CHAPTER 6

TORSIONAL VIBRATIONS OF ROTORS-I:
THE DIRECT AND TRANSFER MATRIX METHODS

In previous chapters, mainly we studied transverse vibrations of simple rotor-bearing systems. It was pointed out that transverse vibrations are very common in rotor systems due residual unbalances, which is the most inherent fault in a rotor. We studied behaviour of rotor due to speed-independent bearing dynamic parameters. Effect of gyroscopic couples on natural whirl frequencies is also investigated in details. In the present chapter, we will extend the analysis of simple rotors to torsional vibrations. We will start with the analysis of torsional vibrations of the single disc rotor, two disc rotor, and three disc rotor systems with the conversional Newton's second law of motion or energy methods. The analysis is extended to the stepped shafts, geared systems, and branched systems. For the multi-DOF system a general procedure of the transfer matrix method (TMM) is discussed for both undamped and damped cases. Advantages and disadvantages of the TMM are outlined. In reciprocating engines large variations of torque take place, however, periodically. This leads to torsional resonances, and to analyse free and forced vibrations of these system a procedure is outline to convert them to an equivalent multi-DOF rotor system, which is relatively easier to analyse. The present chapter will pave the road for the TMM to be extended for the transverse vibrations of multi-DOF rotor systems in subsequent chapters.

The study of torsional vibration of rotors is very important especially in applications where high power transmission and high speed are present. Torsional vibrations are predominant whenever there are large discs on relatively thin shafts (e.g., the flywheel of a punch press). Torsional vibrations may original from the following forcings (i) inertia forces of reciprocating mechanisms (e.g., due to pistons in IC engines), (ii) impulsive loads occurring during a normal machine cycle (e.g., during operations of a punch press), (iii) shock loads applied to electrical machinery (such as a generator line fault followed by fault removal and automatic closure), (iv) torques related to gear mesh frequencies, the turbine blade and compressor fan passing frequencies, etc.; and (v) a rotor rubs with the stator. For machines having massive rotors and flexible shafts (where system natural frequencies of torsional vibrations may be close to, or within, the source frequency range during normal operation) torsional vibrations constitute a potential design problem area. In such cases designers should ensure the accurate prediction of machine torsional frequencies, and frequencies of any torsional load fluctuations should not coincide with torsional natural frequencies. Hence, determination of torsional natural frequencies of the rotor system is very important and in the present chapter we shall deal with it in great detail.

6.1 A Simple Rotor System with a Single Disc Mass

Consider a rotor system as shown Figure 6.1(a). The shaft is considered as mass-less and it provides torsional stiffness. The disc is considered as rigid and has no flexibility. If an initial disturbance is given to the disc in the torsional mode (about its longitudinal or polar axis) and allow it to oscillate its own, it will execute free vibrations. Figure 6.2 shows that rotor is spinning with a nominal speed of ω and executing torsional vibrations, $\varphi_z(t)$, due to this it has actual speed of $\omega + \dot{\varphi}_z(t)$. It should be noted that the spinning speed, ω, remains the same, however, the angular velocity due to torsion have varying direction over a period. In actual practice if we tune a stroboscope (it is a speed/frequency measuring instrument, refer Chapter 15) flashing frequency to the nominal speed of a rotor then free torsional oscillations could be observed. For the present case and in most of our analysis, it is assumed that torsional natural frequency does not depend upon the spin speed of rotor. Hence, in limiting case when the spin speed is zero the natural frequency of the non-spinning rotor will be same as at any other speed. The free oscillation will be simple harmonic motion with a unique frequency, which is called the *torsional natural frequency* of the rotor system.

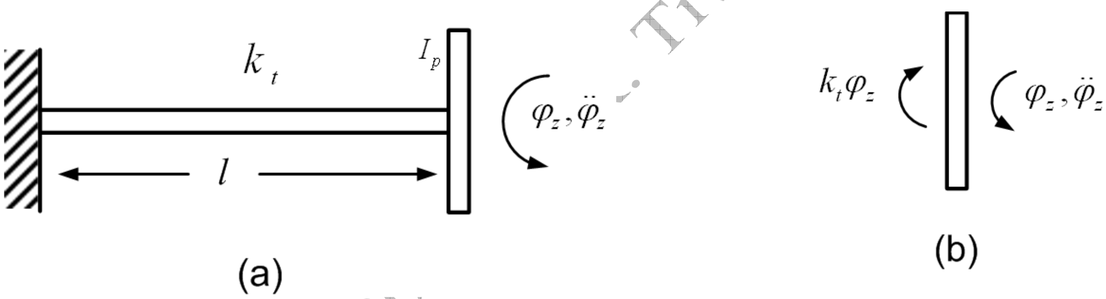


Figure 6.1(a) A single-mass cantilever rotor system (b) A free body diagram of the disc

