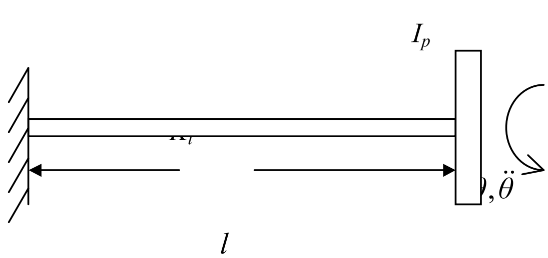
CHAPTER 2

TORSIONAL VIBRATIONS

Torsional vibrations is predominant whenever there is large discs on relatively thin shafts (e.g. flywheel of a punch press). Torsional vibrations may original from the following forcings (i) inertia forces of reciprocating mechanisms (such as pistons in IC engines) (ii) impulsive loads occurring during a normal machine cycle (e.g. during operations of a punch press) (iii) shock loads applied to electrical machinery (such as a generator line fault followed by fault removal and automatic closure) (iv) torques related to gear mesh frequencies, turbine blade passing frequencies, etc. For machines having massive rotors and flexible shafts (where the system natural frequencies of torsional vibrations may be close to, or within, the source frequency range during normal operation) torsional vibrations constitute a potential design problem area. In such cases designers should ensure the accurate prediction of machine torsional frequencies and frequencies of any torsional load fluctuations should not coincide with the torsional natural frequencies. Hence, the determination of torsional natural frequencies of the system is very important.

2.1 Simple System with Single Rotor Mass

Consider a rotor system as shown Figure 2.1(a). The shaft is considered as massless and it provides torsional stiffness only. The disc is considered as rigid and has no flexibility. If an initial disturbance is given to the disc in the torsional mode and allow it to oscillate its own, it will execute the free vibrations as shown in Figure 2.2. It shows that rotor is spinning with a nominal speed of ω and excuting torsional vibrations, $\theta(t)$, due to this it has actual speed of $(\omega + \dot{\theta}(t))$. It should be noted that the spinning speed remains same however angular velocity due to torsion have varying direction over a period. The oscillation will be simple harmonic motion with a unique frequency, which is called the torsional natural frequency of the rotor system.



Fixed end

Figure 2.1a A single-mass cantilever rotor system

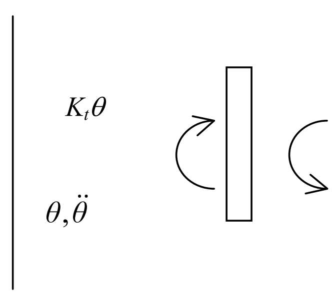


Figure 2.1(b) Free body diagram of disc

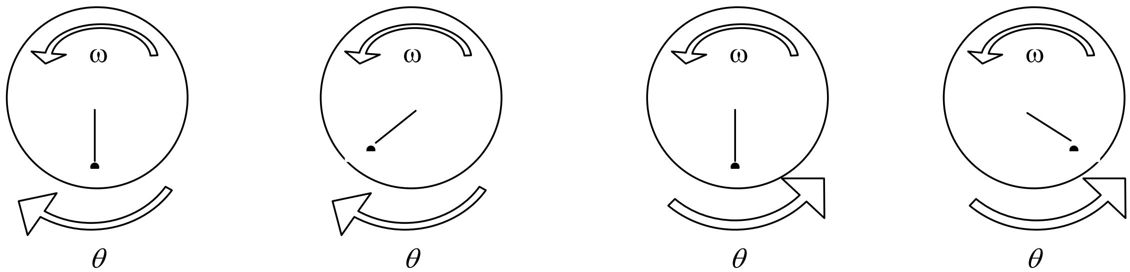


Figure 2.2 Torsional vibrations of a rotor

From the theory of torsion of shaft, we have

$$K_t = \frac{T}{\theta} = \frac{GJ}{l} \tag{1}$$

where, K_t is the torsional stiffness of shaft, I_p is the *rotor* polar moment of inertia, kg-m^2, J is the shaft polar second moment of area, l is the length of the shaft and θ is the angular displacement of the rotor. From the free body diagram of the disc as shown in Figure 2.1(b)

$$\sum \text{External torque of disc} = I_p\ddot{\theta} \quad \Rightarrow \quad -K_t\theta = I_p\ddot{\theta} \tag{2}$$

Equation (2) is the equation of motion of the disc due to free torsional vibrations. The free (or natural) vibration has the simple harmonic motion (SHM). For SHM of the disc, we have

$$\theta(t) = \hat{\theta}\sin\omega_{nf}t \qquad \text{so that} \qquad \ddot{\theta} = -\omega_{nf}^2\hat{\theta}\sin\omega_{nf}t = -\omega_{nf}^2\theta \tag{3, 4}$$

where $\hat{\theta}$ is the amplitude of the torsional vibration and ω_{nf} is the torsional natural frequency. On substituting Eqs. (3) and (4) into Eq. (2), we get

$$-K_t\theta = I_p\left(-\omega_{nf}^2\theta\right) \qquad \text{or} \qquad \omega_{nf} = \sqrt{K_t / I_p} \tag{5}$$

2.2 A Two-Disc Torsional System

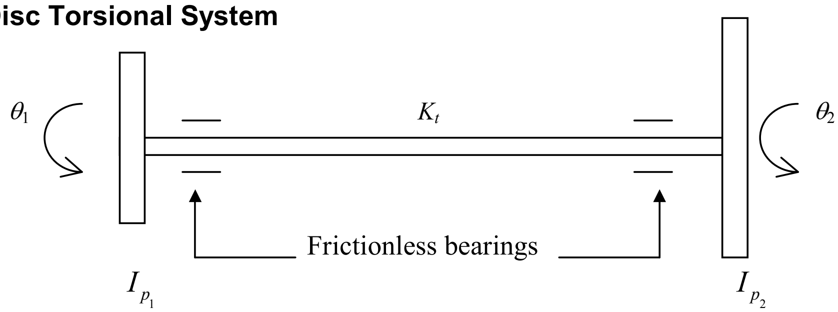


Figure 2.3 A two-disc torsional system

A two-disc torsional system is shown in Figure 2.3. In this case whole of the rotor is free to rotate as the shaft being mounted on frictionless bearings.

