - x_1)\,(x_2 - x_3)} + y_3 \frac{(x - x_1)\,(x - x_2)}{(x_3 - x_1)\,(x_3 - x_2)} \tag{A.30}$$

An advantageous property of the Lagrange interpolation polynomial is that the data points do not need to be arranged in any particular order, as long as they are mutually distinct. Thus, the order of the data points is irrelevant. For an application of the Lagrange interpolation polynomial, say we know y_1

and y_2 or y_1,y_2 and y_3, then we can estimate the function $y(x)$ anywhere in $x \in [x_1, x_2]$ linearly and $x \in [x_1, x_3]$ quadratically. This is what we do in finite element analysis.

A.4.2 Newton Interpolating Polynomial

Suppose there is a known polynomial $p_{n-1}(x)$ that interpolates the data set:

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i), \ldots, (x_n, y_n)\} \quad \text{for} \quad i = 1, 2, \ldots, n \quad (n > 0)$$

When one more data point (x_{n+1}, y_{n+1}), which is distinct from all the old data points, is added to the data set, we can construct a new polynomial that interpolates the new data set. Keep also in mind that the new data point does not need to be at the end of the old data set. Consider the following polynomial of degree n

$$p_n(x) = p_{n-1}(x) + c_n \prod_{i=1}^{n} (x - x_i) \tag{A.31}$$

where c_n is an unknown constant. In the case of $n = 1$, we specify $p_0(x)$ as

$$p_0(x) = y_1$$

where data point 1 does not need to be at the beginning of the data set. If we expand the recursive form, the right-hand-sides of the above equation, we obtain the more familiar form of a polynomial

$$p_{n-1}(x) = c_0 + c_1\ (x - x_1) + c_2\ (x - x_1)\ (x - x_3) + \cdots + c_n\ (x - x_1)\ (x - x_2) \cdots (x - x_n) \tag{A.32}$$

which is called the Newton's interpolation polynomial. Its constants can be determined from the data set:

$$p_0(x_1) = y_1 = c_0$$

$$p_1(x_2) = y_2 = c_0 + c_1\ (x_2 - x_1)$$

$$p_2(x_3) = y_3 = c_0 + c_1\ (x_3 - x_1) + c_2\ (x_3 - x_1)\ (x_3 - x_2)$$

which gives

$$c_0 = y_1$$

$$c_1 = \frac{y_2 - c_0}{x_2 - x_1}$$

$$c_2 = \frac{y_3 - c_0 - c_1\ (x_3 - x_1)}{(x_3 - x_1)\ (x_3 - x_2)}$$

Thus,

$$c_n = \frac{p_n\,(x_{n+1}) - p_{n-1}\,(x_{n+1})}{\displaystyle\prod_{i=1}^{n} (x_{n+1} - x_i)}$$

We should note that forcing the polynomial through data with no regard for rates of change in the data (i.e., derivatives) results in a C^0 continuous interpolating polynomial. Alternatively, each data condition $p(x_i) = y_i$ is called a C^0 constraint. It is important to realize that both the Lagrange and Newton polynomials are C^0 continuous and each would generate the same result.

A.4.3 Hermite Interpolation Polynomial

The Hermite interpolation accounts for the derivatives of a given function. The advantage is that it takes information regarding the slope at the known point.

1. Consider: $y = a\,x^3 + b\,x^2 + c\,x + d$ and $x \in [0, 1]$.
2. Apply conditions

$$\begin{array}{lll} & x = 0 & ;x = 1 \\ \text{Case 1:} & y = 1, y' = 0; & y = 0, y' = 0 \\ \text{Case 2:} & y = 0, y' = 1; & y = 0, y' = 0 \\ \text{Case 3:} & y = 0, y' = 0; & y = 1, y' = 0 \\ \text{Case 4:} & y = 0, y' = 0; & y = 0, y' = 1 \end{array}$$

3. Solve each case for a, b, c, d.

A.5 Numerical Integration

An important aspect of isoparametric finite element analysis is the use of an appropriate numerical integration scheme. Whether a one-, two- or three-dimensional integrals, we usually do not employ exact integration. Although it might seems practical, it is good for only very few cases. In fact, numerical integration with modern computers can be obtained in faster and more convenient manner than using exact close form solutions to finite element equations. The close form solutions are good in understanding how the finite element method is employed, however the computer implementation uses numerical methods rather than close form solutions. In general, the required integrals in the finite element calculations have the form

$$\int \underline{\mathbf{F}}\left(\xi\right)\, d\xi; \quad \iint \underline{\mathbf{F}}\left(\xi,\eta\right)\, d\xi\, d\eta; \quad \iiint \underline{\mathbf{F}}\left(\xi,\eta,\zeta\right)\, d\xi\, d\eta\, d\zeta$$

in the one-, two-, and three-dimensional cases, respectively. These integrals can be approximated numerically by using weighted factors as follows:

$$\int \underline{\mathbf{F}}\left(\xi\right)\, d\xi = \sum_i w_i\, \underline{\mathbf{F}}\left(\xi_i\right) + \underline{\mathbf{R}}_n$$

$$\iint \underline{\mathbf{F}}\left(\xi,\eta\right)\, d\xi\, d\eta = \sum_j \sum_i w_{ij}\, \underline{\mathbf{F}}\left(\xi_i,\eta_j\right) + \underline{\mathbf{R}}_n$$

$$\iiint \underline{\mathbf{F}}\left(\xi,\eta,\zeta\right)\, d\xi\, d\eta\, d\zeta = \sum_k \sum_j \sum_i w_{ijk}\, \underline{\mathbf{F}}\left(\xi_i,\eta_j,\zeta_k\right) + \underline{\mathbf{R}}_n$$

where the summations extend over all i, j, and k specified, the weighting factors are w_i's and $\underline{\mathbf{F}}_i$'s are the matrices evaluated at the i^{th} point. The matrices $\underline{\mathbf{R}}_n$ are the error matrices which in practice we ignore in the calculation.

Here, we present the theory and practical implications of numerical integrations. As the case of most numerical techniques, we are interested in accuracy and hence the number of required integration points plays a big role here. Basically, the finite element method uses widely two integration schemes:

1. Newton-Cotes: Requires $(n+1)$ function evaluations to integrate without error a polynomial of order n.
2. Gauss quadrature: Requires n function evaluations to integrate exactly a polynomial of order $(2n-1)$.

Here, we will discuss the Gauss quadrature because in the finite element analysis a large number of function evaluations directly increases the cost of analysis and hence this method is more attractive for the fewer number of function evaluations. However, we also use the Newton-Cotes formulas because they may be more efficient for nonlinear analysis. Once we choose the appropriate integration scheme, we need to determine the order of numerical integration in evaluated the various finite element integrals. The choice of the order of numerical integration is important because first the cost increases when a higher-order integration is employed, and secondly using different integration orders may vary the results by a large amount.

A.5.1 One-Dimensional Gauss Rules

The numerical integration scheme that is mainly adopted and accepted in most of the finite element applications is known as Gauss-Legendre Quadrature[3], or simply Gaussian quadratures. The main reason to use this scheme is because they use an interval from -1 to $+1$, as is the case of isoparametric finite element formulations. The basic goal behind the Gauss-Legendre formulas is to represent an integral in terms of the sum of product of certain weighting coefficients and the value of the function at some selected points.

To better explain this, consider the integral over a linear domain:

$$I = \int_{x_a}^{x_b} f(x)\, dx$$

Now, we map the local coordinates to the natural coordinates, as follows:

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right) \xi$$

Hence, the isoparametric representation, as shown in Fig. A.2, is

$$I = \int_{x_a}^{x_b} f(x)\, dx = \int_{-1}^{1} f(\xi)\ |\underline{\mathbf{J}}|\ d\xi$$

where the Jacobian is defined as

$$\underline{\mathbf{J}} = \left[\ \frac{\partial x}{\partial \xi}\ \right]$$

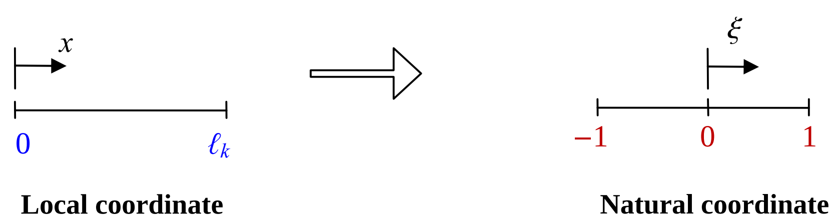


Figure A.2: The k^{th} linear element in local and mapped coordinates.

Since the we are performing a linear mapping, $|\underline{\mathbf{J}}|$ will be a constant number, hence

$$I = \int_{x_a}^{x_b} f(x)\, dx = |\underline{\mathbf{J}}| \int_{-1}^{1} f(\xi)\, d\xi \approx |\underline{\mathbf{J}}| \sum_{i=1}^{n} w_i\, f(\xi_i)$$

we usually use two-, three-, four-, and five-point samplings. The weighted factors are given in Table A.1. Note that as we increase the evaluation points, the accuracy of the integral calculation increases.

[3]The term quadrature means numerical integration.

If n_g is the number of Gauss points, the polynomial of order p that can be integrated exactly is given by

$$p \leq 2\,n_g - 1$$

The reason for this is that the polynomial of order p is defined by $p+1$ parameters. Hence, the number of integration points we need to integrate a polynomial of order p exactly is given by

$$n_g \geq \frac{p+1}{2}$$

We can calculate the Gauss quadrature points and weights for any number of integration points, as given in Table A.1. In the finite element program, we usually program these values once so that we do not have to obtain there values repeatedly. Figure A.3 shows the first five one-dimensional Gauss rules. Sample point locations are marked with black circles. The radii of those circles are proportional to the integration weights. Table A.1 summarizes the one-dimensional Gauss rules.

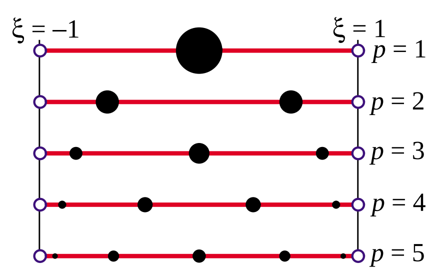


Figure A.3: Gauss one-dimensional numerical integrations sample points over a line segment $\xi \in [-1, +1]$ for Gauss rules $p = 1, 2, 3, 4, 5$.

Table A.1: One-Dimensional Gauss Rules with 1 through 6 sampling points (interval -1 to $+1$).

Points (n)	Order of Polynomial (p)	Weighting factors (w_i)	Evaluation points (ξ_i)
1	$p \leq 1$	$w_1 = 2$	$\xi_1 = 0$
2	$p \leq 3$	$w_1 = 1$ $w_2 = 1$	$\xi_1 = -1/\sqrt{3}$ $\xi_2 = 1/\sqrt{3}$
3	$p \leq 5$	$w_1 = 5/9$ $w_2 = 8/9$ $w_3 = 5/9$	$\xi_1 = -\sqrt{3/5}$ $\xi_2 = 0$ $\xi_3 = \sqrt{3/5}$
4	$p \leq 7$	$w_1 = 1/2 - \sqrt{5/216}$ $w_2 = 1/2 + \sqrt{5/216}$ $w_3 = 1/2 + \sqrt{5/216}$ $w_4 = 1/2 - \sqrt{5/216}$	$\xi_1 = -\sqrt{3/7 + \sqrt{24/245}}$ $\xi_2 = -\sqrt{3/7 - \sqrt{24/245}}$ $\xi_3 = \sqrt{3/7 - \sqrt{24/245}}$ $\xi_4 = \sqrt{3/7 + \sqrt{24/245}}$
5	$p \leq 9$	$w_1 = (322 - 13\sqrt{70})/900$ $w_2 = (322 + 13\sqrt{70})/900$ $w_3 = 512/900$ $w_4 = (322 + 13\sqrt{70})/900$ $w_5 = (322 - 13\sqrt{70})/900$	$\xi_1 = -\sqrt{5/9 + 2\sqrt{10/567}}$ $\xi_2 = -\sqrt{5/9 - 2\sqrt{10/567}}$ $\xi_3 = 0$ $\xi_4 = \sqrt{5/9 - 2\sqrt{10/567}}$ $\xi_5 = \sqrt{5/9 + 2\sqrt{10/567}}$
6	$p \leq 11$	$w_1 = 0.1713244924$ $w_2 = 0.3607615730$ $w_3 = 0.4679139346$ $w_4 = 0.4679139346$ $w_5 = 0.3607615730$ $w_6 = 0.1713244924$	$\xi_1 = -0.9324695142$ $\xi_2 = -0.6612093865$ $\xi_3 = -0.2386191861$ $\xi_4 = 0.2386191861$ $\xi_5 = 0.6612093865$ $\xi_6 = 0.9324695142$

The p are recommended only for polynomial and are not applicable for nonlinear equations not represented by polynomials. For finite element analysis, programming for $n_g = 5$ is good enough for linear finite element analysis.

Example A.14.

Use Gauss-Legendre Quadrature to evaluate the following integral:

$$\int_{2}^{6} (1+x)\,(1-x^2)\,dx$$

First, we proceed to express the integral in its isoparametric representation (map from x to ξ):

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right)\xi \quad \rightarrow \quad dx = \frac{x_b - x_a}{2}\,d\xi$$

$$x = 4 + 2\,\xi \quad \rightarrow \quad dx = 2\,d\xi$$

Hence,

$$f(x) = (1+x)\,(1-x^2) \quad \rightarrow \quad f(\xi) = (2\xi+5)\left(1-(2\xi+4)^2\right)$$

and the Jacobian is:

$$\underline{\mathbf{J}} = \left[\ \frac{dx}{d\xi}\ \right] = [\ 2\ ] \quad \rightarrow \quad |\underline{\mathbf{J}}| = 2$$

Hence, the isoparametric representation is

$$I = \int_a^b f(x)\,dx = |\underline{\mathbf{J}}| \int_{-1}^{1} f(\xi)\,d\xi$$

$$I = \int_2^6 (1+x)\,(1-x^2)\,dx = |\underline{\mathbf{J}}| \int_{-1}^{1} (2\xi+5)\left(1-(2\xi+4)^2\right)\,d\xi$$

The exact solution is (obtained through direct integration):

$$I = -\frac{1108}{3} = -369.333$$

Using Gauss-Legendre Quadrature:

1. One-point

$$\xi_1 = 0 \qquad w_1 = 2 \qquad f_1 = f(\xi_1) = -75.00$$

Hence,

$$I \approx |\underline{\mathbf{J}}|\ (w_1\,f_1) = (2)\Big[(2)(-75)\Big] = -300$$

2. Two-point

$$\begin{aligned} \xi_1 &= -0.57735 & w_1 &= 1 & f_1 &= f(\xi_1) = -27.2852 \\ \xi_2 &= 0.57735 & w_2 &= 1 & f_2 &= f(\xi_2) = -157.381 \end{aligned}$$

Hence,

$$I \approx |\underline{\mathbf{J}}|\ (w_1\,f_1 + w_2\,f_2) = (2)\Big[(1)(-27.2852) + (1)(-157.381)\Big] = -369.333$$

3. Three-point (There is no need to continue since exact solution was achieved. However,

just to highlight the methodology we will complete it.)

$$\begin{array}{lll} \xi_1 = -0.774597 & w_1 = 0.555556 & f_1 = f(\xi_1) = -17.2763 \\ \xi_2 = 0 & w_2 = 0.888889 & f_2 = f(\xi_2) = -75.00 \\ \xi_3 = 0.774597 & w_3 = 0.555556 & f_3 = f(\xi_3) = -195.124 \end{array}$$

Hence,

$$\begin{aligned} I &\approx |\underline{\mathbf{J}}| \, (w_1 \, f_1 + w_2 \, f_2 + w_3 \, f_3) \\ &= (2) \Big[(0.555556)(-17.2763) + (0.888889)(-75.00) + (0.555556)(-195.124)\Big] \\ &= -369.333 \end{aligned}$$

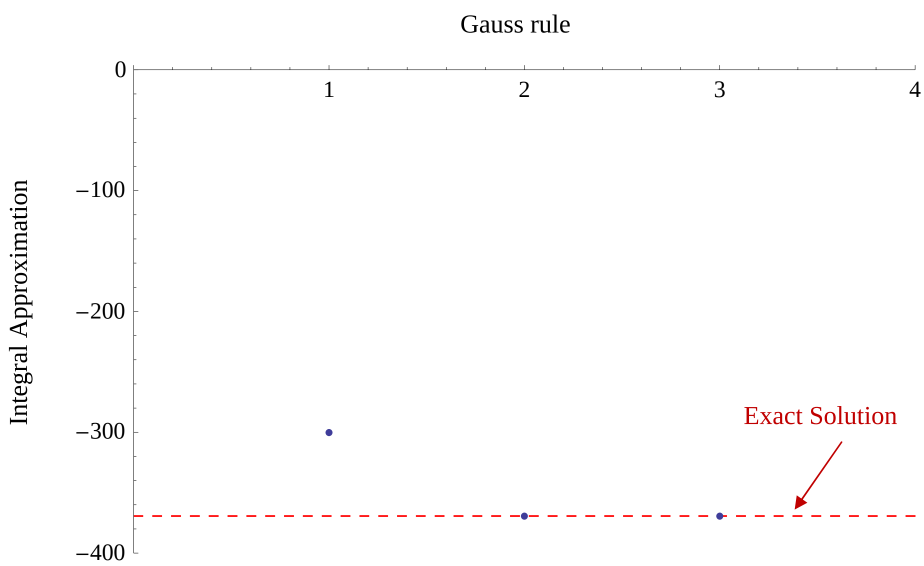


Figure A.4: Convergence plot for Example A.14.