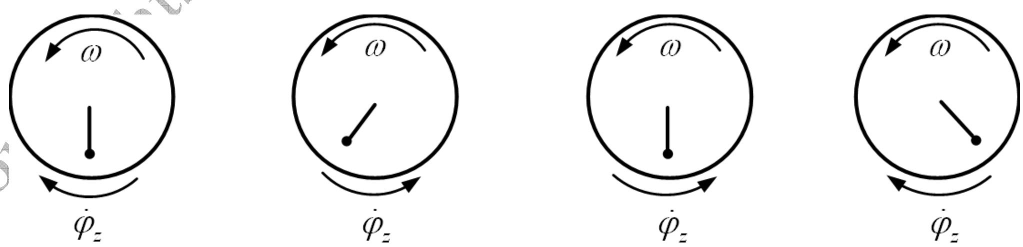


Figure 6.2 Torsional vibrations of a spinning rotor

From the theory of torsion of the shaft (Timoshenko and Young, 1968), we have

$$k_t = \frac{T}{\varphi_z} = \frac{GJ}{l} \qquad \text{with} \qquad J = \frac{\pi}{32} d^4 \tag{6.1}$$

where k_t is the torsional stiffness of shaft, I_p is the polar mass moment of inertia of the disc, J is the polar second moment of area of the shaft cross-section, l is the length of the shaft, d is the diameter of the shaft, and φ_z is the angular displacement of the disc (the counter clockwise direction is assumed as the positive direction). From the free body diagram of the disc as shown in Figure 6.1(b), we have

$$\sum \text{External torque of disc} = I_p \ddot{\varphi}_z \qquad \Rightarrow \qquad -k_t \varphi_z = I_p \ddot{\varphi}_z \tag{6.2}$$

where $\sum$ represents the summation operator. Equation (6.2) is the equation of motion of the disc for free torsional vibrations. The free (or natural) vibration has a *simple harmonic motion* (SHM). For SHM of the disc, we have

$$\varphi_z(t) = \Phi_z \sin \omega_{nf} t \qquad \text{so that} \qquad \ddot{\varphi}_z = -\omega_{nf}^2 \Phi_z \sin \omega_{nf} t = -\omega_{nf}^2 \varphi_z \tag{6.3}$$

where Φ_z is the amplitude of the torsional vibration, and ω_{nf} is the torsional natural frequency. On substituting equation (6.3) into equation (6.2), we get

$$-k_t \varphi_z = I_p \left(-\omega_{nf}^2 \varphi_z\right) \qquad \text{or} \qquad \varphi_z \left(\omega_{nf}^2 I_p - k_t\right) = 0 \tag{6.4}$$

Since $\varphi_z \neq 0$, it gives

$$\omega_{nf} = \sqrt{\frac{k_t}{I_p}} = \sqrt{\frac{GJ}{lI_p}} \tag{6.5}$$

which is similar to the case of single-DOF spring-mass system in where the polar mass moment of inertia and the torsional stiffness replace the mass and the spring stiffness, respectively.

Example 6.1 Obtain the torsional natural frequency of an overhung rotor system as shown in Fig. 6.3. The end B_1 of the shaft has fixed end conditions. The shaft diameter is 10 mm and the length of the span is 0.2 m. The disc D_1 is thin, and has mass of 10 kg and the polar mass moment of inertia equal to 0.02 kg-m^2. Neglect the mass of the shaft.

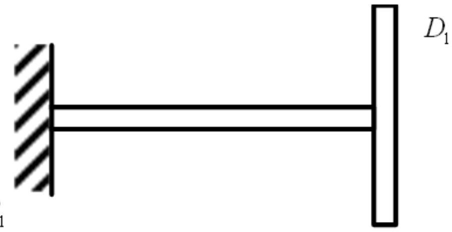


Figure 6.3 An overhung rotor system

Answer: For the present problem the torsional stiffness of the shaft can be obtained as

$$k_t = \frac{GJ}{l} = \frac{0.8\times 10^{11}\times \frac{\pi}{32}(0.01)^4}{0.2} = 392.7 \text{ Nm/rad}$$

Hence, the torsional frequency is given as

$$\omega_{nf} = \sqrt{k_t / I_p} = \sqrt{\frac{392.7}{0.02}} = 140.12 \text{ rad/s } = 22.3 \text{ Hz} \qquad \textit{Answer.}$$