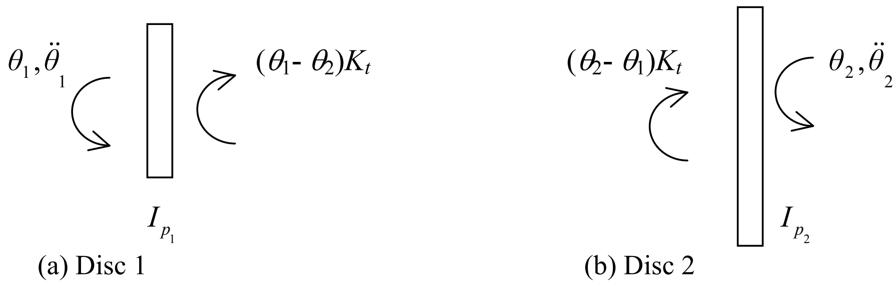


Figure 2.4 Free body diagram of discs

From the free body diagram in Figure 2.4(a)

$\sum$ External torque $= I_{p_1}\ddot{\theta}_1$ and $\sum$ External torque $= I_{p_2}\ddot{\theta}_2$

or $\quad -(\theta_1 - \theta_2)K_t = I_{p_1}\ddot{\theta}_1 \quad$ and $\quad -(\theta_2 - \theta_1)K_t = I_{p_2}\ddot{\theta}_2$

or $\quad I_{p_1}\ddot{\theta}_1 + K_t\theta_1 - K_t\theta_2 = 0 \quad$ and $\quad I_{p_2}\ddot{\theta}_2 + K_t\theta_2 - K_t\theta_1 = 0 \qquad (2)$

For free vibration, we have SHM, so the solution will take the form

$$\ddot{\theta}_1 = -\omega_{nf}^2\theta_1 \quad \text{and} \quad \ddot{\theta}_2 = -\omega_{nf}^2\theta_1 \qquad (3)$$

Substituting equation (3) into equations (1) & (2), it gives

$$-I_{p_1}\omega^2\theta_1 + K_t\theta_1 - K_t\theta_2 = 0 \quad \text{and} \quad -I_{p_2}\omega_{nf}^2\theta_2 + K_t\theta_2 - K_t\theta_1 = 0$$

which can be assembled in a matrix form as

$$\begin{bmatrix} K_t - I_{p_1}\omega_{nf}^2 & -K_t \\ -K_t & K_t - I_{p_2}\omega_{nf}^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad [K]\{\theta\} = \{0\} \qquad (4, 5)$$

The non-trial solution of equation (5) is obtained by taking determinant of the matrix [K] as

$$|K| = 0$$

which gives

$$\left(K_t - I_{p_1}\omega_n^2\right)\left(K_t - I_{p_2}\omega_n^2\right) - K_t^2 = 0 \quad \text{or} \quad I_{p_1}I_{p_2}\omega_{nf}^4 - \left(I_{p_1} + I_{p_2}\right)K_t\omega_{nf}^2 = 0 \tag{6}$$

The roots of equation (6) are given as

$$\omega_{nf_1} = 0 \quad \text{and} \quad \omega_{nf_2} = \left[\left(I_{p_1} + I_{p_2}\right)K_t \big/ \left(I_{p_1}I_{p_2}\right)\right]^{0.5} \tag{7}$$

From equation (4) corresponding to first natural frequency for ω_{nf_1} = 0, we get

$$\theta_1 = \theta_2 \tag{8}$$

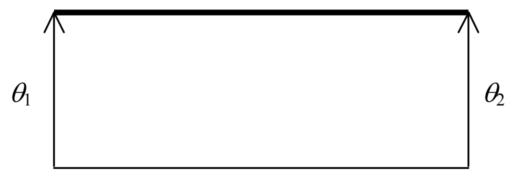


Figure 2.5 First mode shape

From Eq. (8) it can be concluded that, the first root of equation (6) represents the case when both discs simply rolls together in phase with each other as shown in Figure 2.5. It is the rigid body mode, which is of a little practical significance. This mode it generally occurs whenever the system has free-free end conditions (for example aeroplane during flying). From equation (4), for $\omega_{nf} = \omega_{nf_2}$, we get

$$\left(K_t - I_{p_1}\omega_{nf_2}^2\right)\hat{\theta}_1 - K_t\hat{\theta}_2 = 0 \quad \text{or} \quad \left[K_t - I_{p_1}\left(\left(I_{p_1} + I_{p_2}\right)\big/ I_{p_1}I_{p_2}\right)K_t\right]\hat{\theta}_1 - K_t\hat{\theta}_2 = 0$$

which gives relative amplitudes of two discs as

$$\hat{\theta}_1 \big/ \hat{\theta}_2 = -I_{p_2} \big/ I_{p_1} \tag{9}$$

The second mode shape (Eq. 9) represents the case when both masses vibrate in anti-phase with one another. Figure 2.6 shows mode shape of two-rotor system, showing two discs vibrating in opposite directions.