End Example □

Example A.15.

Gauss Rules to evaluate the following elemental stiffness matrix:

$$\underline{\mathbf{K}}_e = \int_0^{80} \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{x}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dx$$

First, we proceed to express the integral in its isoparametric representation (map from x to ξ):

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right)\xi \quad \rightarrow \qquad dx = \frac{x_b - x_a}{2}\, d\xi$$

$$x = 40 + 40\,\xi \quad \rightarrow \qquad dx = 40\, d\xi$$

Hence,

$$\underline{\mathbf{F}}(x) = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{x}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \quad \rightarrow$$

$$\underline{\mathbf{F}}(\xi) = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{40+40\,\xi}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

and the Jacobian is:

$$\underline{\mathbf{J}} = \begin{bmatrix} \frac{dx}{d\xi} \end{bmatrix} = \begin{bmatrix} 40 \end{bmatrix} \quad \rightarrow \quad |\underline{\mathbf{J}}| = 40$$

Hence, the isoparametric representation is

$$\underline{\mathbf{K}}_e = \int_a^b \underline{\mathbf{F}}(x)\, dx = |\underline{\mathbf{J}}| \int_{-1}^{1} \underline{\mathbf{F}}(\xi)\, d\xi$$

$$\underline{\mathbf{K}}_e = \int_0^{80} \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{x}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dx$$

$$= 40 \int_{-1}^{1} \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{40+40\,\xi}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} d\xi$$

The exact solution is (obtained through direct integration):

$$\underline{\mathbf{K}}_e = \int_0^{80} \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{x}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dx = \frac{13\,E}{240} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Using Gauss-Legendre Quadrature:

1. One-point

$$\xi_1 = 0 \qquad w_1 = 2$$

$$\underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1) = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{40+40\,\xi_1}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1+\frac{40}{40}\right)^2 \right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} = \frac{E}{1600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\underline{\mathbf{K}}_e \approx |\underline{\mathbf{J}}| \left(w_1\, \underline{\mathbf{F}}_1\right) = (40) \left((2)\, \frac{E}{1600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

$$= \frac{12\, E}{240} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2. Two-point

$$\xi_1 = -0.57735 \qquad w_1 = 1 \qquad \underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1) = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1 + \frac{40 + 40\,\xi_1}{40}\right)^2\right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

$$\xi_2 = 0.57735 \qquad w_2 = 1 \qquad \underline{\mathbf{F}}_2 = \underline{\mathbf{F}}(\xi_2) = \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[E\left(1 + \frac{40 + 40\,\xi_2}{40}\right)^2\right] \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix}$$

$$\xi_1 = -0.57735 \qquad w_1 = 1 \qquad \underline{\mathbf{F}}_1 = \frac{\left(-6 + \sqrt{3}\right)^2 E}{57600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\xi_2 = 0.57735 \qquad w_2 = 1 \qquad \underline{\mathbf{F}}_2 = \frac{\left(6 + \sqrt{3}\right)^2 E}{57600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\underline{\mathbf{K}}_e \approx |\underline{\mathbf{J}}|\; (w_1\, \underline{\mathbf{F}}_1 + w_2\, \underline{\mathbf{F}}_2)$$

$$= (40) \left\{(1)\left(\frac{\left(-6 + \sqrt{3}\right)^2 E}{57600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) + (1)\left(\frac{\left(6 + \sqrt{3}\right)^2 E}{57600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\right\}$$

$$= \frac{13\, E}{240} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3. Three-point (There is no need to continue since exact solution was achieved.)

End Example □

Example A.16.

Gauss Rules to evaluate the following elemental mass matrix:

$$\underline{\mathbf{M}}_e = \int_0^{80} \begin{bmatrix} 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \left[\rho \left(1 + \frac{x}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} dx$$

First, we proceed to express the integral in its isoparametric representation (map from x to ξ):

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right)\xi \quad \rightarrow \quad dx = \frac{x_b - x_a}{2}\, d\xi$$

$$x = 40 + 40\,\xi \quad \rightarrow \quad dx = 40\, d\xi$$

Hence,

$$\underline{\mathbf{F}}(x) = \begin{bmatrix} 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \left[\rho \left(1 + \frac{x}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} \quad \rightarrow$$

$$\underline{\mathbf{F}}(\xi) = \begin{bmatrix} 1 - \frac{40+40\,\xi}{80} \\ \frac{40+40\,\xi}{80} \end{bmatrix} \left[\rho \left(1 + \frac{40 + 40\,\xi}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{40+40\,\xi}{80} & \frac{40+40\,\xi}{80} \end{bmatrix}$$

and the Jacobian is:

$$\underline{\mathbf{J}} = \begin{bmatrix} \frac{dx}{d\xi} \end{bmatrix} = \begin{bmatrix} 40 \end{bmatrix} \quad \rightarrow \quad |\underline{\mathbf{J}}| = 40$$

Hence, the isoparametric representation is

$$\underline{\mathbf{M}}_e = \int_a^b \underline{\mathbf{F}}(x)\, dx = |\underline{\mathbf{J}}| \int_{-1}^{1} \underline{\mathbf{F}}(\xi)\, d\xi$$

$$\underline{\mathbf{M}}_e = \int_0^{80} \begin{bmatrix} 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \left[\rho \left(1 + \frac{x}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} dx$$

$$= 40 \int_{-1}^{1} \begin{bmatrix} 1 - \frac{40+40\,\xi}{80} \\ \frac{40+40\,\xi}{80} \end{bmatrix} \left[\rho \left(1 + \frac{40 + 40\,\xi}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{40+40\,\xi}{80} & \frac{40+40\,\xi}{80} \end{bmatrix} d\xi$$

The exact solution is (obtained through direct integration):

$$\underline{\mathbf{M}}_e = \int_0^{80} \begin{bmatrix} 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \left[\rho \left(1 + \frac{x}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} dx = \frac{\rho}{6} \begin{bmatrix} 384 & 336 \\ 336 & 1024 \end{bmatrix}$$

Using Gauss-Legendre Quadrature:

1. One-point

$$\xi_1 = 0 \qquad w_1 = 2$$

$$\underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1) = \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} \\ \frac{40+40\,\xi_1}{80} \end{bmatrix} \left[\rho \left(1 + \frac{40 + 40\,\xi_1}{40}\right)^2 \right] \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} & \frac{40+40\,\xi_1}{80} \end{bmatrix}$$

$$= \rho \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\underline{\mathbf{M}}_e \approx |\underline{\mathbf{J}}| \left(w_1\, \underline{\mathbf{F}}_1\right) = (40)\left((2)\,\rho \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \frac{\rho}{6}\begin{bmatrix} 480 & 480 \\ 480 & 480 \end{bmatrix}$$

2. Two-point

$$\xi_1 = -0.57735 \qquad w_1 = 1 \qquad \underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1)$$
$$\xi_2 = 0.57735 \qquad w_2 = 1 \qquad \underline{\mathbf{F}}_2 = \underline{\mathbf{F}}(\xi_2)$$

$$\underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1) = \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} \\ \frac{40+40\,\xi_1}{80} \end{bmatrix} \left[\rho\left(1 + \frac{40+40\,\xi_1}{40}\right)^2\right] \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} & \frac{40+40\,\xi_1}{80} \end{bmatrix}$$

$$\underline{\mathbf{F}}_2 = \underline{\mathbf{F}}(\xi_2) = \begin{bmatrix} 1 - \frac{40+40\,\xi_2}{80} \\ \frac{40+40\,\xi_2}{80} \end{bmatrix} \left[\rho\left(1 + \frac{40+40\,\xi_2}{40}\right)^2\right] \begin{bmatrix} 1 - \frac{40+40\,\xi_2}{80} & \frac{40+40\,\xi_2}{80} \end{bmatrix}$$

$$\xi_1 = -0.57735 \qquad w_1 = 1 \qquad \underline{\mathbf{F}}_1 = \frac{\rho}{6}\begin{bmatrix} 7.55342 & 7.55342 \\ 2.02393 & 2.02393 \end{bmatrix}$$

$$\xi_2 = 0.57735 \qquad w_2 = 1 \qquad \underline{\mathbf{F}}_2 = \frac{\rho}{6}\begin{bmatrix} 1.77992 & 1.77992 \\ 6.64273 & 6.64273 \end{bmatrix}$$

Hence,

$$\begin{aligned}\underline{\mathbf{K}}_e &\approx |\underline{\mathbf{J}}|\ (w_1\, \underline{\mathbf{F}}_1 + w_2\, \underline{\mathbf{F}}_2) \\ &= (40)\left\{(1)\left(\frac{\rho}{6}\begin{bmatrix} 7.55342 & 7.55342 \\ 2.02393 & 2.02393 \end{bmatrix}\right) + (1)\left(\frac{\rho}{6}\begin{bmatrix} 1.77992 & 1.77992 \\ 6.64273 & 6.64273 \end{bmatrix}\right)\right\} \\ &= \frac{\rho}{6}\begin{bmatrix} 373.3 & 346.7 \\ 346.7 & 1013.3 \end{bmatrix}\end{aligned}$$

3. Three-point

$$\xi_1 = -\sqrt{3/5} \qquad w_1 = 5/9 \qquad \underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1)$$
$$\xi_2 = 0.0 \qquad w_2 = 8/9 \qquad \underline{\mathbf{F}}_2 = \underline{\mathbf{F}}(\xi_2)$$
$$\xi_3 = \sqrt{3/5} \qquad w_3 = 5/9 \qquad \underline{\mathbf{F}}_3 = \underline{\mathbf{F}}(\xi_3)$$

$$\underline{\mathbf{F}}_1 = \underline{\mathbf{F}}(\xi_1) = \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} \\ \frac{40+40\,\xi_1}{80} \end{bmatrix} \left[\rho\left(1 + \frac{40+40\,\xi_1}{40}\right)^2\right] \begin{bmatrix} 1 - \frac{40+40\,\xi_1}{80} & \frac{40+40\,\xi_1}{80} \end{bmatrix}$$

$$\underline{\mathbf{F}}_2 = \underline{\mathbf{F}}(\xi_2) = \begin{bmatrix} 1 - \frac{40+40\,\xi_2}{80} \\ \frac{40+40\,\xi_2}{80} \end{bmatrix} \left[\rho\left(1 + \frac{40+40\,\xi_2}{40}\right)^2\right] \begin{bmatrix} 1 - \frac{40+40\,\xi_2}{80} & \frac{40+40\,\xi_2}{80} \end{bmatrix}$$

$$\underline{\mathbf{F}}_3 = \underline{\mathbf{F}}(\xi_3) = \begin{bmatrix} 1 - \frac{40+40\,\xi_3}{80} \\ \frac{40+40\,\xi_3}{80} \end{bmatrix} \left[\rho\left(1 + \frac{40+40\,\xi_3}{40}\right)^2\right] \begin{bmatrix} 1 - \frac{40+40\,\xi_3}{80} & \frac{40+40\,\xi_3}{80} \end{bmatrix}$$

$$\xi_1 = -\sqrt{3/5} \qquad w_1 = 5/9 \qquad \underline{\mathbf{F}}_1 = \frac{\rho}{6}\begin{bmatrix} 7.09331 & 7.09331 \\ 0.900968 & 0.900968 \end{bmatrix}$$

$$\xi_2 = 0.0 \qquad w_2 = 8/9 \qquad \underline{\mathbf{F}}_2 = \frac{\rho}{6}\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$$

$$\xi_3 = \sqrt{3/5} \qquad w_3 = 5/9 \qquad \underline{\mathbf{F}}_3 = \frac{\rho}{6}\begin{bmatrix} 0.586694 & 0.586694 \\ 4.61903 & 4.61903 \end{bmatrix}$$

Hence,

$$\begin{aligned}
\underline{\mathbf{K}}_e &\approx |\underline{\mathbf{J}}| \left(w_1\, \underline{\mathbf{F}}_1 + w_2\, \underline{\mathbf{F}}_2 + w_3\, \underline{\mathbf{F}}_3\right) \\
&= (40) \left\{ \left(\frac{5}{9}\right) \left(\frac{\rho}{6} \begin{bmatrix} 7.09331 & 7.09331 \\ 0.900968 & 0.900968 \end{bmatrix}\right) + \left(\frac{8}{9}\right) \left(\frac{\rho}{6} \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}\right) \right. \\
&\qquad \left. + \left(\frac{5}{9}\right) \left(\frac{\rho}{6} \begin{bmatrix} 0.586694 & 0.586694 \\ 4.61903 & 4.61903 \end{bmatrix}\right) \right\} \\
&= \frac{\rho}{6} \begin{bmatrix} 384 & 336 \\ 336 & 1024 \end{bmatrix}
\end{aligned}$$

Note that for this mass matrix we require three-point integration. We need a higher-order integration in calculating the mass matrix because it is obtained from the displacement interpolation functions, whereas the stiffness matrix is calculated using derivatives of the displacement functions. It is interesting to note that with too low an order of integration the total mass of the element and the total load to which the element is subject is not taken fully into account.

End Example □

Although we obtained the exact value of the integrals in the previous examples, this is absolutely not true. In fact:

1. numerical integration is not always exact but an approximation
2. numerical integration converges to exactness as we increase the number of integration points.

Usually, when the integrand is a polynomial, one can achieve exact solution; however, in all other cases, only approximations are achieved. The following example will help illustrate this point.

Example A.17.

Use Gauss-Legendre Quadrature to evaluate the following integral:

$$I = \int_{-1}^{1} \frac{\xi^2 - 1}{(\xi + 3)^2}\, d\xi$$

Note that this integral is expressed in its isoparametric representation. Hence, we we can proceed to directly use Gauss Rules at this point. The integrand is:

$$f(\xi) = \frac{\xi^2 - 1}{(\xi + 3)^2}$$

The exact solution is (obtained through direct integration):

$$I = \int_{-1}^{1} \frac{\xi^2 - 1}{(\xi + 3)^2}\, d\xi = 4 - \ln(64) = -0.158883$$

Using Gauss-Legendre Quadrature:

1. One-point

   $$\xi_1 = 0 \qquad w_1 = 2 \qquad f_1 = f(\xi_1) = -0.111111$$

   Hence,

   $$I \approx w_1\, f_1 = (2)(-0.111111) = -0.222222$$

2. Two-point

   $$\begin{aligned} &\xi_1 = -0.57735 &\quad w_1 = 1 &\quad f_1 = f(\xi_1) = -0.113587 \\ &\xi_2 = 0.57735 &\quad w_2 = 1 &\quad f_2 = f(\xi_2) = -0.0520938 \end{aligned}$$

   Hence,

   $$I \approx w_1\, f_1 + w_2\, f_2 = (1)(-0.113587) + (1)(-0.0520938) = -0.16568$$

3. Three-point

   $$\begin{aligned} &\xi_1 = -0.774597 &\quad w_1 = 0.555556 &\quad f_1 = f(\xi_1) = -0.0807686 \\ &\xi_2 = 0 &\quad w_2 = 0.888889 &\quad f_2 = f(\xi_2) = -0.111111 \\ &\xi_3 = 0.774597 &\quad w_3 = 0.555556 &\quad f_3 = f(\xi_3) = -0.0280749 \end{aligned}$$

   Hence,

   $$\begin{aligned} I &\approx w_1\, f_1 + w_2\, f_2 + w_3\, f_3 \\ &= (0.555556)(-0.0807686) + (0.888889)(-0.111111) + (0.555556)(-0.0280749) = -0.159234 \end{aligned}$$

4. Four-point

$$\begin{aligned} &\xi_1 = -0.861136 && w_1 = 0.347855 && f_1 = f(\xi_1) = -0.0564938 \\ &\xi_2 = -0.339981 && w_2 = 0.652145 && f_2 = f(\xi_2) = -0.124993 \\ &\xi_3 = 0.339981 && w_3 = 0.652145 && f_3 = f(\xi_3) = -0.0792806 \\ &\xi_4 = 0.861136 && w_4 = 0.347855 && f_4 = f(\xi_4) = -0.0564938 \end{aligned}$$

Hence,

$$\begin{aligned} I &\approx w_1\, f_1 + w_2\, f_2 + w_3\, f_3 + w_4\, f_4 \\ &= (0.347855)(-0.0564938) + (0.652145)(-0.124993) \\ &\quad + (0.652145)(-0.0792806) + (0.347855)(-0.0173355) = -0.158898 \end{aligned}$$

As we can see that we never reach the exact value, instead we convergence to the exact value. We can increase the convergence by increasing the number of sampling or Gauss points.