Hence, if the rotor has cyclic torque variation with a period of 1/22.3 sec then the rotor might undergo to the resonance under torsional vibrations. To have a comparison with the transverse natural frequency, the bending stiffness is given as

$$k_b = \frac{3EI}{l^3} = \frac{3\times 2.1\times 10^{11}\times \frac{\pi}{64}(0.01)^4}{0.2^3} = 3.87\times 10^4 \text{ N/m}$$

Hence, the transverse natural frequency is given as

$$\omega_{nf} = \sqrt{k_b / m} = \sqrt{\frac{3.87\times 10^4}{10}} = 62.21 \text{ rad/s} = 9.9 \text{ Hz} \qquad \textit{Answer.}$$

If the same rotor has small amount of unbalance and if rotor is spinning around 9.9 Hz speed, then the rotor might undergo to the resonance under transverse vibrations. For the present case, the transverse natural frequency is much lower than the torsional natural frequency.

6.2 A Two-Disc Torsional Rotor System

A two-disc torsional system is shown in Figure 6.4. In this case the whole of the rotor is free to rotate as the shaft is mounted on frictionless bearings. Hence, it is a free-free end condition, and the application of which can be found in an aircraft when it is flying and whole structure has torsional vibrations due to aerodynamic forces.

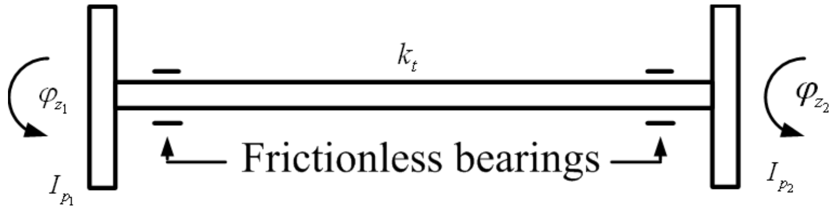


Figure 6.4 A two-disc torsional system

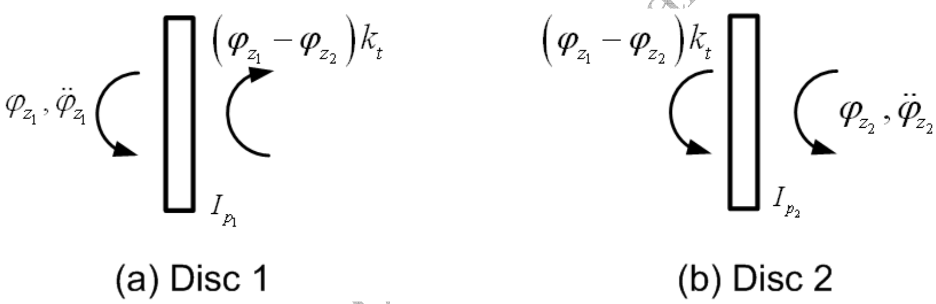


Figure 6.5 Free body diagrams of discs

Let φ_{z_1} and φ_{z_2} are angular displacements of the disc 1 and 2, respectively. For both angular displacements the counter clockwise direction is chosen as positive direction. Let I_{p_1} and I_{p_2} are polar mass moment of inertia of the disc 1 and 2, respectively. From the free body diagram of discs as shown in Figure 6.5, we have

$$\sum \text{External torque} = I_{p_1}\ddot{\varphi}_{z_1} \qquad \Rightarrow \qquad -\left(\varphi_{z_1}-\varphi_{z_2}\right)k_t = I_{p_1}\ddot{\varphi}_{z_1}$$

and

$$\sum \text{External torque} = I_{p_2}\ddot{\varphi}_{z_2} \qquad \Rightarrow \qquad \left(\varphi_{z_1}-\varphi_{z_2}\right)k_t = I_{p_2}\ddot{\varphi}_{z_2}$$

where k_t is the torsional stiffness of the shaft, and let $\left(\varphi_{z_1} - \varphi_{z_2}\right)$ be the relative twist of the shaft ends. Above expressions give the following equations of motion

$$I_p \, \ddot{\varphi}_{z_1} + k_t \varphi_{z_1} - k_t \varphi_{z_2} = 0 \qquad \text{and} \qquad I_{p_2} \ddot{\varphi}_{z_2} + k_t \varphi_{z_2} - k_t \varphi_{z_1} = 0 \tag{6.6}$$

For free vibrations, we have SHM, so the solution will take the form

$$\ddot{\varphi}_{z_1} = -\omega_{nf}^2 \varphi_{z_1} \qquad \text{and} \qquad \ddot{\varphi}_{z_2} = -\omega_{nf}^2 \varphi_{z_1} \tag{6.7}$$