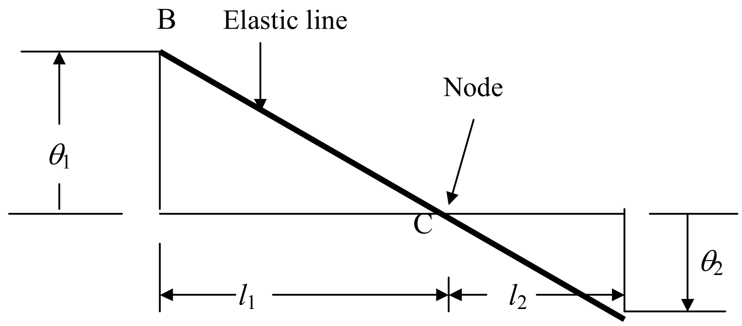


Figure 2.6 Second mode shape

From mode shapes, we have

$$\frac{\theta_1}{l_1} = \frac{\theta_2}{l_2} \Rightarrow \frac{\theta_1}{\theta_2} = \frac{l_1}{l_2} \tag{10}$$

Since both the masses are always vibrating in opposite direction, there must be a point on the shaft where torsional vibration is not taking place i.e. a torsional node. The location of the node may be established by treating each end of the real system as a separate single-disc cantilever system as shown in Figure 2.6. The node being treated as the point where the shaft is rigidly fixed. Since value of natural frequency is known (the frequency of oscillation of each of the single-disc system must be same), hence we write

$$\omega_{nf_2}^2 = K_{t_1} / I_{p_1} = K_{t_2} / I_{p_2} \tag{11}$$

where ω_{nf_2} is defined by equation (7), K_{t_1} and K_{t_2} are torsional stiffness of two (equivalent) single-rotor system, which can be obtained from equation (11), as

$$K_{t_1} = \omega_{nf_2}^2 I_{p_1} \quad \text{and} \quad K_{t_2} = \omega_{nf_2}^2 I_{p_2}$$

The length l_1 and l_2 then can be obtained by (from equation 1)

$$l_1 = GJ/K_{t_1} \qquad \text{and} \qquad l_2 = GJ/K_{t_2} \qquad \text{with} \qquad l_1 + l_2 = l \tag{12}$$

2.3 System with a Stepped Shaft

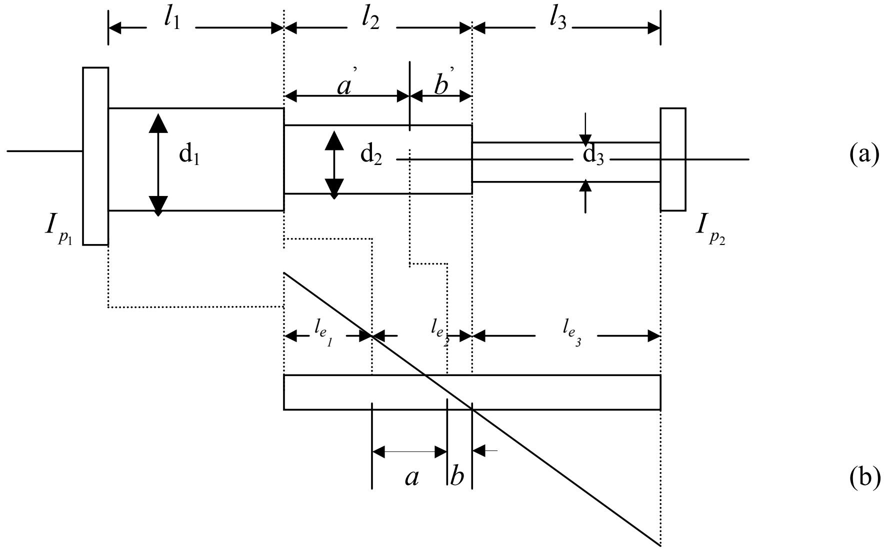


Figure 2.7(a) Two discs with stepped shaft (b) Equivalent uniform shaft

Figure 2.7(a) shows a two-disc steped shaft. In such cases the actual shaft should be replaced by an unstepped equivalent shaft for the purpose of the analysis as shown in Fig. 2.7(b). The equivalent shaft diameter may be same as the smallest diameter of the real shaft. The equivalent shaft must have the same torsional stiffness as the real shaft, since the torsional springs are connected in series. The equivalent torsional spring can be written as

$$\frac{1}{K_{t_e}} = \frac{1}{K_{t_1}} + \frac{1}{K_{t_2}} + \frac{1}{K_{t_3}}$$

Nothing equation (1), we have

$$l_e/J_e = l_1/J_1 + l_2/J_2 + l_3/J_3$$

which gives

$$l_e = l_1\, J_e/J_1 + l_2\, J_e/J_2 + l_3\, J_e/J_3 = l_{e_1} + l_{e_2} + l_{e_3} \tag{13}$$

with $\quad l_{e_1} = l_1 J_e / J_1, \qquad l_{e_1} = l_2 J_e / J_2, \qquad l_{e_1} = l_3 J_e / J_3$

where $l_{e_1}, l_{e_2}, l_{e_3}$ are equivalent lengths of shaft segments having equivalent shaft diameter d_3 and l_e is the total equivalent length of unstepped shaft having diameter d_3 as shown in Figure 2.7(b). From Figure 2.7(b) and noting equations (11) and (12), in equivalent shaft the node location can be obtained as

$$l_{e_1} + a = GJ_e \big/ \left(\omega_{nf_2}^2 I_{p_1}\right) \quad \text{and} \quad l_{e_3} + b = GJ_e \big/ \left(\omega_{n_2}^2 I_{p_2}\right) \tag{14}$$

where $\omega_{n_2} = \left[\dfrac{\left(I_{p_1} + I_{p_2}\right) K_{t_e}}{I_{p_1} I_{p_2}}\right]^{1/2}$ and $K_{t_e} = \dfrac{1}{l_1/GJ_1 + l_2/GJ_2 + l_3/GJ_3}$