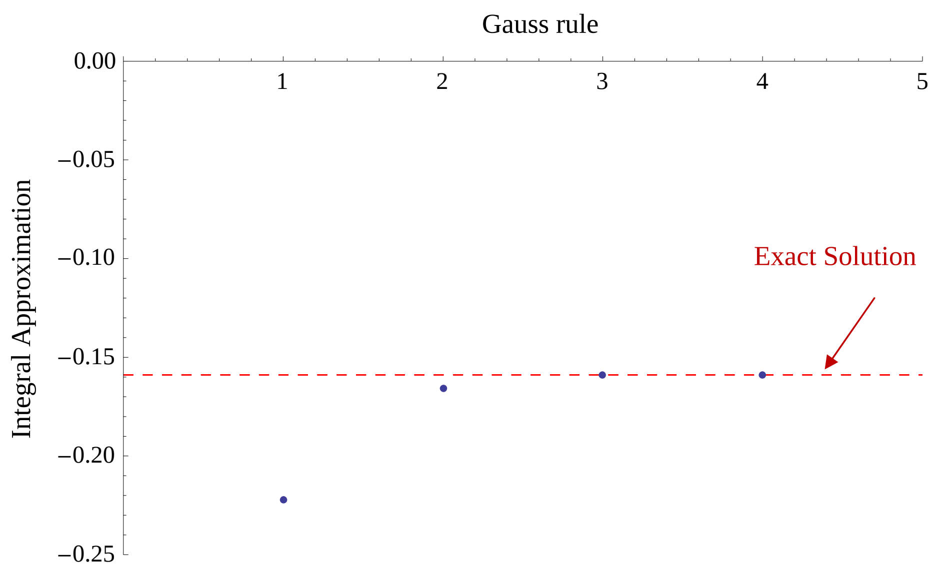


Figure A.5: Convergence plot for Example A.17.

End Example □

A.5.2 2D Gauss Rules for a Quadrilateral Domain

To better explain this, consider the surface integral over a quadrilateral domain:

$$I = \int_{y_a}^{y_b} \int_{x_a}^{x_b} f(x,y)\, dx\, dy$$

Now, we map the local coordinates to the natural coordinates, as follows:

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right)\xi, \qquad y = \left(\frac{y_b + y_a}{2}\right) + \left(\frac{y_b - y_a}{2}\right)\eta$$

Hence, the isoparametric representation, as shown in Fig. A.6, is

$$I = \int_{y_a}^{y_b} \int_{x_a}^{x_b} f(x,y)\, dx\, dy = \int_{-1}^{1} \int_{-1}^{1} f(\xi,\eta)\; |\underline{\mathbf{J}}|\; d\xi\, d\eta$$

where the Jacobian is defined as

$$\underline{\mathbf{J}} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Since the we are performing a linear mapping, $|\underline{\mathbf{J}}|$ will be a constant number, hence

$$I = \int_{y_a}^{y_b} \int_{x_a}^{x_b} f(x,y)\, dx\, dy = |\underline{\mathbf{J}}| \int_{-1}^{1} \int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta$$

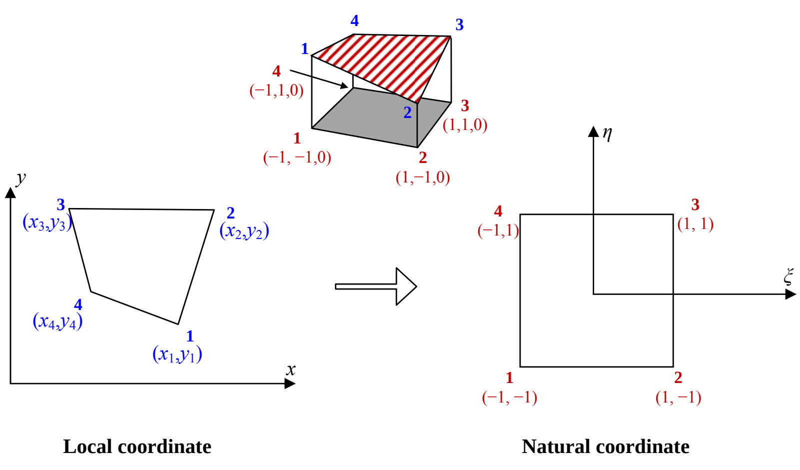


Figure A.6: The k^{th} four-node bilinear quadrilateral element in local and mapped coordinates.

Integration on square elements usually relies on tensor products of the one-dimensional formulas, which we illustrated in Section A.5.1. Thus, the application of 1D Gauss Rules to a two-dimensional integral on a canonical $[-1,1] \times [-1,1]$ square domain yields the approximation:

$$I = |\underline{\mathbf{J}}| \int_{-1}^{1} \left\{ \int_{-1}^{1} f(\xi,\eta)\, d\xi \right\} d\eta \approx |\underline{\mathbf{J}}| \int_{-1}^{1} \left\{ \sum_{i=1}^{n} w_i\, f(\xi_i,\eta) \right\} d\eta \approx |\underline{\mathbf{J}}| \sum_{j=1}^{m} w_j \left\{ \sum_{i=1}^{n} w_i\, f(\xi_i,\eta_j) \right\}$$
$$\approx |\underline{\mathbf{J}}| \sum_{i=1}^{n} \sum_{j=1}^{m} w_i\, w_j\, f(\xi_i,\eta_j)$$

where n and m are the number of Gauss points in the ξ and η directions, respectively. Usually, we choose $n = m$ if we choose the same shape function in the ξ and η directions. The weighted factors are those given for one-dimensional Gauss rules in Table A.1. Figure A.7 shows the first four two-dimensional Gauss product rules for quadrilateral regions. Sample point locations are marked with black circles.

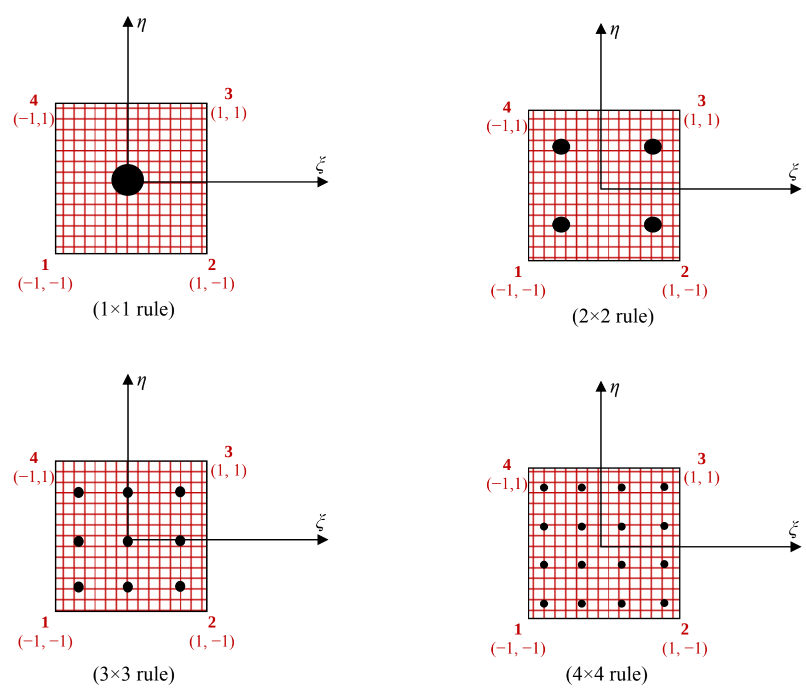


Figure A.7: Gauss two-dimensional numerical integration sample points over a straight-sided quadrilateral region ($\xi \in [-1,+1], \eta \in [-1,+1]$) for Gauss product rules 1×1, 2×2, 3×3, 4×4.

Example A.18.

Using 1 through 3 product rules in Gauss-Legendre Quadrature, evaluate the following two-dimensional integral defined over a quadrilateral region:

$$\int_0^2 \int_2^6 \left(1 + \frac{1}{x^2}\right) \left(1 - (x\,y)^2\right) dx\,dy$$

First, we proceed to express the integral in its isoparametric representation (map from x to ξ):

$$x = \left(\frac{x_b + x_a}{2}\right) + \left(\frac{x_b - x_a}{2}\right)\xi \quad \rightarrow \quad x = 4 + 2\,\xi$$
$$y = \left(\frac{y_b + y_a}{2}\right) + \left(\frac{y_b - y_a}{2}\right)\eta \quad \rightarrow \quad y = 1 + \eta$$

Hence,

$$f(x,y) = \left(1 + \frac{1}{x^2}\right) \left(1 - (x\,y)^2\right) \quad \rightarrow \quad f(\xi,\eta) = \left(1 + \frac{1}{(2\xi+4)^2}\right) \left(1 - (\eta+1)^2(2\xi+4)^2\right)$$

and the Jacobian is:

$$\underline{\mathbf{J}} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \rightarrow \quad |\underline{\mathbf{J}}| = 2$$

Hence, the isoparametric representation is

$$I = |\underline{\mathbf{J}}| \int_{-1}^{1} \left\{ \int_{-1}^{1} f(\xi,\eta)\, d\xi \right\} d\eta \approx |\underline{\mathbf{J}}| \sum_{i=1}^{n} \sum_{j=1}^{m} w_i\, w_j\, f(\xi_i, \eta_j)$$

The exact solution is (obtained through direct integration):

$$I = \int_0^2 \int_2^6 \left(1 + \frac{1}{x^2}\right) \left(1 - (x\,y)^2\right) dx\,dy = -\frac{1682}{9} = -186.889$$

Using Gauss-Legendre Quadrature:

1. One-point

$$\left.\begin{array}{ll} \xi_1 = 0 & \rightarrow\ w_{\xi_1} = 2 \\ \eta_1 = 0 & \rightarrow\ w_{\eta_1} = 2 \end{array}\right\} \quad \rightarrow \quad f_{11} \;= f(\xi_1, \eta_1) = -15.9375$$

Hence,

$$\begin{aligned} I &\approx |\underline{\mathbf{J}}|\ (w_{\xi_1}\, w_{\eta_1}\, f_{11}) \\ &= (2)\left[(2)(2)(-15.9375)\right] = -127.5 \end{aligned}$$

2. Two-point

$$\left.\begin{array}{ll}\xi_1=-0.57735 & \rightarrow\ w_{\xi_1}=1\\ \eta_1=-0.57735 & \rightarrow\ w_{\eta_1}=1\end{array}\right\}\ \rightarrow\ f_{11}=f(\xi_1,\eta_1)=-0.501274$$

$$\left.\begin{array}{ll}\xi_1=-0.57735 & \rightarrow\ w_{\xi_1}=1\\ \eta_2=0.57735 & \rightarrow\ w_{\eta_2}=1\end{array}\right\}\ \rightarrow\ f_{12}=f(\xi_1,\eta_2)=-21.507$$

$$\left.\begin{array}{ll}\xi_2=0.57735 & \rightarrow\ w_{\xi_2}=1\\ \eta_1=-0.57735 & \rightarrow\ w_{\eta_1}=1\end{array}\right\}\ \rightarrow\ f_{21}=f(\xi_2,\eta_1)=-3.88744$$

$$\left.\begin{array}{ll}\xi_2=0.57735 & \rightarrow\ w_{\xi_2}=1\\ \eta_2=0.57735 & \rightarrow\ w_{\eta_2}=1\end{array}\right\}\ \rightarrow\ f_{22}=f(\xi_2,\eta_2)=-67.5598$$

Hence,

$$\begin{aligned}I &\approx |\underline{\mathbf{J}}|\ (w_{\xi_1}\,w_{\eta_1}\,f_{11}+w_{\xi_1}\,w_{\eta_2}\,f_{12}+w_{\xi_2}\,w_{\eta_1}\,f_{21}+w_{\xi_2}\,w_{\eta_2}\,f_{22})\\ &=(2)\Big[(1)(1)(-0.501274)+(1)(1)(-21.507)+(1)(1)(-3.88744)+(1)(1)(-67.5598)\Big]\\ &=-186.911\end{aligned}$$

3. Three-point

$$\left.\begin{array}{ll}\xi_1=-0.774597 & \rightarrow\ w_{\xi_1}=0.555556\\ \eta_1=-0.774597 & \rightarrow\ w_{\eta_1}=0.555556\end{array}\right\}\ \rightarrow\ f_{11}=f(\xi_1,\eta_1)=0.810513$$

$$\left.\begin{array}{ll}\xi_1=-0.774597 & \rightarrow\ w_{\xi_1}=0.555556\\ \eta_2=0.00000 & \rightarrow\ w_{\eta_2}=0.888889\end{array}\right\}\ \rightarrow\ f_{12}=f(\xi_1,\eta_2)=-5.83997$$

$$\left.\begin{array}{ll}\xi_1=-0.774597 & \rightarrow\ w_{\xi_1}=0.555556\\ \eta_3=0.774597 & \rightarrow\ w_{\eta_3}=0.555556\end{array}\right\}\ \rightarrow\ f_{13}=f(\xi_1,\eta_3)=-20.8982$$

$$\left.\begin{array}{ll}\xi_2=0.00000 & \rightarrow\ w_{\xi_2}=0.888889\\ \eta_1=-0.774597 & \rightarrow\ w_{\eta_1}=0.555556\end{array}\right\}\ \rightarrow\ f_{21}=f(\xi_2,\eta_1)=0.198787$$

$$\left.\begin{array}{ll}\xi_2=0.00000 & \rightarrow\ w_{\xi_2}=0.888889\\ \eta_2=0.00000 & \rightarrow\ w_{\eta_2}=0.888889\end{array}\right\}\ \rightarrow\ f_{22}=f(\xi_2,\eta_2)=-15.9375$$

$$\left.\begin{array}{ll}\xi_2=0.00000 & \rightarrow\ w_{\xi_2}=0.888889\\ \eta_3=0.774597 & \rightarrow\ w_{\eta_3}=0.555556\end{array}\right\}\ \rightarrow\ f_{23}=f(\xi_2,\eta_3)=-52.4738$$

$$\left.\begin{array}{ll}\xi_3=0.774597 & \rightarrow\ w_{\xi_2}=0.555556\\ \eta_1=-0.774597 & \rightarrow\ w_{\eta_1}=0.555556\end{array}\right\}\ \rightarrow\ f_{31}=f(\xi_3,\eta_1)=-0.58285$$

$$\left.\begin{array}{ll}\xi_3=0.774597 & \rightarrow\ w_{\xi_2}=0.555556\\ \eta_2=0.00000 & \rightarrow\ w_{\eta_2}=0.888889\end{array}\right\}\ \rightarrow\ f_{32}=f(\xi_3,\eta_2)=-30.7611$$

$$\left.\begin{array}{ll}\xi_3=0.774597 & \rightarrow\ w_{\xi_2}=0.555556\\ \eta_3=0.774597 & \rightarrow\ w_{\eta_3}=0.555556\end{array}\right\}\ \rightarrow\ f_{33}=f(\xi_3,\eta_3)=-99.0916$$