From above equations the node position *a* & *b* can be obtained in the equivalent shaft length. Now the node location in real shaft system can be obtained as follows:

From equation (13), we have

$$l_{e_2} = l_2 \frac{J_e}{J_2}, \quad J_e = \frac{\pi}{64} d_3^4, \quad J_2 = \frac{\pi}{4} d_2^4$$

Since above equation is for shaft segment in which node is assumed to be present, we can write

$$a = a' J_e / J_2 \quad \text{and} \quad b = b' J_e / J_2$$

above equations can be combined as

$$\frac{a}{b} = \frac{a'}{b'} \tag{15}$$

So once *a* & *b* are obtained from equation (14) the location of node in actual shaft can be obtained equation (15) i.e. the final location of node on the shaft in real system is given in the same proportion along the length of shaft in equivalent system in which the node occurs.

2.4 MODF Systems

When there are several number of discs in the rotor system it becomes is multi-DOF system. When the mass of the shaft itself may be significant then the analysis described in previous sections (i.e. single or two-discs rotor systems) is inadequate to model such system, however, they could be extended to allow for more number of lumped masses (i.e. rigid discs) but resulting mathematics becomes cumbersome. Alternative methods are: (i) transfer matrix methods (ii) methods of mechanical impedance and (iii) finite element methods.

2.4.1 Transfer matrix method: A multi-disc rotor system, supported on frictionless supports, is shown in Fig. 7. Fig. 8 shows the free diagram of a shaft and a disc, separately. At particular station in the system, we have two state variables: the angular twist θ and Torque T. Now in subsequent sections we will develop relationship of these state variables between two neighbouring stations and which can be used to obtain governing equations of motion of the whole system.

1.**Point matrix**:

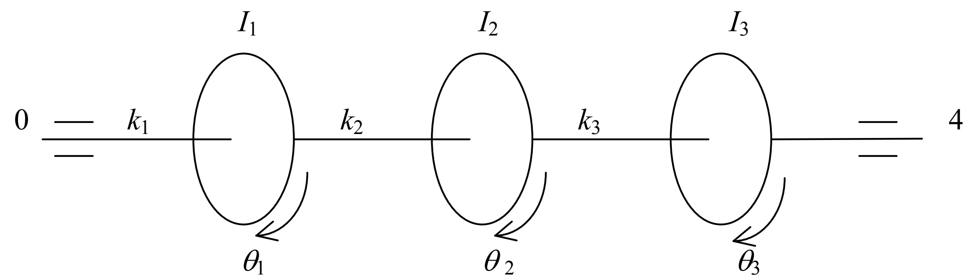


Figure 2.8 A multi-disc rotor system

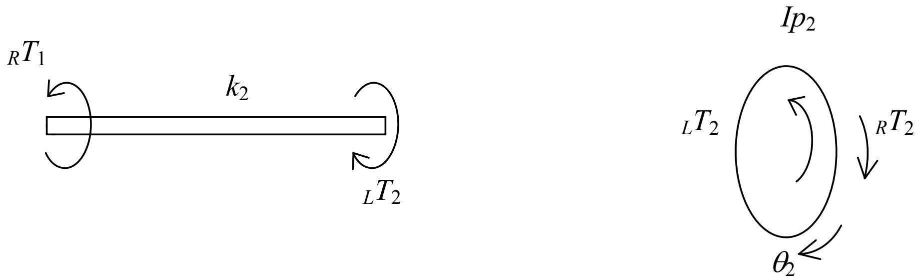


Fig. 2.9(a) Free body diameter of shaft section 2 (b) Free body diagram of rotor section 2

The equation of motion for the disc 2 is given by (see Figure 2.9(b))

$$_RT_2 - {}_LT_2 = I_{p_2}\ddot{\theta}_2 \tag{16}$$

For free vibrations, angular oscillations of the disc is given by

$$\theta_2 = \hat{\theta}\sin\omega t \qquad \text{so that} \qquad \ddot{\theta}_2 = -\omega_{nf}^2\hat{\theta}\sin\omega t = -\omega_{nf}^2\theta_2 \tag{17}$$

Substituting back into equation (16), we get

$$_RT_2 - {}_LT_2 = -\omega_{nf}^2 I_{p_2}\theta_2 \tag{18}$$

Angular displacements on the either side of the rotor are equal, hence

$$_R\theta_2 = {_L\theta_2} \tag{19}$$

Equations (18) and (19) can be combined as

$$_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2 = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_p & 1 \end{bmatrix}_L \begin{Bmatrix} \theta \\ T \end{Bmatrix}_2 \qquad \text{or} \qquad {_R\{S\}_2} = [P]_2\ {_L\{S\}_2} \tag{20, 21}$$

where $\{S\}_2$ is the state vector at station 2 and $[P]_2$ is the point matrix for station 2.

2. Field matrix:

For shaft element 2 as shown in Figure 2.9(a), the angle of twist is related to its torsional stiffness and to the torque, which is transmitted through it, as

$$\theta_2 - \theta_1 = \frac{T}{K_2} \tag{22}$$

Since the torque transmitted is same at either end of the shaft, hence

$$_LT_2 = {_RT_1} \tag{23}$$

Combining (22) and (23), we get

$$_L\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2 = \begin{bmatrix} 1 & 1/k \\ 0 & 1 \end{bmatrix}_2 {_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2} \tag{24}$$

which can be written as

$$_L\{S\}_2 = [F]_2\ {_R\{S\}_1} \tag{25}$$

where $[F]_2$ is the field matrix for the shaft element 2. Now we have

$$_R\{S\}_2 = [P]_2\{S\}_2 = [P_2][F]_2\ {_R\{S\}_1} == [U]_2\ {_R\{S\}_1}$$

where $[U]_2$ is the transfer matrix, which relates the state vector at right of station 2 to the state vector at right of station 1. On the same lines, we can write

$$\begin{aligned}
&_R\{S\}_1 = [U]_1\ {_R\{S\}_0} \\
&_R\{S\}_2 = [U]_2\ {_R\{S\}_1} = [U]_2[U]_1\{S\}_0 \\
&_R\{S\}_3 = [U]_3\ {_R\{S\}_2} = [U]_3[U]_2[U]_1\{S\}_0 \\
&\vdots \\
&_R\{S\}_n = [U]_n\ {_R\{S\}_{n-1}} = [U]_n[U]_{n-1}\cdots[U]_1\{S\}_0 = [T]\{S\}_0
\end{aligned} \tag{26}$$

where $[T]$ is the overall system transfer matrix. The overall transfermation can be written as

$$_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_n = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}_R \begin{Bmatrix} \theta \\ T \end{Bmatrix}_0 \tag{27}$$

For free-free boundary conditions, the each end of the machine torque transmitted through the shaft is zero, hence

$$_RT_n = {}_RT_0 = 0 \tag{28}$$

On using equation (28) into equation(27), the second set of equation gives

$$t_{21}\, {}_R\theta_0 = 0 \text{ which gives } t_{21}\left(\omega_{nf}\right) = 0 \text{ since } {}_R\theta_0 \neq 0 \tag{29}$$

which is satisfied for some values of ω_{nf}, which are system natural frequencies. These roots ω_n may be found by any root-searching technique. Angular twists can be determined for each value of ω_{nf} from first set of equation of equation (27), as

$$_R\theta_n = t_{11}\, {}_RT_0$$

On taking $_R\theta_0 = 1$, we get

$$_R\theta_0 = 1 \quad \text{we} \quad \text{get} \quad {}_R\theta_0 = t_{11}\left(\omega_{nf}\right) \tag{30}$$

In Eq. (30), t_{11} contains ω_n so for each value of ω_n different value of $_R\theta_4$ is obtained and using Eq. (27) relative displacements of all other stations can be obtained, by which mode shapes can be plotted.

Example 2.1. Obtain the torsional natural frequency of the system shown in Figure 2.10 using the transfer matrix method. Check results with closed form solution available. Take $G = 0.8\times10^{11}$ N/m^2.

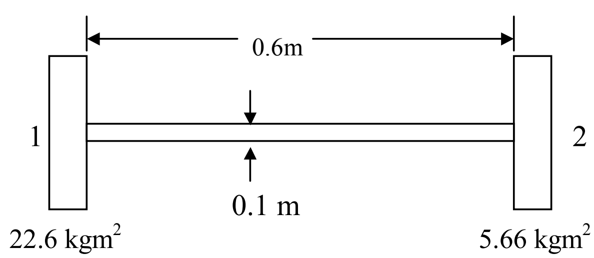


Figure 2.10 Example 2.1

Solution: We have following properties of the rotor

$$G = 0.8\times 10^{11}\ \text{N/m}^2;\quad l = 0.6\ \text{m};\quad J = \frac{\pi}{32}(0.1)^4 = 9.82\times 10^{-6}\text{m}^4$$

The torsional stiffness is given as

$$k_t = \frac{GJ}{l} = \frac{0.8\times 10^{11}}{0.6}9.82\times 10^{-6} = 1.31\times 10^6\ \text{Nm/rad}$$

Analytical method: The natural frequencies in the closed form are given as

$$\omega_{n_2} = 0;\quad \text{and}\quad \omega_{n_2} = \sqrt{\frac{(I_{p_1}+I_{p_2})k_t}{I_{p_1}I_{p_2}}} = \sqrt{\frac{(22.6+5.66)1.31\times 10^6}{22.6\times 5.66}} = 537.97\ \text{rad/sec}$$

Mode shapes are given as

For $\omega_{n_1} = 0$ $\quad \{\theta\}_2^R = \{\theta\}_0$

and $\omega_{n_2} = 537.77$ rad/s $\quad \{\theta\}_2^R = -\dfrac{I_{p_1}}{I_{p_2}}\{\theta\}_0 = -4.0\{\theta\}_0$

Transfer matrix method: State vectors can be related between stations 0 & 1 and 1 & 2, as

$$\{S\}_1^R = [P]_1\{S\}_0$$

$$\{S\}_2^R = [P]_2[F]_2\{S\}_1^R = [P]_2[F]_2[P]_1\{S\}_0$$

The overall transformation of state vectors between 2 & 0 is given as

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{bmatrix}1 & 0\\ -\omega_n^2 I_{p_2} & 1\end{bmatrix}\begin{bmatrix}1 & 1/k_t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\omega_n^2 I_{p_1} & 1\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_0 = \begin{bmatrix}1 & 1/k_t\\ -\omega_n^2 I_{p_2} & \left(1-\omega_n^2 I_{p_2}/k_t\right)\end{bmatrix}\begin{bmatrix}1 & 0\\ -\omega_n^2 I_{p_1} & 1\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_0$$

$$= \begin{bmatrix}1-\omega_n^2 I_{p_1}/k_t & 1/k_t\\ -\omega_n^2 I_{p_2} - \omega_n^2 I_{p_1}\left(1-\omega^2 I_{p_2}/k_t\right) & \left(1-\omega_n^2 I_{p_2}/k_t\right)\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_0$$