Hence,

$$\begin{aligned}I \approx |\underline{\mathbf{J}}|\ (&w_{\xi_1}\,w_{\eta_1}\,f_{11}+w_{\xi_1}\,w_{\eta_2}\,f_{12}+w_{\xi_1}\,w_{\eta_3}\,f_{13}+w_{\xi_2}\,w_{\eta_1}\,f_{21}+w_{\xi_2}\,w_{\eta_2}\,f_{22}+w_{\xi_2}\,w_{\eta_3}\,f_{23}\\ &+w_{\xi_3}\,w_{\eta_1}\,f_{31}+w_{\xi_3}\,w_{\eta_2}\,f_{32}+w_{\xi_3}\,w_{\eta_3}\,f_{33})\\ =-186.891&\end{aligned}$$

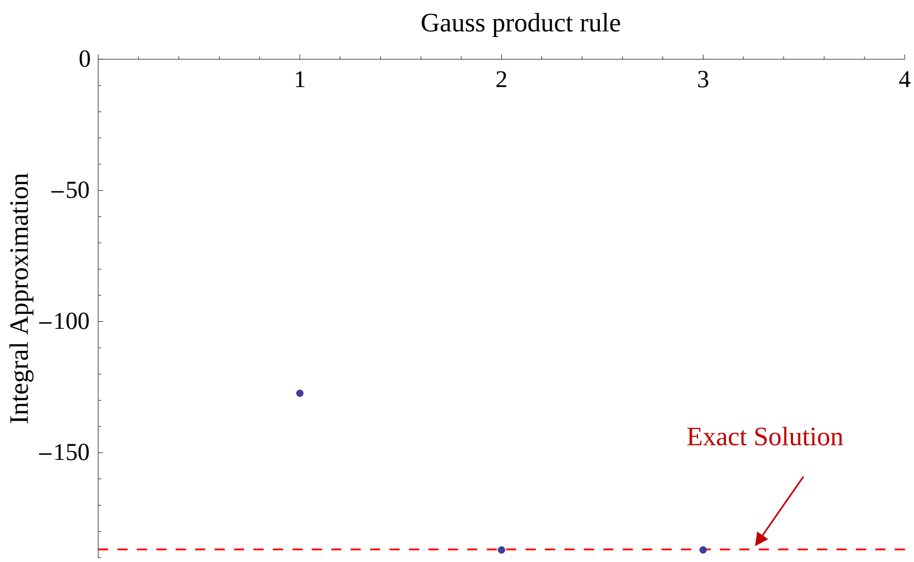


Figure A.8: Convergence plot for Example A.18.

End Example □

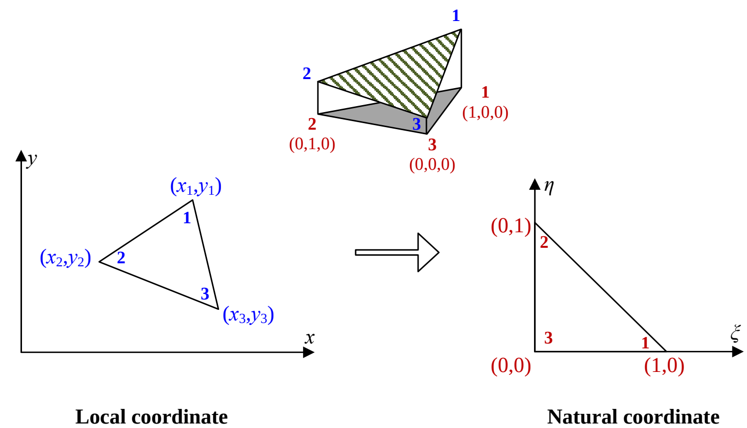


Figure A.9: The k^{th} three-node linear triangular element in local and mapped coordinates.

A.5.3 2D Gauss Rules for a Triangular Domain

The Gauss points for a triangular region differ from the square regions. For a triangular region, we can show that

$$\iint_A dA = \int_0^1 \int_0^{1-\eta} d\xi \, d\eta$$

We can further show, that we can represent surface integrals over a triangular region in their isoparametric representation, as shown in Fig. A.9, as follows:

$$\int_0^1 \int_0^{1-\eta} \underline{\mathbf{F}}\,(\xi,\eta)\ |\underline{\mathbf{J}}|\ d\xi \, d\eta$$

where the Jacobian is defined as

$$\underline{\mathbf{J}} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

This type of surface integrals can be approximated using Gauss rules as follows:

$$\int_0^1 \int_0^{1-\eta} \underline{\mathbf{F}}\,(\xi,\eta)\ |\underline{\mathbf{J}}|\ d\xi \, d\eta = \frac{1}{2}\ |\underline{\mathbf{J}}| \sum_{i=1}^{n} w_i \, \underline{\mathbf{F}}\,(\xi_i,\eta_i)$$

where the summations extend over all i specified, the weighting factors are w_i's and $\underline{\mathbf{F}}_i$'s are the matrices evaluated at the i^{th} point, n are the number of Gauss points in the triangular region. Note that the enclosed area for a straight triangle is given by:

$$A_e = \frac{1}{2}\ |\underline{\mathbf{J}}|$$

The procedure is similar to that of quadrilateral region; however, we use use Table A.2, Gauss rules for a triangle, and express the integrals are expressed from: $0 < \eta < 1$ and $0 < \xi < 1 - \eta$. Figure A.10 shows the first six two-dimensional Gauss rules for triangular regions. Sample point locations are marked with black circles.

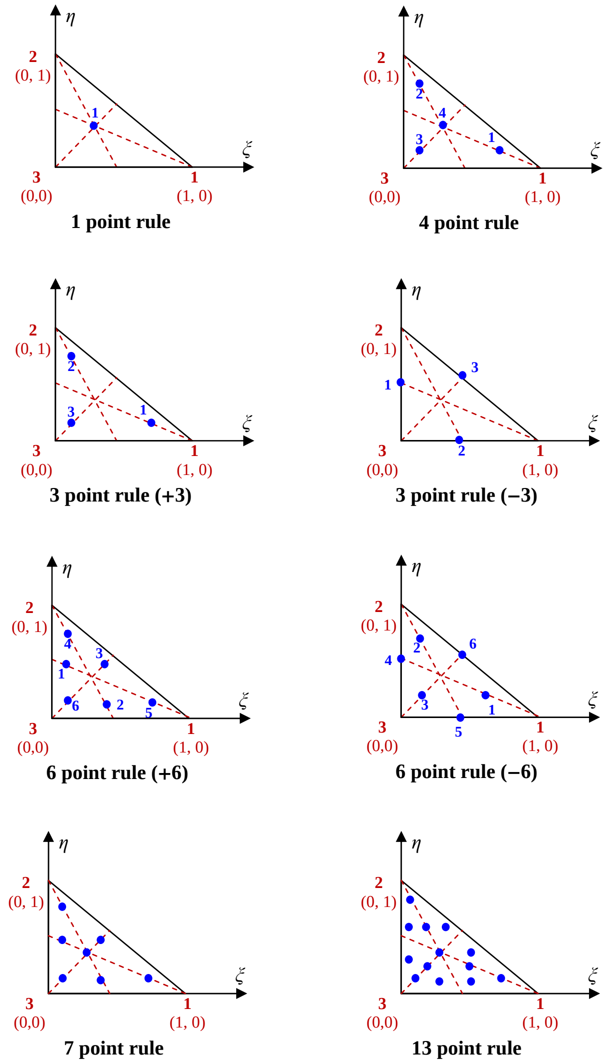


Figure A.10: Gauss two-dimensional numerical integrations sample points over a triangular region $(0 < \eta < 1,\ 0 < \xi < 1 - \eta)$.

Table A.2: Gauss rules for a triangular region ($0 < \eta < 1$, $0 < \xi < 1 - \eta$).

Gauss Rule (n)	Degree of precision	Weighting factors (w_i)	Evaluation points (ξ_i)	(η_i)
1	1	1	1/3	1/3
3	2	1/3 1/3 1/3	2/3 1/6 1/6	1/6 2/3 1/6
−3	2	1/3 1/3 1/3	0 1/2 1/2	1/2 0 1/2
4	3	−27/48 25/48 25/48 25/48	1/3 0.6 0.2 0.2	1/3 0.2 0.6 0.2
6	4	0.22338158967801146570 0.22338158967801146570 0.22338158967801146570 0.10995174365532186764 0.10995174365532186764 0.10995174365532186764	0.10810301816807022736 0.44594849091596488632 0.44594849091596488632 0.81684757298045851308 0.091576213509770743460 0.091576213509770743460	0.44594849091596488632 0.10810301816807022736 0.44594849091596488632 0.091576213509770743460 0.81684757298045851308 0.091576213509770743460
−6	4	3/10 3/10 3/10 1/30 1/30 1/30	2/3 1/6 1/6 0 1/2 1/2	1/6 2/3 1/6 1/2 0 1/2
7	5	0.12593918054482715260 0.12593918054482715260 0.12593918054482715260 0.13239415278850618074 0.13239415278850618074 0.13239415278850618074 9/40	0.79742698535308732240 0.10128650732345633880 0.10128650732345633880 0.059715871789769820459 0.47014206410511508977 0.47014206410511508977 1/3	0.10128650732345633880 0.79742698535308732240 0.10128650732345633880 0.47014206410511508977 0.059715871789769820459 0.47014206410511508977 1/3
13	6	0.050844906370206816921 0.050844906370206816921 0.050844906370206816921 0.11678627572637936603 0.11678627572637936603 0.11678627572637936603 0.082851075618373575194 0.082851075618373575194 0.082851075618373575194 0.082851075618373575194 0.082851075618373575194 0.082851075618373575194	0.87382197101699554332 0.063089014491502228340 0.063089014491502228340 0.50142650965817915742 0.24928674517091042129 0.24928674517091042129 0.059715871789769820459 0.053145049844816947353 0.31035245103378440542 0.053145049844816947353 0.31035245103378440542 0.63650249912139864723 0.63650249912139864723	0.063089014491502228340 0.87382197101699554332 0.063089014491502228340 0.24928674517091042129 0.50142650965817915742 0.24928674517091042129 0.47014206410511508977 0.31035245103378440542 0.053145049844816947353 0.63650249912139864723 0.63650249912139864723 0.053145049844816947353 0.31035245103378440542

Gauss Rules are 1, 3, −3, 6, −6, 7, and 13, the order of the rule. As we can see, there are two rules of order 3 and 6: hence, the negative value returns the second one.

Example A.19.

Using 1 and 3 sampling points in Gauss-Legendre Quadrature, evaluate the following two-dimensional integral defined over a triangular region:

$$\int_0^1 \int_0^{1-\eta} \Big(4\,\xi - 1\Big)\Big(6\,\xi - 4\,\eta\Big)\;|\underline{\mathbf{J}}|\;d\xi d\eta$$

where the Jacobian is

$$\underline{\mathbf{J}} = \begin{bmatrix} -2 & -6 \\ 5 & -4 \end{bmatrix}$$

First, let us calculate the determinant of the Jacobian:

$$|\underline{\mathbf{J}}| = \begin{vmatrix} -2 & -6 \\ 5 & -4 \end{vmatrix} = 38$$

The integrand is

$$f(\xi,\eta) = \Big(4\,\xi - 1\Big)\Big(6\,\xi - 4\,\eta\Big)$$

Using Gauss Quadrature

$$I = \int_0^1 \int_0^{1-\eta} \Big(4\,\xi - 1\Big)\Big(6\,\xi - 4\,\eta\Big)\;|\underline{\mathbf{J}}|\;d\xi d\eta \approx \frac{1}{2}\,|\underline{\mathbf{J}}|\sum_{i=1}^{n} w_i\,f(\xi_i,\eta_i)$$

The exact solution is (obtained through direct integration):

$$I = \int_0^1 \int_0^{1-\eta} \Big(4\,\xi - 1\Big)\Big(6\,\xi - 4\,\eta\Big)(38)\,d\xi d\eta = 38$$

Using Gauss-Legendre Quadrature:

1. One-point

$$\left.\begin{array}{l} \xi_1 = 0.333333 \\ \eta_1 = 0.333333 \\ w_1 = 1 \end{array}\right\} \quad \rightarrow \quad f_1 \;\; = f(\xi_1,\eta_1) = 0.222222$$

   Hence,

$$\begin{aligned} I &\approx \frac{1}{2}\,|\underline{\mathbf{J}}|\;(w_1\,f_1) \\ &= \frac{1}{2}(38)\,\Big[(1)(0.222222)\Big] = 4.22222 \end{aligned}$$

2. Three-point (+3)

$$\left.\begin{array}{l} \xi_1 = 0.666667 \\ \eta_1 = 0.1666667 \\ w_1 = 0.333333 \end{array}\right\} \quad \rightarrow \quad f_1 \;\; = f(\xi_1,\eta_1) = 5.55556$$

$$\left.\begin{aligned} \xi_2 &= 0.1666667 \\ \eta_2 &= 0.666667 \\ w_2 &= 0.333333 \end{aligned}\right\} \quad \rightarrow \quad f_2 \quad = f(\xi_2, \eta_2) = 0.555556$$

$$\left.\begin{aligned} \xi_3 &= 0.1666667 \\ \eta_3 &= 0.1666667 \\ w_3 &= 0.333333 \end{aligned}\right\} \quad \rightarrow \quad f_3 \quad = f(\xi_3, \eta_3) = -0.111111$$

Hence,

$$\begin{aligned} I &\approx \frac{1}{2} \left|\underline{\mathbf{J}}\right| \, (w_1 \, f_1 + w_2 \, f_2 + w_3 \, f_3) \\ &= \frac{1}{2}(38) \Big[(0.333333)(5.55556) + (0.333333)(0.555556) + (0.333333)(-0.111111)\Big] \\ &= 38 \end{aligned}$$

3. Three-point (-3)

$$\left.\begin{aligned} \xi_1 &= 0 \\ \eta_1 &= 0.5 \\ w_1 &= 0.333333 \end{aligned}\right\} \quad \rightarrow \quad f_1 \quad = f(\xi_1, \eta_1) = 2$$

$$\left.\begin{aligned} \xi_2 &= 0.5 \\ \eta_2 &= 0 \\ w_2 &= 0.333333 \end{aligned}\right\} \quad \rightarrow \quad f_2 \quad = f(\xi_2, \eta_2) = 3$$

$$\left.\begin{aligned} \xi_3 &= 0.5 \\ \eta_3 &= 0.5 \\ w_3 &= 0.333333 \end{aligned}\right\} \quad \rightarrow \quad f_3 \quad = f(\xi_3, \eta_3) = 1$$

Hence,

$$\begin{aligned} I &\approx \frac{1}{2} \left|\underline{\mathbf{J}}\right| \, (w_1 \, f_1 + w_2 \, f_2 + w_3 \, f_3) \\ &= \frac{1}{2}(38) \Big[(0.333333)(2) + (0.333333)(3) + (0.333333)(1)\Big] \\ &= 38 \end{aligned}$$

This example shows how bad the approximate is when using only one Gauss point. We can also note that it does not matter what 3-point (either +3 or −3) we use, we will get the same answer in both cases.

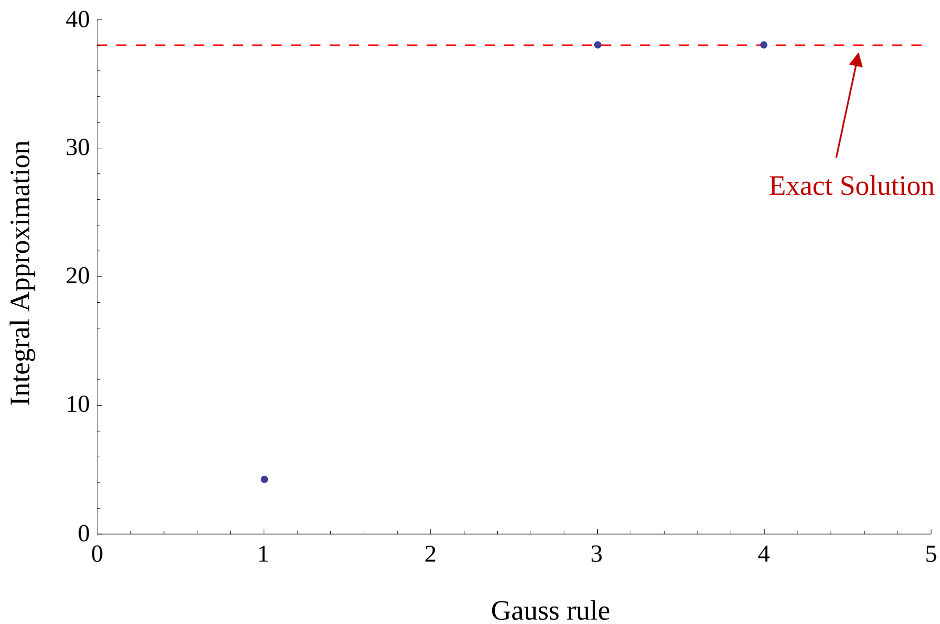


Figure A.11: Convergence plot for Example A.19.

End Example □

A.5.4 Gaussian Quadrature Code

Following is an explanation of the Gaussian Quadrature function we developed in MATLAB:

```
function [I]= GaussQuadratures(Type,Np,f)
```

The function `GaussianQuadrature.m` approximates line integrals as well as two-dimensional integrals over both triangular and quadrilateral regions. We must express the integrals in their natural coordinates in order to use the function. The function requires the following input: `Type`, `Np`, `f`.

Type: The type of Gauss Quadrature to perform: "1" for a one-dimensional region; "2" for a two-dimensional quadrilateral region; "3" for a two-dimensional triangular region.

Np: The number of Gauss Points to approximate the integral. For triangular elements, the Gauss Points and their corresponding weight factors are given for the following cases: 1, 3, −3, 4, 6, −6, 7 and 13. The function will always return values for −3 or −6 for triangular regions.

f: The integrand (the function to integrate). We must define the function (which may be a scalar, vector, or a matrix) as follows:

```
f = @(xi) [integrand function goes here]      % for one-dim
f = @(xi,eta) [integrand function goes here]  % for two-dim
```

Now, the function chooses the appropriate Gauss Rule, as requested by the user. First, determines what type of integral we are approximating, i.e.,

```
switch(Type)
case(1)              %One Dimensional Gaussian Quadrature
.
.
.
case(2)              %Two Dimensional Quadrilateral Quadrature
.
.
.
case(3)              %Two Dimensional Triangular Quadrature
.
.
.
end
```

Then it chooses the number of Gauss rules we are interested in and reads the Gauss rules and points, i.e.,

```
switch(abs(Np))
```

```
    case(1)
       w(1)=2;
       xi(1)=0;
    case(2)
       w(1)=1;
       w(2)=1;
       xi(2)=1/sqrt(3);
       xi(1)=-xi(2);
   .
   .
   .
end
```

The function returns the approximate value of the integral. For one-dimensional region:

```
I=0;
for ii=1:abs(Np);     %Subscript for xi
    I = I + w(ii)*f(xi(ii));
end
```

For a two-dimensional quadrilateral region:

```
I=0;
for ii=1:abs(Np);     %Subscript for xi
    for jj=1:abs(Np);  %Subscript for eta
        I = I + w(ii)*w(jj)*f(xi(ii),eta(jj));
    end
end
```

For a two-dimensional triangular region:

```
I=0;
for ii=1:abs(Np);     %Subscript for xi and eta
    I = I + 0.5*w(ii)*f(xi(ii),eta(ii));
end
```

Example A.20.

Redo all Gauss Quadrature examples using MATLAB:

1. Example B.14

   ```
   f=@(xi) 2*(2*xi+5)*(1-(2*xi+4)^2)
   I = GaussQuadratures(1,5,f);
   ```

   Output is

   ```
   The approximation is I =
    -369.3333
   ```

2. Example B.15 (Note that E is taken as one)

   ```
   f=@(xi) 40*[-1/80; 1/80]*[(1+(40+40*xi)/40)^2]*[-1/80  1/80]
   I = GaussQuadratures(1,5,f);
   ```

   Output is

   ```
   The approximation is I =
       0.0542   -0.0542
      -0.0542    0.0542
   ```

3. Example B.16 (Note that ρ is taken as one)

   ```
   f=@(xi) 40*[1-(40+40*xi)/80; (40+40*xi)/80]...
   *[(1+(40+40*xi)/40)^2]*[1-(40+40*xi)/80  (40+40*xi)/80]
   I = GaussQuadratures(1,5,f);
   ```

   Output is

   ```
   The approximation is I =
      64.0000   56.0000
      56.0000  170.6667
   ```

4. Example B.17

   ```
   f=@(xi) (xi^2-1)/(xi+3)^2
   I = GaussQuadratures(1,5,f);
   ```

   Output is

   ```
   The approximation is I =
      -0.1589
   ```

5. Example B.18

   ```
   f=@(xi,eta) 2*(1+1/(2*xi+4)^2)*(1-(eta+1)^2*(2*xi +4)^2)
   I = GaussQuadratures(2,5,f);
   ```

   Output is

   ```
   The approximation is I =
    -186.8889
   ```

6. Example B.19

```
f=@(xi,eta) 38*(4*xi-1)*(6*xi-4*eta)
I = GaussQuadratures(3,5,f);
```

Output is

```
The approximation is I =
     38
```

End Example □

A.6 Roots of polynomials

In MATLAB®, roots of polynomials of the form

$$c_n\,x^n + \cdots + c_2\,x^2 + c_1\,x + c_0 = 0$$

can be obtained as follows:

```
>> c=[c_n ... c_2  c_1  c_0]

>> roots(c)
```

A.6.1 Linear Equations

The general form of a linear equation is given by:

$$c_1\,\alpha + c_0 \quad = \quad 0$$

and the solution to the above equation is:

$$\alpha_1 = -\frac{c_0}{c_1} \tag{A.33}$$

A.6.2 Quadratic Equations

The general form of a quadratic equation is given by:

$$c_2\,\alpha^2 + c_1\,\alpha + c_0 = 0 \tag{A.34}$$

and the solution to the above equation is:

$$\alpha = \frac{-c_1 \pm \sqrt{c_1^2 - 4\,c_2\,c_0}}{2\,c_2}$$

$$\alpha_1 = \frac{-c_1 + \sqrt{c_1^2 - 4\,c_2\,c_0}}{2\,c_2} \qquad \alpha_2 = \frac{-c_1 - \sqrt{c_1^2 - 4\,c_2\,c_0}}{2\,c_2} \tag{A.35}$$

A.6.3 Cubic Equations

The general form of a Cubic equation is given by:

$$\alpha^3 + c_2\,\alpha^2 + c_1\,\alpha + c_0 = 0 \tag{A.36}$$

and the solution to the above equation is (note that $c_3 = +1$, *positive one*):

$$\alpha_1 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3}\right) \tag{A.37a}$$

$$\alpha_2 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) \tag{A.37b}$$

$$\alpha_3 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) \tag{A.37c}$$

where:

$$\beta = \cos^{-1}\left\{\frac{-2\,c_2^3 + 9\,c_1\,c_2 - 27\,c_0}{2\,\left(c_2^2 - 3\,c_1\right)^{3/2}}\right\} \qquad \text{(must be in radians)}$$

Example A.21.

Using MATLAB®, find the roots of

$$\alpha^2 + 7\,\alpha + 6 = 0$$

```
>> c=[1 7 6]

c =

 1     7     6

>> roots(c)

ans =

-6
-1
```

End Example □

Example A.22.

Find the roots of

$$-8 + 13\,\alpha + 6.5\,\alpha^2 - \alpha^3 = 0$$

First, ensure that $c_3 = 1$:

$$\alpha^3 - 6.5\,\alpha^2 - 13\,\alpha + 8 = 0$$

Using the cubic equation we get (from course handout):

$$c_2 = -6.5 \qquad c_1 = -13 \qquad c_0 = 8$$

and the solution to the above equation is (note that $c_3 = 1$):

$$\beta = \cos^{-1}\left\{\frac{-2\,c_2^3 + 9\,c_1\,c_2 - 27\,c_0}{2\,\left(c_2^2 - 3\,c_1\right)^{3/2}}\right\} \qquad \text{(must be in radians)}$$
$$= 0.72769 \text{ rads}$$

$$\alpha_1 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3}\right) = -2.0$$

$$\alpha_2 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3} + \frac{2\,\pi}{3}\right) = 0.5$$

$$\alpha_3 = -\frac{c_2}{3} + \frac{2}{3}\sqrt{c_2^2 - 3\,c_1}\;\cos\left(\frac{\beta}{3} + \frac{4\,\pi}{3}\right) = 8.0$$

End Example □

Example A.23.

Using MATLAB®, find the roots of

$$\alpha^3 - 6\alpha^2 - 15\alpha + 29 = 0$$

```
>> c=[1 -6 -15 29]
c =
 1    -6   -15    29

>> roots(c)
ans =
7.4862
   -2.8469
1.3607
```

End Example □

A.7 The Eigenvalue Problem

The term *eigen* is German and means "proper" or "characteristic". One could think of characteristic of your professor, a friend and, most likely, it is unique of that person. The same way when talking about the eigenvalue problem, we are talking about the characteristic of a matrix. Here, we will discuss an algebraic eigenvalue problem. This information will be crucial in solving optimization problems.

Derivation

A linear system of equations can be expressed as follows:

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}} \tag{A.38}$$

where $\underline{\mathbf{A}}$ is an $n \times n$ square matrix, $\underline{\mathbf{x}}$ an $n \times 1$ column vector, and $\underline{\mathbf{b}}$ an $n \times 1$ column vector. When $\underline{\mathbf{b}}$ is a nonzero vector, the linear equations are commonly referred as a nonhomogeneous system. For a nonhomogeneous system, unique solutions exist as long as the determinant of matrix $\underline{\mathbf{A}}$ is nonzero.

We may express the linear system of equations one where the vector $\underline{\mathbf{b}}$ is a scalar multiplied by the $\underline{\mathbf{x}}$, e.g.,

$$\underline{\mathbf{b}} = \alpha\,\underline{\mathbf{x}} \tag{A.39}$$

and this leads to a set of linear equations of the form

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \alpha\,\underline{\mathbf{x}} \tag{A.40}$$

Now, using matrix algebra, we know that

$$\underline{\mathbf{x}} = \underline{\mathbf{I}}\,\underline{\mathbf{x}} \tag{A.41}$$

where $\underline{\mathbf{I}}$ is an identity matrix of order n. Thus, Eq. (A.40) can be rewritten as

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \alpha\,\underline{\mathbf{I}}\,\underline{\mathbf{x}} \tag{A.42}$$

It is of common practice to express Eq. (A.42) as

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} - \alpha\,\underline{\mathbf{I}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

or

$$\left[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}\right]\,\underline{\mathbf{x}} = \underline{\mathbf{0}} \tag{A.43}$$

where $\underline{\mathbf{A}}$ is an $n \times n$ square matrix, $\underline{\mathbf{I}}$ the identity matrix of order n, $\underline{\mathbf{x}}$ an $n \times 1$ unknown vector, and α an unknown scalar. Eq. (A.43) is called an eigenvalue problem.

This type of problems are very useful to engineers in solving buckling problems, vibration of elastic structures, electrical systems, principal stresses, and optimization problems, among other applications.

Solution

The matrix eigenvalue problem is

$$[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}]\;\underline{\mathbf{x}} = \underline{\mathbf{0}} \tag{A.44}$$

Here, both vector $\underline{\mathbf{x}}$ and scalar α are unknown and our goal is to determine both.

Before we proceed with the solution to the eigenvalue problem, review the statements, which are equivalent for any $n \times n$ matrix $\underline{\mathbf{A}}$, in solving linear system of equations, provided in Section A.3. These statements imply that the eigenvalue problem, Eq. (A.44), has two possible solutions: (i) when $\underline{\mathbf{x}} = \underline{\mathbf{0}}$, (ii) and when $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$. It is very important to note that in order to solve an eigenvalue problem the square matrix $\underline{\mathbf{A}}$ should be nonsingular.

When $\underline{\mathbf{x}} = \underline{\mathbf{0}}$ is a solution of Eq. (A.44) for any value of α, this is of no practical interest because it gives no information regarding the problem in hand. This is often known as the trivial solution and we wish to study the nonzero solutions corresponding to the eigenvalue problems.

However, when $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$ is the solution we are interested in. A value of α for which Eq. (A.44) has a solution $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$ is called an eigenvalue, or characteristic value, of the matrix $\underline{\mathbf{A}}$. The corresponding solutions $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$ of Eq. (A.44) are called eigenvectors, or characteristic vectors, of $\underline{\mathbf{A}}$ corresponding to that eigenvalue of α.

Eigenvalues

Here, we shall outline the basic steps in obtaining eigenvalues. Let us begin with the eigenvalue problem:

$$[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}]\;\underline{\mathbf{x}} = \underline{\mathbf{0}} \tag{A.45}$$

Since we are only interested in nonzero values of $\underline{\mathbf{x}}$, we want

$$\det\left[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}\right] = 0 \tag{A.46a}$$

$$\left|\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \alpha \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}\right| = 0 \tag{A.46b}$$

$$\left|\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha \end{bmatrix}\right| = 0 \tag{A.46c}$$

$$\begin{vmatrix} (a_{11}-\alpha) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\alpha) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-\alpha) \end{vmatrix} = 0 \tag{A.46d}$$

Thus, we define the characteristic determinant as:

$$p(\alpha) = \begin{vmatrix} (a_{11}-\alpha) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\alpha) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-\alpha) \end{vmatrix} \tag{A.47}$$

Then, the characteristic equation of matrix $\underline{\mathbf{A}}$ is

$$p(\alpha) = 0 \tag{A.48}$$

and it is in terms of α. If the size of matrix $\underline{\mathbf{A}}$ is $n \times n$ then the characteristic equations will produce a n^{th} degree polynomial. Consequently, has at most n distinct roots, some of which may be complex. In short,

- The roots of the characteristic equation of $\underline{\mathbf{A}}$ are the eigenvalues of the square matrix $\underline{\mathbf{A}}$.
- An $n \times n$ matrix has at least one eigenvalue and at most n numerically different eigenvalues.
- The eigenvalues must be determined first.

If a matrix has zero eigenvalues, the matrix is a singular matrix.

In practice, eigenvalues of large matrices are not computed using the characteristic polynomial. Faster and more numerically stable methods are available, for instance the QR/QZ decompositions. MATLAB® uses robust and stable algorithms based on LAPACK libraries to obtain eigenvalues:

```
E = EIG(A,B)
```

The vector $\underline{\mathbf{E}}$ is a vector containing the generalized eigenvalues of square matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$.

```
[V,D] = EIG(A,B)
```

The above statement produces a diagonal matrix $\underline{\mathbf{D}}$ of generalized eigenvalues and a full matrix $\underline{\mathbf{V}}$ whose columns are the corresponding eigenvectors so that $\underline{\mathbf{A}}\,\underline{\mathbf{V}} = \underline{\mathbf{B}}\,\underline{\mathbf{V}}\,\underline{\mathbf{D}}$.

```
EIG(A,B,'chol')
```

The above is the same as EIG(A,B) for symmetric A and symmetric positive definite $\underline{\mathbf{B}}$. It computes the generalized eigenvalues of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ using the Cholesky factorization of $\underline{\mathbf{B}}$.

```
EIG(A,B,'qz')
```

The above ignores the symmetry of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ and uses the QZ algorithm. In general, the two algorithms return the same result, however using the QZ algorithm may be more stable for certain problems. The flag is ignored when $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are not symmetric.

Example A.24.

Obtain the eigenvalues for the following matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}]\;\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

Since we are only interested in nonzero values of $\underline{\mathbf{x}}$, we want

$$\begin{aligned}
\det\,[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}] &= 0 \\
\left|\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\right| &= 0 \\
\begin{vmatrix} (-5-\alpha) & 2 \\ 2 & (-2-\alpha) \end{vmatrix} &= 0
\end{aligned}$$

Thus, we define the characteristic determinant as:

$$p(\alpha) = \begin{vmatrix} (-5-\alpha) & 2 \\ 2 & (-2-\alpha) \end{vmatrix} = (-2-\alpha)(-5-\alpha) - 4 = 0$$

Then, the characteristic equation of matrix $\underline{\mathbf{A}}$ is

$$p(\alpha) = \alpha^2 + 7\,\alpha + 6 = (\alpha + 6)(\alpha + 1) = 0$$

and the eigenvalues are then:

$$\alpha = -1, -6$$

Using MATLAB®,

```
>> A=[-5,2;2,-2]

A =

-5     2
 2    -2

>> alpha =eig(A)

alpha =

-6
-1
```

End Example □

Example A.25.