On substituting values of various rotor parameters, it gives

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{bmatrix}\left(1-1.73\times 10^{-5}\omega_n^2\right) & 7.64\times 10^{-7}\\ \left(-5.66\omega_n^2 + 9.77\times 10^{-5}\omega_n^4 - 22.6\omega_n^2\right) & 9.77\times 10^{-7}\omega_n^2 + 1\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_0 \qquad \text{(A)}$$

Since ends of the rotor are free, the following boundary conditions will apply

$$T_0 = T_2^R = 0$$

On application of boundary conditions, we get the following condition

$$t_{21} = [-28.26\omega_n^2 + 9.77\times10^{-5}\omega_n^4]\{\theta\}_0 = 0$$

Since $\{\theta\}_0 \neq 0$, we have

$$\omega_n^2[9.77\times10^{-5}\omega_n^2 - 28.26] = 0$$

which gives the natural frequency as

$$\omega_{n_1} = 0 \text{ and } \omega_{n_2} = 537.77 \text{ rad/sec}$$

which are exactly the same as obtained by the closed form solution. Mode shapes can be obtained by substituting these natural frequencies one at a time into equation (A), as

For $\omega_{n_1} = 0$ $\quad \{\theta\}_2^R = \{\theta\}_0$ $\quad$ rigid body mode

and $\omega_{n_2} = 537.77$ rad/s $\quad \{\theta\}_2^R = -4.0\{\theta\}_0$ $\quad$ anti-phase mode

which are also exactly the same as obtained by closed form solutions.

Example 2.2. Find torsional natural frequencies and mode shapes of the rotor system shown in Figure 1. B is a fixed end and D_1 and D_2 are rigid discs. The shaft is made of steel with modulus of rigidity $G = 0.8\ (10)^{11}$ N/m^2 and uniform diameter d = 10 mm. The various shaft lengths are as follows: BD_1 = 50 mm, and D_1D_2 = 75 mm. The polar mass moment of inertia of discs are: I_{p1} = 0.08 kg-m^2 and I_{p2} = 0.2 kg-m^2. Consider the shaft as massless and use (i) the analytical method and (ii) the transfer matrix method.

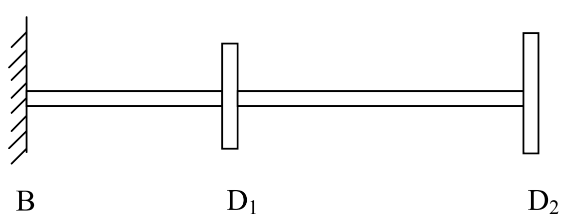


Figure 2.11 Example 2.2

Solution:

Analytical method: From free body diagrams of discs as shown in Figure 2.12, equations of motion can be written as

$$I_{p_1}\ddot{\theta}_1 + k_1\theta_1 + k_2(\theta_1 - \theta_2) = 0$$

$$I_{p_2}\ddot{\theta}_2 + k_2(\theta_2 - \theta_1) = 0$$

The above equations for free vibrations and they are homogeneous second order differential equations. In free vibrations discs will execute simple harmonic motions.

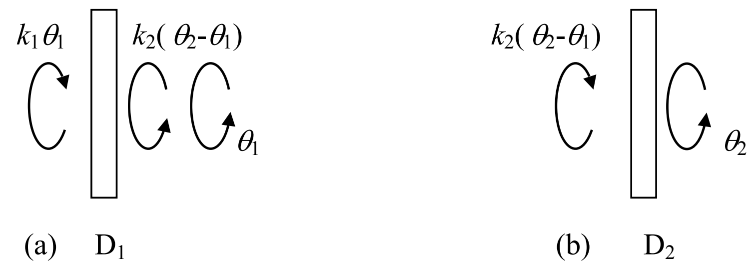


Figure 2.12 Free body diagram of discs

For the simple harmonic motion $\ddot{\theta} = -\omega_n^2\theta$, hence equations of motion take the form

$$\begin{bmatrix} k_1 + k_2 - I_{p_1}\omega_n^2 & -k_2 \\ -k_2 & k_2 - I_{p_2}\omega_n^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

On taking determinant of the above matrix, it gives the frequency equation as

$$I_{p_1}I_{p_2}\omega_n^4 - (I_{p_1}k_2 + I_{p_2}k_1 + I_{p_2}k_2)\omega_n^2 + k_1k_2 = 0$$

which can be solved for ω_n^2, as

$$\omega_n^2 = \frac{I_{p_1}k_2 + I_{p_2}k_1 + I_{p_2}k_2 \pm \sqrt{\left(I_{p_1}k_2 + I_{p_2}k_1 + I_{p_2}k_2\right)^2 - 4k_1k_2I_{p_1}I_{p_2}}}{2I_{p_1}I_{p_2}}$$

For the present problem the following properties are gives

$$k_1 = \frac{GJ_1}{l_1} = 1878\ \text{N/m} \quad \text{and} \quad k_2 = \frac{GJ_2}{l_2} = 523.598\ \text{N/m}$$

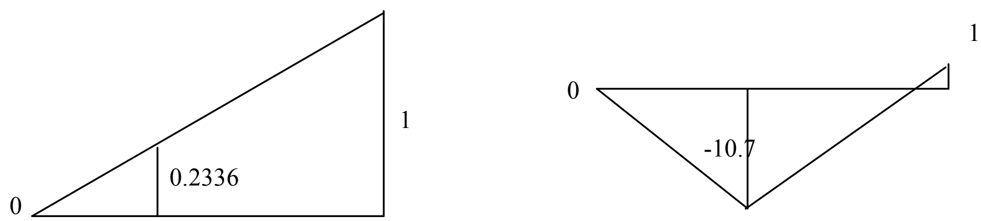
$$I_{p_1} = 0.08\ \text{kgm}^2 \quad \text{and} \quad I_{p_2} = 0.2\ \text{kgm}^2$$