Obtain the eigenvalues for the following matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

$$[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}]\ \underline{\mathbf{x}} = \underline{\mathbf{0}}$$

Since we are only interested in nonzero values of $\underline{\mathbf{x}}$, we want

$$\det\left[\underline{\mathbf{A}} - \alpha\,\underline{\mathbf{I}}\right] = 0$$

$$\left| \begin{bmatrix} -2 & 0 & 3 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix} - \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} -2 & 0 & 3 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (-2-\alpha) & 0 & 3 \\ 1 & (4-\alpha) & 3 \\ 2 & 3 & (4-\alpha) \end{vmatrix} = 0$$

Thus, we define the characteristic determinant as:

$$p(\alpha) = \begin{vmatrix} (-2-\alpha) & 0 & 3 \\ 1 & (4-\alpha) & 3 \\ 2 & 3 & (4-\alpha) \end{vmatrix} = -\alpha^3 + 6\alpha^2 + 15\alpha - 29 = 0$$

Then, the characteristic equation of matrix $\underline{\mathbf{A}}$ is

$$p(\alpha) = \alpha^3 - 6\alpha^2 - 15\alpha + 29 = 0$$

and the eigenvalues are then:

$$\alpha = 7.4862, -2.8469, 1.3607$$

Using MATLAB®,

```
>> A=[-2,0,3;1,4,3;2,3,4]

A =

-2     0     3
 1     4     3
 2     3     4

>> alpha =eig(A)

alpha =

7.4862
   -2.8469
1.3607
```

End Example □

A.8 References

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Thomas, G. B., R. L. Finney, M. D. Weir and F. R. Giordano, *Thomas' Calculus, Early Transcendentals Update*, Tenth Edition, Addison-Wesley, Massachusetts, 2003. Entire book.

A.9 Suggested Problems

Problem A.1.

You are asked to create a program using MATLAB®. The program should consist in: a script file, a function file, and an output file. All input and output must be done from the script file only. All output must be done to an external file. The files must be well documented. The following is a description of what the program should do:

1. The function `cadprog` takes as an input three arbitrary real numbers a, b and c. The output should be:

   (a) $z_1 = a\, e^b$

   (b) $z_2 = \ln c$

   (c) $z_3 = c^b$

   (d) $z_4 = a/c$

   (e) z_5 a condition for not running the function if $c = 0$.

   Make sure the function does not print anything.

2. The script file should:

   (a) Contain all the input variables.

   (b) Print the input variables (with an explanation).

   (c) Call the function.

   (d) Place a condition, where the output of the variables is only printed if $c \neq 0$. Otherwise use your condition z_5 to print a line saying there is an error.

3. Use good programming skills.

□

Problem A.2.

Solve this problem *by hand* and verify your answers using MATLAB®. Consider the following matrices and vectors:

$$\underline{\mathbf{A}} = \begin{bmatrix} x & -y & 1 & 1 \\ x\,z & z^2 & y & x \\ -y & z & y^3\,z & 2 \end{bmatrix} \quad \underline{\mathbf{B}} = \begin{bmatrix} 8 & -2 & 0 \\ -2 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix} \quad \underline{\mathbf{C}} = \begin{bmatrix} a & b\,c & 0 \\ b & a^3 & -a^4 \\ 0 & -c\,b^2 & a\,c\,b^2 \end{bmatrix} \quad \underline{\mathbf{d}} = \begin{Bmatrix} 20 \\ 0 \\ -4 \end{Bmatrix}$$

Taking

$$x = y = z = 1, \qquad a = 2\,b = -3\,c = 6$$

determine the following:

a) $\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{B}}$

b) det[$\underline{\mathbf{C}}$]. Is matrix $\underline{\mathbf{C}}$ singular? Justify your answer.

c) Use Gauss-Jordan elimination to determine the inverse of matrix $[\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{A}}]$. Verify your answer using MATLAB.

d) $\underline{\mathbf{A}}\,\underline{\mathbf{d}}$

e) $\underline{\mathbf{d}}^{\mathrm{T}}$

f) $\underline{\mathbf{d}}^{\mathrm{T}}\,\underline{\mathbf{A}}$

g) $\underline{\mathbf{d}}\,\underline{\mathbf{d}}^{\mathrm{T}}$

h) $\underline{\mathbf{d}}^{\mathrm{T}}\,\underline{\mathbf{d}}$

i) $\underline{\mathbf{B}} + \underline{\mathbf{d}}^{\mathrm{T}}$

j) $\underline{\mathbf{B}} + \underline{\mathbf{A}}^{\mathrm{T}}$

k) $\underline{\mathbf{B}} + \underline{\mathbf{C}}^{\mathrm{T}}$

l) Eigenvalues of $\underline{\mathbf{A}}$

m) Eigenvalues of $\underline{\mathbf{C}}$

n) Eigenvalues of $\underline{\mathbf{d}}$

□

Problem A.3.

Solve this problem *by hand* and verify your answers using MATLAB®. Consider the following matrices and vectors:

$$\underline{\mathbf{A}} = \begin{bmatrix} -y^2 & -y^2 & 1 & 0 \\ x\,z\,y & x\,z\,y & z^2 & y \\ -y & -y & z & 10\,z \end{bmatrix} \qquad \underline{\mathbf{B}} = \begin{bmatrix} -3 & 2 & -1 \\ -2 & 1 & -3 \\ 0 & -3 & 3 \\ 10 & 3 & 2 \end{bmatrix}$$

$$\underline{\mathbf{C}} = \begin{bmatrix} a & -b & c^2 \\ -b & a^3 & a^2\,b \\ c^2 & a^2\,b & a \end{bmatrix} \qquad \underline{\mathbf{d}} = \left\{\ 1 \quad 1 \quad -2 \quad 1\ \right\}$$

Taking

$$x = y = 2\,z = 2, \qquad a = 2 \quad b = 1 \quad c = 1$$

determine the following:

a) $\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{A}}$. Is the matrix symmetric?

b) $\underline{\mathbf{d}}\,\underline{\mathbf{I}}$

c) $(\underline{\mathbf{A}}\,\underline{\mathbf{B}})^{\mathrm{T}}$

d) $\det[\underline{\mathbf{C}}]$. Is matrix $\underline{\mathbf{C}}$ singular? Justify your answer.

e) Use Gauss-Jordan elimination to determine the inverse of matrix $[\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{A}}]$. Verify your answer using MATLAB.

f) $\|\gamma\,\underline{\mathbf{d}}^{\mathrm{T}}\|_2$ where $\gamma = 2$

g) $\underline{\mathbf{A}}^{\mathrm{T}}\,\underline{\mathbf{d}}^{\mathrm{T}}$

h) $\underline{\mathbf{A}}\,\underline{\mathbf{d}}^{\mathrm{T}}$

i) $\underline{\mathbf{d}}\,\underline{\mathbf{A}}^{\mathrm{T}}$

j) $\underline{\mathbf{d}}\,\underline{\mathbf{d}}^{\mathrm{T}}$

k) $\underline{\mathbf{d}}^{\mathrm{T}}\,\underline{\mathbf{d}}$

l) $\underline{\mathbf{B}} + \underline{\mathbf{A}}^{\mathrm{T}}$

m) $\underline{\mathbf{A}}\,\underline{\mathbf{I}}$

n) Eigenvalues of $\underline{\mathbf{C}}$

□

Problem A.4.

Solve this problem *by hand* (using Gauss-Jordan elimination) and verify your answers using MATLAB®. Solve the given system of equations:

(a)
$$\begin{aligned} 3\,x_1 + 2\,x_2 &= 18 \\ 18\,x_1 + 17\,x_2 &= 123 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 4\,x_2 + 2\,x_3 &= 81 \\ -\,4\,x_1 + 25\,x_2 + 4\,x_3 &= -153 \\ 2\,x_1 + 4\,x_2 + 24\,x_3 &= 7 \end{aligned}$$

(c)
$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ -27 \\ 5 \end{Bmatrix}$$

□

Problem A.5.

Consider the following linear system of equations:

$$\begin{bmatrix} \epsilon & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

a) For what values of ϵ, does the system of equations have a solution? Verify your answer using MATLAB® and Gauss-Jordan elimination.

b) For what values of ϵ, the system of equations does not have a finite number of solutions? Justify your answer using MATLAB®.

□

Problem A.6.

Create the MATLAB® function `cadcourse` and script file, that will take as input the following matrices, vectors, and variables:

$$\underline{\mathbf{A}} = \begin{bmatrix} -y^2 & -y^2 & 1 & 0 \\ x\,z\,y & x\,z\,y & z^2 & y \\ -y & -y & z & 10\,z \end{bmatrix} \qquad \underline{\mathbf{B}} = \begin{bmatrix} -3 & 2 & -1 \\ -2 & 1 & -3 \\ 0 & -3 & 3 \\ 10 & 3 & 2 \end{bmatrix}$$

$$\underline{\mathbf{C}} = \begin{bmatrix} a & -b & c^2 \\ -b & a^3 & a^2\,b \\ c^2 & a^2\,b & a \end{bmatrix} \qquad \underline{\mathbf{d}} = \left\{ \begin{matrix} 1 & 1 & -2 & 1 \end{matrix} \right\}$$

$$x = y = 2\,z = 2, \qquad a = 2 \quad b = 1 \quad c = 1$$

and the function should output the following:

a) $\underline{\mathbf{Z1}} = \underline{\mathbf{A}}^{\mathrm{T}}\underline{\mathbf{A}}$

b) $\underline{\mathbf{Z2}} = (\underline{\mathbf{A}}\,\underline{\mathbf{B}})^{\mathrm{T}}$

c) $mm = \det[\underline{\mathbf{C}}]$

d) $nn = \|\underline{\mathbf{d}}\|_2$

e) $\underline{\mathbf{Z3}} = \underline{\mathbf{d}}^{\mathrm{T}}\,\underline{\mathbf{d}}$

f) $\underline{\mathbf{Phi}}$ = Eigenvalues of $\underline{\mathbf{C}}$. ($\underline{\mathbf{Phi}}$ must be a row vector and not a diagonal matrix.)

Provide a printout of the M-Files. Print the output to a different file and print this output-file.

□

Problem A.7.

Solve by hand and use the MATLAB function to evaluate the following one-dimensional integrals using Gauss quadrature. Provide a convergence plot using the first six Gauss rules for one-dimensional domains: one-, two-, three-, four-, five-, and six- point.

a)

$$\int_2^4 \left(x^2+1\right)\, \cos x\, dx$$

b)

$$\int_0^3 \left(3\, e^x + x^2 + \frac{1}{x+2}\right)\, \cos x\, dx$$

□

Problem A.8.

Solve by hand and use the MATLAB function to evaluate the following two-dimensional quadrilateral regions integrals using Gauss quadrature. Provide a convergence plot using the first five Gauss product rules for quadrilateral domains.

a)

$$\int_0^1 \int_0^1 e^{-\left(x^2+y^2\right)}\, dy\, dx$$

b)

$$\int_0^1 \int_0^1 \tan^{-1}(x\, y)\, dy\, dx$$

c)

$$\int_\pi^{2\pi} \int_0^\pi \left(\sin x + \cos y\right)\, dx\, dy$$

□

Problem A.9.

Solve by hand and use the MATLAB function to evaluate the following two-dimensional triangular regions integrals using Gauss quadrature. Provide a convergence plot using the first five Gauss rules for triangular domains: one-, three- (+3 or −3), four-, six- (+6 or −6), and seven- point.

a)

$$\int_0^1 \int_0^{1-\eta} \left(1 - \xi - \frac{1}{\sqrt{34}}\,\eta\right)\left(1 + \xi - \frac{1}{\sqrt{34}}\,\eta\right) |\underline{\mathbf{J}}|\; d\xi\, d\eta$$

$$\underline{\mathbf{J}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

b)

$$\int_0^1 \int_0^{1-\eta} \left(1 + \xi - \frac{1}{\sqrt{34}}\,\eta\right) |\underline{\mathbf{J}}|\; d\xi\, d\eta$$

$$\underline{\mathbf{J}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

□

Appendix B
Overview Mohr's Circle

B.1 Mohr's Circle in Stress Analysis

In order to better understand the Mohr's cirlce, let the stress vector $\underline{\mathbf{T}}^{(\mathrm{n})}$ on an arbitrary plane at **P** be decomposed into a normal component to the plane (σ_{nn}) and a shear component which acts in the plane (σ_{tt}). As previously shown:

$$\sigma_{\mathrm{nn}} = \underline{\mathbf{T}}^{(\mathrm{n})}\,\underline{\hat{\mathbf{n}}}$$
$$\sigma_{\mathrm{tt}} = \underline{\mathbf{T}}^{(\mathrm{n})}\cdot\underline{\mathbf{T}}^{(\mathrm{n})} - \sigma_{\mathrm{nn}}^2$$

Let the state of stress at **P** be referenced to principal planes and the principal stresses be ordered according to

$$\sigma_1 > \sigma_2 > \sigma_3 \tag{B.1}$$

As a consequence,

$$\begin{aligned} \sigma_{\mathrm{nn}} &= \sigma_1\,n_1^2 + \sigma_2\,n_2^2 + \sigma_3\,n_3^2 \\ \sigma_{\mathrm{nn}}^2 + \sigma_{\mathrm{tt}}^2 &= \sigma_1^2\,n_1^2 + \sigma_2^2\,n_2^2 + \sigma_3^2\,n_3^2 \end{aligned} \tag{B.2}$$

which along with the condition

$$n_1^2 + n_2^2 + n_3^2 = 1$$

provide us with three equations for the three direction cosines n_1, n_2, and n_3. For simplicity let:

$$\sigma = \sigma_{\mathrm{nn}} \qquad \text{and} \qquad \tau = \sigma_{\mathrm{tt}}$$

Solving the three equations in terms of n_1, n_2, and n_3 we get

$$n_1^2 = \frac{(\sigma-\sigma_2)\,(\sigma-\sigma_3) - \tau^2}{(\sigma_1-\sigma_2)\,(\sigma_1-\sigma_3)} \tag{B.3}$$

$$n_2^2 = \frac{(\sigma-\sigma_3)\,(\sigma-\sigma_1) - \tau^2}{(\sigma_2-\sigma_3)\,(\sigma_2-\sigma_1)} \tag{B.4}$$

$$n_3^2 = \frac{(\sigma-\sigma_1)\,(\sigma-\sigma_2) - \tau^2}{(\sigma_3-\sigma_1)\,(\sigma_3-\sigma_2)} \tag{B.5}$$

Note that the principal stresses

$$\sigma_1 > \sigma_2 > \sigma_3$$

are known and σ and τ are functions of the direction cosines n_i. Our intention here is to interpret these

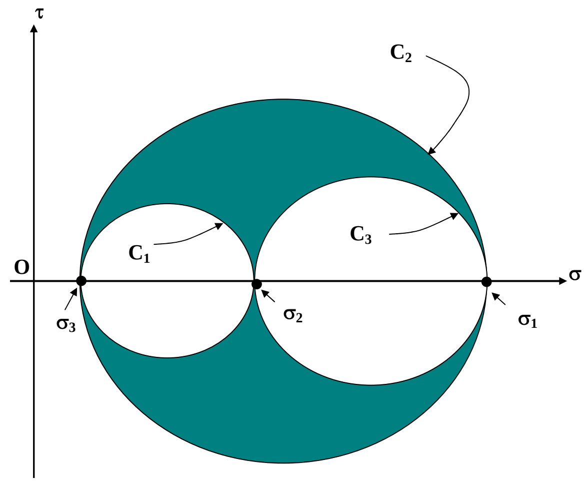


Figure B.1: Typical Mohr's circles for a given state of stress.

equations graphically by representing conjugate pairs of σ-τ values, which satisfy Eqs. (B.3) to (B.5), as a point in the stress plane having σ as abscissa and τ as ordinate (see Fig. B.1).

To develop this interpretation of the three-dimensional stress state in terms of σ and τ, first note that the denominator of Eq. (B.3) is positive since both $\sigma_1 - \sigma_2 > 0$ and $\sigma_1 - \sigma_3 > 0$ (from Eq. (B.1)), and since $n_1^2 > 0$, the numerator of the right-hand side satisfies the relationship:

$$(\sigma - \sigma_2)\ (\sigma - \sigma_3) - \tau^2 \geq 0 \tag{B.6}$$

For the case in which the equality sign holds, this equation may be rewritten, after some simple algebraic manipulations, to read

$$\left[\sigma - \left(\frac{\sigma_2 + \sigma_3}{2}\right)\right]^2 + \tau^2 = \left[\left(\frac{\sigma_2 - \sigma_3}{2}\right)\right]^2 \tag{B.7}$$

which is the equation of a circle in the σ-τ plane, with its center at the point

$$C(\sigma, \tau) = C\left(0, \frac{\sigma_2 + \sigma_3}{2}\right)$$

and a radius of

$$R = \frac{\sigma_2 - \sigma_3}{2}$$

We label this circle C_1 and it is shown in Fig. B.1. For the case in which the inequality sign holds for Eq. B.6, we observe that conjugate pairs of values of σ and τ which satisfy this relationship result in stress points having coordinates exterior to circle C_1. Thus, combinations of σ and τ which satisfy Eq. (B.3) lie on, or exterior to, circle C_1 in Fig. B.1.

Examining Eq. (B.4), we note that the denominator is negative since $\sigma_2 - \sigma_3 > 0$ and $\sigma_2 - \sigma_1 < 0$ (from Eq. (B.1)). The direction cosines are real numbers, so that $n_2^2 > 0$ and thus the numerator of the right-hand side satisfies the relationship:

$$(\sigma - \sigma_3)\ (\sigma - \sigma_1) - \tau^2 \leq 0 \tag{B.8}$$

For the case of the equality sign defines the circle

$$\left[\sigma - \left(\frac{\sigma_1 + \sigma_3}{2}\right)\right]^2 + \tau^2 = \left[\left(\frac{\sigma_1 - \sigma_3}{2}\right)\right]^2 \tag{B.9}$$

which is the equation of a circle in the σ-τ plane, with its center at the point

$$C(\sigma, \tau) = C\left(0, \frac{\sigma_1 + \sigma_3}{2}\right)$$

and a radius of

$$R = \frac{\sigma_1 - \sigma_3}{2}$$

This circle is labeled C_2 in Fig. B.1, and the stress points which satisfy the inequality of Eq. (B.8) lie interior to it.

Following the same general procedure, we can rearrange Eq. (B.5) into an expression from which we extract the equation of the third circle C_3 in Fig. B.1. Examining Eq. (B.5), we note that the denominator is positive since $\sigma_3 - \sigma_1 < 0$ and $\sigma_3 - \sigma_2 < 0$ (from Eq. (B.1)). The direction cosines are real numbers, so that $n_3^2 > 0$ and thus the numerator of the right-hand side satisfies the relationship:

$$(\sigma - \sigma_1)\ (\sigma - \sigma_2) - \tau^2 \geq 0 \tag{B.10}$$

For the case of the equality sign defines the circle

$$\left[\sigma - \left(\frac{\sigma_1 + \sigma_2}{2}\right)\right]^2 + \tau^2 = \left[\left(\frac{\sigma_1 - \sigma_2}{2}\right)\right]^2 \tag{B.11}$$

which is the equation of a circle in the σ-τ plane, with its center at the point

$$C(\sigma, \tau) = C\left(0, \frac{\sigma_1 + \sigma_2}{2}\right)$$

and a radius of

$$R = \frac{\sigma_1 - \sigma_2}{2}$$

This circle is labeled C_3 in Fig. B.1. Admissible stress points in the σ-τ which satisfy the inequality of Eq. (B.10) lie on or exterior to this circle.

The three circles defined above and shown in Fig. B.1 are called Mohr's circles for a stress point. All possible pairs of values of σ and τ at **P** which satisfy Eqs. (B.3) to (B.5) lie on these circles or within the shaded areas enclosed by them. We see that the sign of the shear component is arbitrary so sometimes only the top half of the symmetrical circle diagram is drawn. In addition, it should be clear from the Mohr's circles diagram that the maximum shear stress value at **P** is the radius of circle C_2.

B.2 Procedure for the Mohr's Circle

Six specific steps can be identified in using Mohr's circle approach:

1. Calculate the radius and center

2. Draw the circle and locate the points

3. Calculate all angles

4. Determine the normal and shear stresses on the inclined plane(s)

5. Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

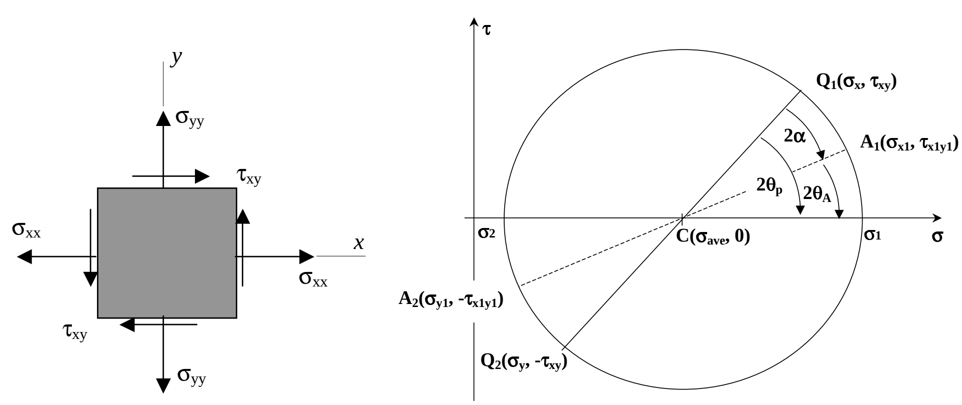
6. Show all results on sketches of properly oriented elements

Step I. Calculate the radius and center of the Mohr's circle

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{xx}} + \sigma_{\text{yy}}}{2} \qquad \sigma_{\text{diff}} = \frac{\sigma_{\text{xx}} - \sigma_{\text{yy}}}{2} \tag{B.12}$$

$$R = \sqrt{\tau_{\text{xy}}^2 + \sigma_{\text{diff}}^2} \qquad C = C(\sigma_{\text{ave}}, 0) \tag{B.13}$$

Step II. Draw the circle and locate all points
Note that the positive convention.



(a) Positive stresses on a two dimensional element

(b) Mohr's circle for plane stress in the xy plane. Point **Q** is the location of the defined state of stress. Point **A** is the location in the Mohr's circle where information is desired.

Figure B.2: Sketch of the given information on the Mohr's circle.

Locate the following points:

$$Q_1 = Q_1(\sigma_{\text{xx}}, \tau_{\text{xy}}) \qquad Q_2 = Q_2(\sigma_{\text{yy}}, -\tau_{\text{xy}}) \qquad C = C(\sigma_{\text{ave}}, 0) \tag{B.14}$$

Step III. Calculate angles:
(All measured positive clockwise from $\overline{Q_1 C}$)

First calculate $2\,\theta'_p$,

$$\tan 2\,\theta'_p = \frac{\tau_{\text{xy}}}{\sigma_{\text{diff}}} \qquad 2\,\theta'_p = \tan^{-1}\left\{\frac{\tau_{\text{xy}}}{\sigma_{\text{diff}}}\right\} \tag{B.15}$$

Now consider the location of Q_1:

CASE A: $Q_1 \rightarrow$ first quadrant $(\sigma_{\text{xx}} > 0, \quad \tau_{\text{xy}} > 0)$ $\quad 2\,\theta_p = 2\,\theta'_p$

CASE B: $Q_1 \rightarrow$ second quadrant $(\sigma_{\text{xx}} < 0, \quad \tau_{\text{xy}} > 0)$ $\quad 2\,\theta_p = 180^\circ - \left|2\,\theta'_p\right|$

CASE C: $Q_1 \rightarrow$ third quadrant $(\sigma_{\text{xx}} < 0, \quad \tau_{\text{xy}} < 0)$ $\quad 2\,\theta_p = 180^\circ + \left|2\,\theta'_p\right|$

CASE D: $Q_1 \rightarrow$ fourth quadrant $(\sigma_{\text{xx}} > 0, \quad \tau_{\text{xy}} < 0)$ $\quad 2\,\theta_p = 360^\circ - \left|2\,\theta'_p\right|$

Principal stresses act on an element inclined at an angle θ_p are

$$\theta_p \quad = \quad \frac{1}{2}\,(2\,\theta_p) \tag{B.16}$$

Minimum/maximum in-plane shear stresses act on an element inclined at an angle θ_s

$$2\,\theta_s \quad = \quad 2\,\theta_p \pm 90^\circ \tag{B.17}$$

Note that at a rotation of $2\,\theta_s = 2\,\theta_p + 90^\circ$ from $\overline{Q_1 C}$ the value of the in-plane shear stress is negative thus it gives the minimum shear stress, the maximum is obtained by taking $2\,\theta_s = 2\,\theta_p - 90^\circ$. In short,

Maximum Shear Stress: $2\,\theta_s \;= 2\,\theta_p - 90^\circ \rightarrow \quad \tau_{\text{max}}$

Minimum Shear Stress: $2\,\theta_s \;= 2\,\theta_p + 90^\circ \rightarrow \quad \tau_{\text{min}}$

If $2\,\theta_s > 360^\circ$ then let

$$2\,\theta_s \quad = \quad (360^\circ - 2\,\theta_p) \pm 90^\circ \tag{B.18}$$

Transformed stresses act on an element inclined at an angle α

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha$$

If $2\,\theta_A < 0$, then let

$$2\,\theta_A = (2\,\theta_p - 2\,\alpha) + 360^\circ \qquad \text{(to measure clockwise)}$$

Note: When working in the x–y and y–z plane, all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element. When working in the x–z plane, all angles are measured positive clockwise in the Mohr's circle and in the rotation of the differential element. Also, note that $2\,\theta_p$ is measured from $\overline{Q_1 C}$ to positive σ-axis.

Step IV. Determine the normal and shear stresses on the inclined plane

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} = \sigma_{\text{ave}} + R\cos(2\,\theta_A) \tag{B.19}$$

$$\sigma_{y_1} = \sigma_{\text{ave}} + R\cos(2\,\theta_A + 180^\circ) = \sigma_{\text{ave}} - R\cos(2\,\theta_A) \tag{B.20}$$

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1 y_1} = R\sin(2\,\theta_A) \tag{B.21}$$

Step V. Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \to 2\,\theta_A = 0^\circ$, therefore the principal normal stresses are found as follows

$$\begin{aligned} \lambda_1 &= \sigma_{\text{ave}} + R \\ \lambda_2 &= \sigma_{\text{ave}} - R \\ \lambda_3 &= 0 \end{aligned}$$

The principal stresses are chosen as:

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3]$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3]$$

Thus the principal stresses are given as follows

$$\sigma_1 > \sigma_2 > \sigma_3$$

The maximum and minimum normal stresses acting on an element inclined at an angle θ_p are

$$\sigma_{\max} = \sigma_1 \tag{B.22}$$
$$\sigma_{\min} = \sigma_3 \tag{B.23}$$

The in-plane maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{\max}\Big|_{\text{in-plane}} = R = \frac{\sigma_1 - \sigma_2}{2} \tag{B.24}$$

The overall maximum shear stress acting on an element inclined at an angle θ_s is

$$\tau_{\max} = \left|\frac{\sigma_{\max} - \sigma_{\min}}{2}\right| \tag{B.25}$$

Step VI. Show all results on sketches of properly oriented elements

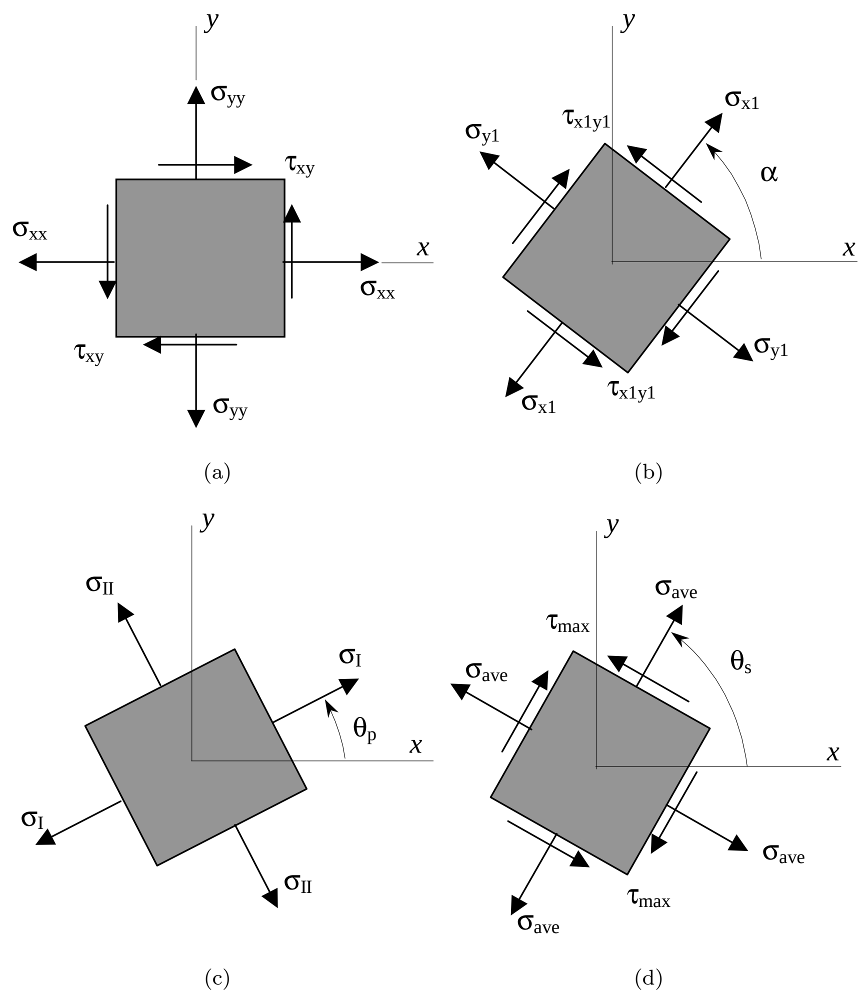


Figure B.3: a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum in-plane shear stresses.

B.3 Mohr's Circle in Three-Dimensional Stresses

Mohr's circle can be generated for triaxial stress state, but it is often unnecessary. In most cases it is not necessary to know the orientations of the principal stresses but it is sufficient to know their values. Thus, Eq. (2.26) is usually all that is needed:

$$\lambda^3 - I_{\sigma_1}\,\lambda^2 + I_{\sigma_2}\,\lambda - I_{\sigma_3} = 0$$

In such cases, the Mohr's circle will consist of three circles, two externally tangent and inscribed within the third circle. Recall: that the principal stresses chosen such that $\sigma_1 > \sigma_2 > \sigma_3$.

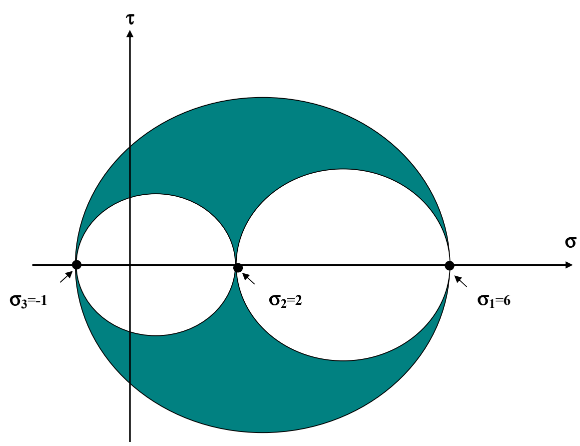
Opposed to the case of plane stress, where $\lambda_3 = 0$, for general state of stress this might not be the case. As for an example, at a point the state of stress is

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and we would like to draw the Mohr's circle. First of all, note that in the above equation all are principal stress thus

$$\sigma_1 = 6 \qquad \sigma_2 = 2 \qquad \sigma_3 = -1$$

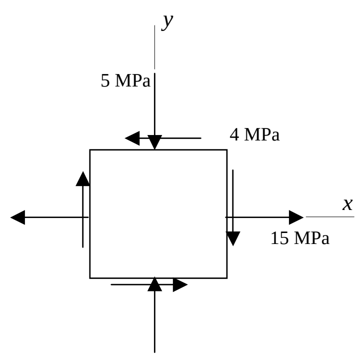
The principal stresses are point in the σ-axis. Thus the Mohr's Circle is



Example B.1.