Natural frequencies are obtained as

$$\omega_{n_1} = 44.792 \text{ rad/s} \quad \text{and} \quad \omega_{n_2} = 175.02 \text{ rad/s}$$

The relative amplitude ratio can be obtained as (Figure 2.13)

$$\frac{\theta_1}{\theta_2} = \frac{k_2 - I_{p_2}\omega_n^2}{k_2} = 0.2336 \quad \text{for} \quad \omega_{n_1} \quad \text{and} \quad -10.700 \qquad \text{for}\, \omega_{n_2}$$



(a) For ω_{n_1} (b) For ω_{n_2}

Figure 2.13 Mode shapes

Transfer matrix method

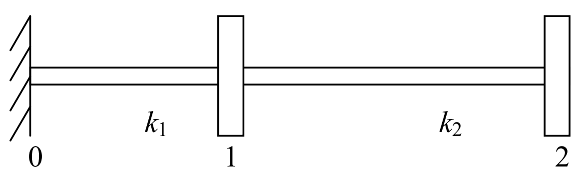


Figure 2.14 Two-discs rotor system with station numbers

For Figure 2.14 state vectors can be related as

$$\{\theta\}_2^R = [P]_2 [F]_2 [P]_1 [F]_1 \{\theta\}_0$$

The above state vector at various stations can be related as

$${}_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1 = \begin{bmatrix} 1 & 1/k_1 \\ -\omega_n^2 I_{p_1} & \dfrac{-\omega_n^2 I_{p_1}}{k_1} + 1 \end{bmatrix} {}_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0 \quad \text{and} \quad {}_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2 = \begin{bmatrix} 1 & 1/k_2 \\ -\omega_n^2 I_{p_2} & \dfrac{-\omega_n^2 I_{p_2}}{k_1} + 1 \end{bmatrix} {}_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1$$

which can be combined to give

$$ {}_R\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2 = \begin{bmatrix} \left(1 - \dfrac{\omega_n^2 I_{p_1}}{k_2}\right) & \dfrac{1}{k_1}\left(1 - \dfrac{\omega_n^2 I_{p_1}}{k_2}\right) + \dfrac{1}{k_2} \\ p & \left(\dfrac{p}{k_1} - \dfrac{\omega_n^2 I_{p_2}}{k_2} + 1\right) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_0 \qquad \text{(A)} $$

with

$$ p = -\omega_n^2 I_{p_2} - \omega_n^2 I_{p_1}\left(\frac{-\omega_n^2 I_{p_2}}{k_2} + 1\right) $$

Boundary conditions are given as

At station 0 $\Rightarrow \theta = 0$ and $T = 1$ (assumed)

and at right of station 2 $\Rightarrow T = 0$

On application of boundary conditions the second equation of equation (A), we get

$$ 0 = p \times 0 + \left(\frac{p}{k_1} - \frac{\omega_n^2 I_{p_2}}{k_2} + 1\right) {}_R T_0 $$

since ${}_R T_0 \neq 0$ and on substituting for p, we get

$$ \frac{1}{k_1}\left[-\omega_n^2 I_{p_2} - \omega_n^2 I_{p_1}\left(\frac{-\omega_n^2 I_{p_2}}{k_2} + 1\right)\right] - \frac{\omega_n^2 I_{p_2}}{k_2} + 1 = 0 $$

which can be solved to give

$$ \omega_n^2 = \frac{1}{2}\left(\frac{k_2}{I_{p_2}} + \frac{2k_1}{I_{p_1}}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_2}{I_{p_2}} + \frac{2k_1}{I_{p_1}}\right) - 4\frac{k_1 k_2}{I_{p_1} I_{p_2}}} $$

It should be noted that it is same as obtained by the analytical method.

Exercise 2.1. Obtain the torsional critical speed of a rotor system as shown in Figure E.2.1 Take the polar mass moment of inertia, I_p = 0.04 kg-m^2. Take shaft length a = 0.3 m and b = 0.7 m; modulus of rigidity $G = 0.8 \times 10^{11}$ N/m^2. The diameter of the shaft is 10 mm. Bearing A is flexible and provides a torsional spring of stiffness equal to 5 percent of the stiffness of the shaft segment having length a and bearing B is a fixed bearing. Use either the finite element method or the transfer matrix method.

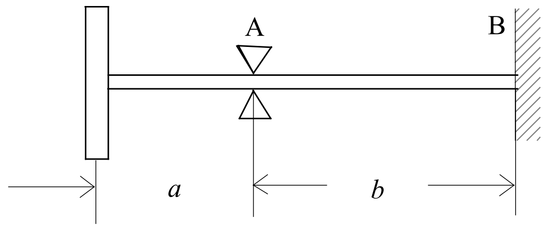


Figure E2.1 An overhang rotor system

Exercise 2.2. Find the torsional critical speeds and the mode shapes of the rotor system shown in Figure E2.2 by transfer matrix method. B_1 and B_2 are frictionless bearings and D_1 and D_2 are rigid discs. The shaft is made of steel with modulus of rigidity $G = 0.8\ (10)^{11}$ N/m^2 and uniform diameter d = 10 mm. The various shaft lengths are as follows: B_1D_1 = 50 mm, D_1D_2 = 75 mm, and D_2B_2 = 50 mm. The polar mass moment of inertia of discs are: J_{d1} = 0.0008 kg-m2 and J_{d2} = 0.002 kg-m^2. Consider shaft as massless.