An element in plane stress at the surface of a wing panel is subjected to the following stresses

$$\underline{\boldsymbol{\sigma}} = \begin{bmatrix} 15 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$



Considering only the in-plane stresses and using Mohr's Circle determine:

1. Stresses acting on a element inclined at an angle $\theta = 40°$.
2. Principal stresses and maximum shear stresses.

B.1a) Calculate the radius and center of the Mohr's circle

The average stress acting on the differential element will be:

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{xx}} + \sigma_{\text{yy}}}{2} = \frac{(15) + (-5)}{2} \text{ MPa} = 5 \text{ MPa}$$

The difference in stresses acting on the differential element will be:

$$\sigma_{\text{diff}} = \frac{\sigma_{\text{xx}} - \sigma_{\text{yy}}}{2} = \frac{(15) - (-5)}{2} \text{ MPa} = 10 \text{ MPa}$$

The radius of the inplane state of stress is:

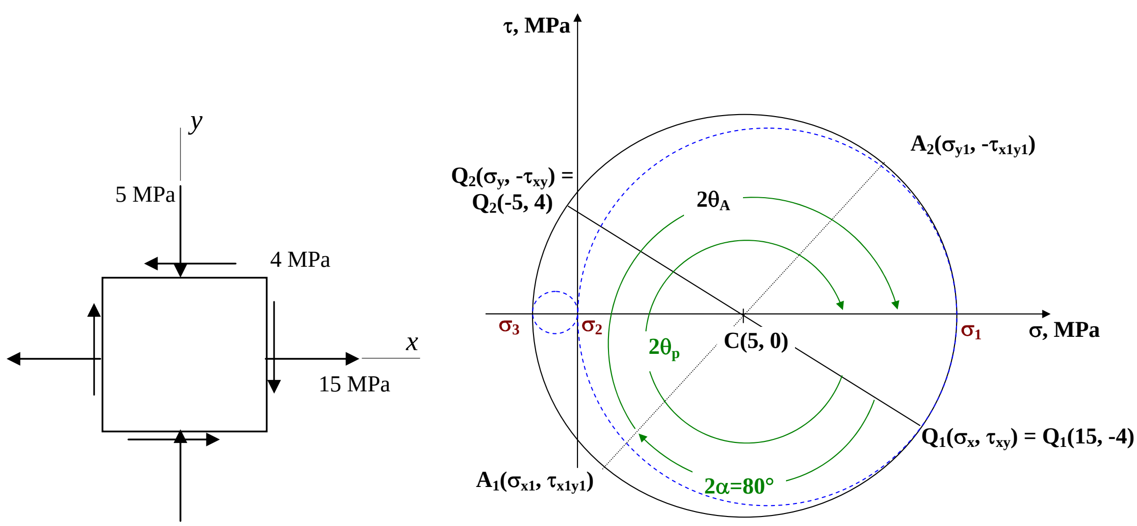
$$R = \sqrt{\tau_{\text{xy}}^2 + \sigma_{\text{diff}}^2} = \sqrt{(-4)^2 + (10)^2} \text{ MPa} = 10.7703 \text{ MPa}$$

The center of the circle is:

$$C = C(\sigma_{\text{ave}}, 0) = C(5 \text{ MPa}, 0)$$

B.1b) Draw the circle and locate all points

$$Q_1 = Q_1(15, -4) \qquad Q_2 = Q_2(-5, 4) \qquad C = C(5, 0)$$



(a) stresses on a two dimensional element (b) Mohr's circle for plane stress in the x-y plane

Figure B.4: Mohr's circle for plane stress in the x-y plane.

B.1c) Calculate angles:

Principal stresses act on an element inclined at an angle θ_p

$$2\,\theta_p' = \tan^{-1}\left[\frac{\tau_{\text{xy}}}{\sigma_{\text{diff}}}\right] = \tan^{-1}\left[\frac{(-4)}{(10)}\right] = -21.8014^\circ$$

$$2\theta_p = 360^\circ - \left|2\theta_p'\right| = 338.199^\circ$$

$$\theta_p = 169.099^\circ$$

Note that we are in CASE D (See Appendix B) because $2\,\theta_p$ is measured from $\overline{Q_1C}$ to positive σ-axis. Minimum and maximum inplane shear stresses act on an element inclined at an angle θ_s

$$2\theta_s = 2\theta_p \pm 90^\circ = 338.199^\circ \pm 90^\circ$$

$$\theta_s = \theta_p \pm 45^\circ = 169.099^\circ \pm 45^\circ$$

Transformed stresses act on an element inclined at an angle $\alpha = 40^\circ$

$$2\,\theta_A = 2\,\theta_p - 2\,\alpha = 338.199^\circ - 80^\circ = 258.199^\circ$$

Note that all angles are measured positive clockwise in the Mohr's circle but are positive counterclockwise in the rotation of the differential element.

B.1d) Determine the normal and shear stresses on the inclined plane(s)

The normal stresses acting on an element inclined at an angle α are

$$\sigma_{x_1} = \sigma_{\text{ave}} + R\,\cos(2\,\theta_A) = (5) + (10.7703)\,\cos(258.199^\circ) = 2.79725 \text{ MPa}$$

$$\sigma_{y_1} = \sigma_{\text{ave}} - R\,\cos(2\,\theta_A) = (5) - (10.7703)\,\cos(258.199^\circ) = 7.20275 \text{ MPa}$$

The shear stresses acting on an element inclined at an angle α are

$$\tau_{x_1y_1} = R\,\sin(2\,\theta_A) = (10.7703)\,\sin(258.199^\circ) = -10.5427 \text{ MPa}$$

B.1e) Determine the maximum normal stresses, the in-plane maximum shear and the overall maximum shear

Note that when calculating principal stresses $2\,\alpha = 2\,\theta_p \rightarrow 2\,\theta_A = 0^\circ$, therefore the principal stresses are

$$\lambda_1 = \sigma_{\text{ave}} + R = (5) + (10.7703) = 15.7703 \text{ MPa}$$
$$\lambda_2 = \sigma_{\text{ave}} - R = (5) - (10.7703) = -5.77033 \text{ MPa}$$
$$\lambda_3 = 0 \text{ MPa}$$

The principal stresses are chosen as:

$$\sigma_1 = \max[\lambda_1, \lambda_2, \lambda_3] = 15.7703 \text{ MPa}$$
$$\sigma_3 = \min[\lambda_1, \lambda_2, \lambda_3] = -5.77033 \text{ MPa}$$
$$\sigma_2 = 0$$

Note $\sigma_1 > \sigma_2 > \sigma_3$.

The maximum and minimum normal stresses acting on an element inclined at an angle θ_p are

$$\sigma_{\max} = \sigma_1 = 15.7703 \text{ MPa}$$
$$\sigma_{\min} = \sigma_3 = -5.77033 \text{ MPa}$$

The in-plane maximum shear stresses acting on an element inclined at an angle θ_s are

$$\tau_{\max}\Big|_{\text{in-plane}} = R = \frac{\sigma_1 - \sigma_2}{2} = 10.7703 \text{ MPa}$$

The maximum inplane shear stresses will be:

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = 7.885 \text{ MPa}$$
$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = 2.885 \text{ MPa}$$
$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = 10.770 \text{ MPa}$$

The overall maximum shear stress acting on an element inclined at an angle θ_s is

$$\tau_{\max} = \left|\frac{\sigma_{\max} - \sigma_{\min}}{2}\right| = 10.7703 \text{ MPa}$$

B.1f) Show all results on sketches of properly oriented elements

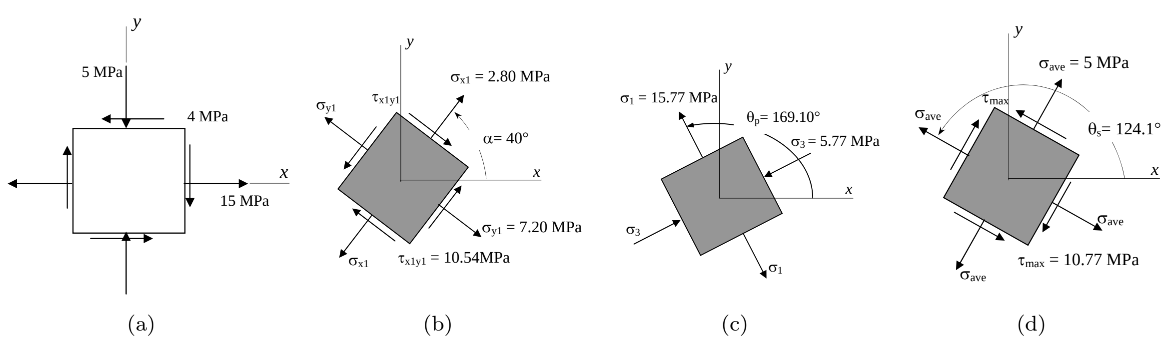


Figure B.5: a) Stresses acting on an element in plane stress. b) Stresses acting on an element oriented at an angle $\theta = \alpha$. c) Principal normal stresses. d) Maximum in-plane shear stresses.

End Example □

B.4 Final Remarks

Mohrs circle displays in a graphical manner many important features characterizing the state of stress at a point:

1. The stress components acting on two mutually orthogonal faces are represented by two diametrically opposite points on Mohrs circle. Since the center of the circle is on the horizontal axis, the shear stresses on those two faces are equal in magnitude and opposite in sign, as required by the principal of reciprocity of shear stresses.

2. The faces corresponding to the principal stress orientation are represented by the points at the intersection of Mohrs circle with the horizontal axis. Clearly, the shear stresses vanish on the principal stress faces.

3. The faces on which the maximum shear stresses occurs are represented by the points at the intersection of Mohrs circle with a vertical line passing through its center. It is clear that the magnitude of the maximum shear stress equals the radius of Mohrs circle. The angle between the principal stress directions and those of the face of maximum shear is 45 degrees. Finally, the normal stresses acting on the faces of maximum shear equal the average of the principal stresses,

4. Finally, it must be noted that all the points on Mohrs circle represent the same state of stress at one point of the solid. Of course, this state of stress is represented by stress components that depend on the orientation of the face on which they act. Mohrs circle is a graphical representation of all the stress components corresponding to a single state of stress.

B.5 References

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B.6 Suggested Problems

Problem B.1.

□

Appendix C
Strain-Gradient Matrix Expressions

In chapter 2 expressions for the Green-Lagrange strain components were given as

$$\begin{aligned}
\epsilon_1 &= e_{xx} = g_1 + \frac{1}{2}\left(g_1^2 + g_2^2 + g_3^2\right)\\
\epsilon_2 &= e_{yy} = g_5 + \frac{1}{2}\left(g_4^2 + g_5^2 + g_6^2\right)\\
\epsilon_3 &= e_{zz} = g_9 + \frac{1}{2}\left(g_7^2 + g_8^2 + g_9^2\right)\\
\epsilon_4 &= 2\,e_{yz} = g_6 + g_8 + g_4\,g_7 + g_5\,g_8 + g_6\,g_9\\
\epsilon_5 &= 2\,e_{xz} = g_3 + g_7 + g_1\,g_7 + g_2\,g_8 + g_3\,g_9\\
\epsilon_6 &= 2\,e_{xy} = g_2 + g_4 + g_1\,g_4 + g_2\,g_5 + g_3\,g_6
\end{aligned}$$

and were rewritten in the quadratic form

$$\epsilon_i = \underline{\mathbf{h}}_i^{\mathrm{T}}\,\underline{\mathbf{g}} + \frac{1}{2}\,\underline{\mathbf{g}}^{\mathrm{T}}\,\underline{\mathbf{H}}_i\,\underline{\mathbf{g}}$$

where the displacements gradients in vector form are

$$\underline{\mathbf{g}}^{\mathrm{T}} = \left\{\begin{matrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & g_9 \end{matrix}\right\}$$

The vectors $\underline{\mathbf{h}}_i$'s are sparse 9×1 vectors:

$$\begin{aligned}
\underline{\mathbf{h}}_1^{\mathrm{T}} &= \left\{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right\}\\
\underline{\mathbf{h}}_2^{\mathrm{T}} &= \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{matrix}\right\}\\
\underline{\mathbf{h}}_3^{\mathrm{T}} &= \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right\}\\
\underline{\mathbf{h}}_4^{\mathrm{T}} &= \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{matrix}\right\}\\
\underline{\mathbf{h}}_5^{\mathrm{T}} &= \left\{\begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}\right\}\\
\underline{\mathbf{h}}_6^{T} &= \left\{\begin{matrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right\}
\end{aligned}$$

The matrices $\underline{\mathbf{H}}_i$'s are very sparse 9×9 symmetric matrices:

$$\underline{\mathbf{H}}_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \underline{\mathbf{H}}_2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\underline{\mathbf{H}}_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \quad \underline{\mathbf{H}}_4 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{bmatrix}$$

$$\underline{\mathbf{H}}_5 = \begin{bmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0 \end{bmatrix} \quad \underline{\mathbf{H}}_6 = \begin{bmatrix} 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Appendix D
Distance Perpendicular to the Contour

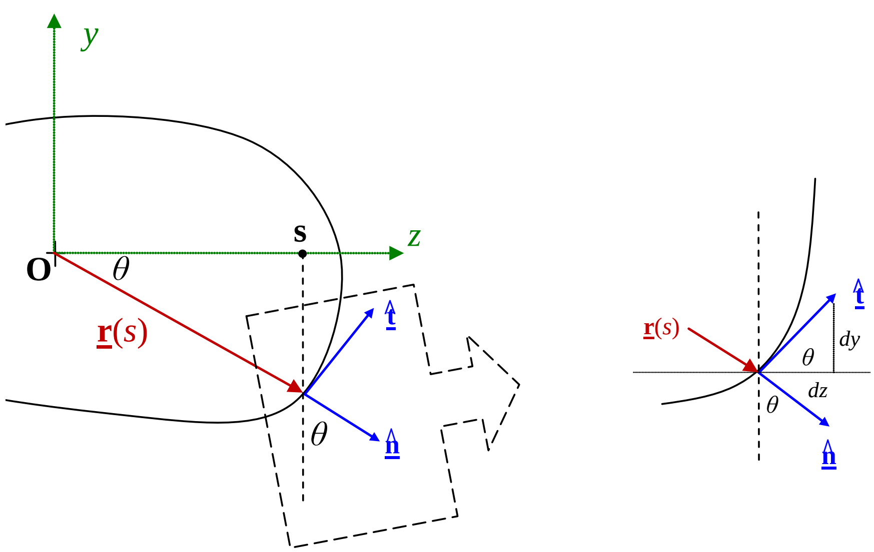

Figure D.1: Distance perpendicular from a reference point to the contour at s.

In order to derive the distance perpendicular from a reference point to the contour, let us consider Fig. D.1. The reference point **O** can be any point within or outside the cross-sectional domain. The distance from point **O** to the location s of the contour is given by:

$$\underline{\mathbf{r}}(s) = y(s)\,\hat{\underline{\mathbf{j}}} + z(s)\,\hat{\underline{\mathbf{k}}}$$

where $y(s)$ and $z(s)$ are the parametric equation in terms of the distance s along the contour. The unit normal at the point of interest on the contour is:

$$\underline{\mathbf{n}} = -\cos\theta\,\hat{\underline{\mathbf{j}}} + \sin\theta\,\hat{\underline{\mathbf{k}}}$$

Now the magnitude of the distance perpendicular from a reference point to the contour can be found taking the dot product:

$$r(s) = \underline{\mathbf{r}} \cdot \underline{\mathbf{n}} = \left\{ \begin{array}{cc} y(s) & z(s) \end{array} \right\} \left\{ \begin{array}{c} -\cos\theta \\ \sin\theta \end{array} \right\} = -y\,\cos\theta + z\,\sin\theta$$

At the contour,

$$\cos\theta = \frac{dz}{ds}, \qquad \sin\theta = \frac{dy}{ds}$$

Hence,

$$r(s) = -y\,\frac{dz}{ds} + z\,\frac{dy}{ds}$$

Or more generally written as:

$$r_i(s_i) = -\frac{\partial z_i}{\partial s_i}\, y_i(s_i) + \frac{\partial y_i}{\partial s_i}\, z_i(s_i) \qquad i = 1, 2, \ldots, n \tag{D.1}$$

where i represents the various sections of the cross-section.