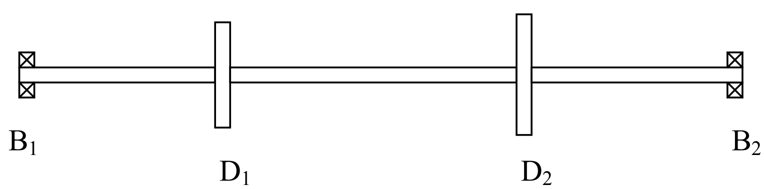


Figure E2.2

Exercise 2.3. Obtain the torsional critical speed of an overhang rotor system as shown in Figure E2.3. The end B_1 of the shaft is having fixed end conditions. The disc is thin and has 0.02 kg-m^2 of polar mass moment of inertia. Neglect the mass of the shaft. Use (i) the finite element and (ii) the transfer matrix method.

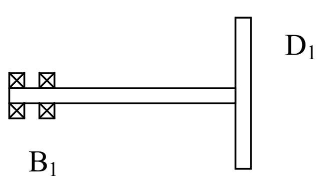


Figure E2.3

Exercise 2.4 Find the torsional natural frequencies and the mode shapes of the rotor system a shown in Figure E2.4 by ONLY *transfer matrix method*. B_1 and B_2 are fixed supports and D_1 and D_2 are rigid discs. The shaft is made of steel with modulus of rigidity $G = 0.8\ (10)^{11}$ N/m^2 and uniform diameter d

= 10 mm. The various shaft lengths are as follows: B_1D_1 = 50 mm, D_1D_2 = 75 mm, and D_2B_2 = 50 mm. The polar mass moment of inertia of discs are: J_{d1} = 0.08 kg-m2 and J_{d2} = 0.2 kg-m^2. Consider shaft as massless.

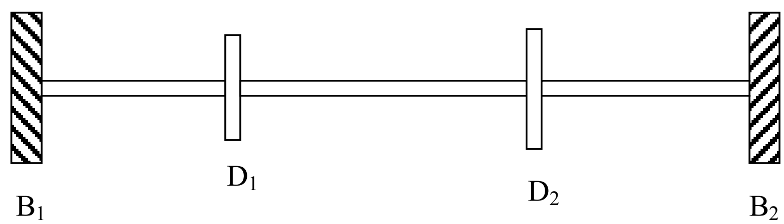


Figure E2.4

Exercise 2.5 Find all the torsional natural frequencies and draw corresponding mode shapes of the rotor system shown in Figure E2.5. B and D represent bearing and disc respectively. B_1 is fixed support (with zero angular displacement about shaft axis) and B_2 and B_3 are simply supported (with non-zero angular displacement about shaft axis). The shaft is made of steel with modulus of rigidity G = 0.8 $(10)^{11}$ N/m^2 and uniform diameter d = 10 mm. The various shaft lengths are as follows: B_1D_1 = 50 mm, D_1B_2 = 50 mm, B_2D_2 = 25 mm, D_2B_3 = 25 mm, and B_3D_3 = 30 mm. The polar mass moment of inertia of the discs are: I_{p1} = 2 kg-m^2, I_{p2} = 1 kg-m^2, and I_{p3} = 0.8 kg-m^2. Use both the transfer matrix method and the finite element method so as to verify your results. Give all the detailed steps in obtaining the final system equations and application of boundary conditions. Consider the shaft as massless and discs as lumped masses.

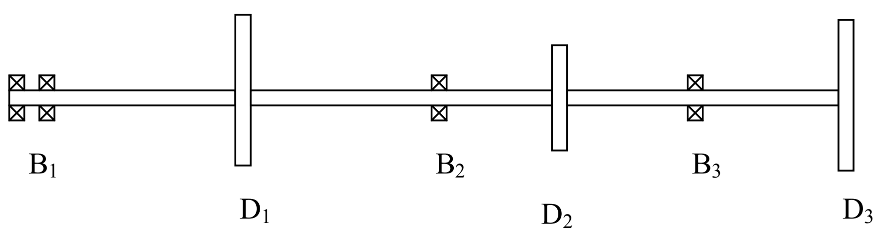


Figure E2.5

Exercise 2.6 Obtain the torsional critical speed of turbine-coupling-generator rotor as shown in Figure E2.6 by the transfer matrix and finite element methods. The rotor is assumed to be supported on frictionless bearings. The polar mass moment of inertias are Ip_T = 25 kg-m^2, Ip_C = 5 kg-m^2 and Ip_G = 50 kg-m^2. Take modulus of rigidity $G = 0.8 \times 10^{11}$ N/m^2. Assume the shaft diameter throughout is 0.2 m and lengths of shaft between bearing-turbine-coupling-generator-bearing are 1 m each so that the total span is 5 m. Consider shaft as massless.

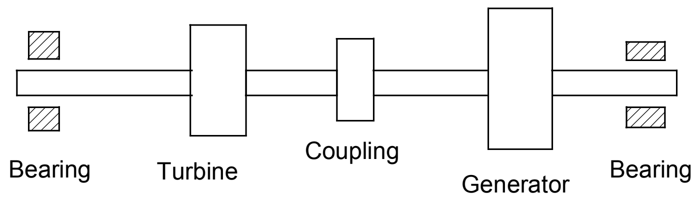


Figure E2.6 A turbine-generator set

Exercise 2.7 In a laboratory experiment one small electric motor drives another through a long coil spring (n turns, wire diameter d, coil diameter D). The two motor rotors have inertias I_1 and I_2 and are distance l apart, (a) Calculate the lowest torsional natural frequency of the set-up (b) Assuming the ends of the spring to be "built-in" to the shafts, calculate rotational speed (assume excitation frequency will be at the rotational frequency of the shaft) of the assembly at which the coil spring bows out at its center, due to whirling.