Substituting equation (6.7) into equation (6.6), it gives

$$-I_{p_1} \omega^2 \varphi_{z_1} + k_t \varphi_{z_1} - k_t \varphi_{z_2} = 0 \quad \text{and} \qquad -I_{p_2} \omega_{nf}^2 \varphi_{z_2} + k_t \varphi_{z_2} - k_t \varphi_{z_1} = 0 \tag{6.8}$$

Noting equation (6.3), equation (6.8) can be assembled in a matrix form as

$$[D]\{\Phi_z\} = \{0\} \tag{6.9}$$

with

$$[D] = \begin{bmatrix} k_t - I_{p_1} \omega_{nf}^2 & -k_t \\ -k_t & k_t - I_{p_2} \omega_{nf}^2 \end{bmatrix}; \qquad \{\Phi_z\} = \begin{Bmatrix} \Phi_{z_1} \\ \Phi_{z_2} \end{Bmatrix}$$

The non-trial solution of equation (6.9) is obtained by taking determinant of the matrix [D] equal to zero, as

$$|D| = 0$$

which gives

$$\left(k_t - I_{p_1} \omega_n^2\right)\left(k_t - I_{p_2} \omega_n^2\right) - k_t^2 = 0 \qquad \text{or} \qquad I_{p_1} I_{p_2} \omega_{nf}^4 - \left(I_{p_1} + I_{p_2}\right) k_t \omega_{nf}^2 = 0 \tag{6.10}$$

Roots of equation (6.10) are given as

$$\omega_{nf_1} = 0 \qquad \text{and} \qquad \omega_{nf_2} = \sqrt{\frac{\left(I_{p_1} + I_{p_2}\right) k_t}{I_{p_1} I_{p_2}}} \tag{6.11}$$

Hence, the system has two torsional natural frequencies and one of them is zero. From equation (6.9) corresponding to first natural frequency for $\omega_{nf_1} = 0$, we get

$$\Phi_{z_1} = \Phi_{z_2} \tag{6.12}$$

Figure 6.6 The first mode shape

From equation (6.12), it can be concluded that, the first root of equation (6.10) represents the case when both discs simply rolls together in phase with each other as shown in Figure 6.6. The representation of the relative angular displacement of two discs in this form is called the *mode shape*. The mode shape shown in Fig. 6.6 is called the rigid body mode, which is of a little practical significance because no stresses develop in the shaft. This mode it generally occurs whenever the system has free-free boundary conditions (for example an aeroplane during flying).

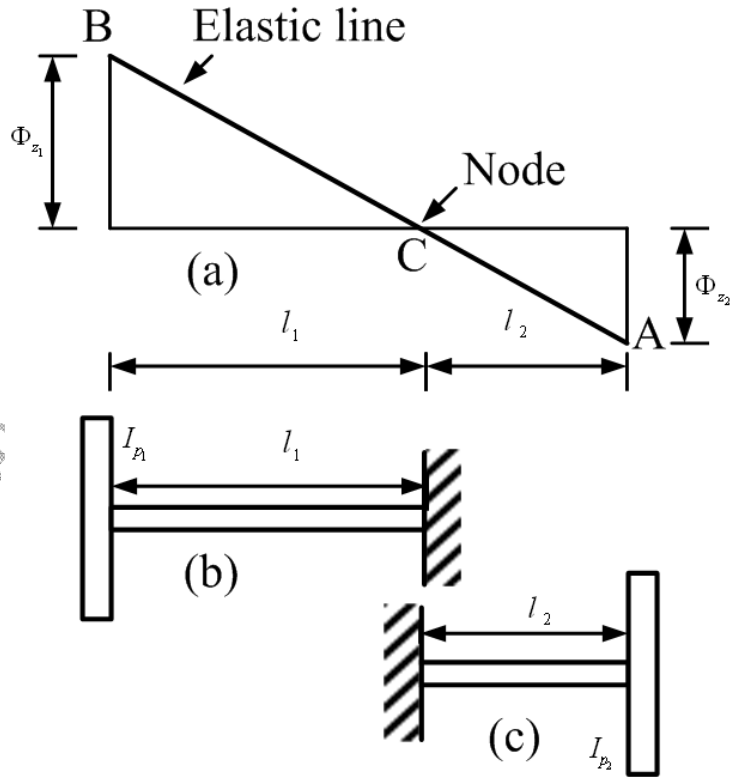


Figure 6.7 (a) The second mode (b) equivalent system 1 (c) equivalent system 2

Now from first set of equation (6.9), for $\omega_{nf} = \omega_{nf_2}$, we get

$$\left(k_t - I_{p_1}\omega_{nf_2}^2\right)\Phi_{z_1} - k_t\Phi_{z_2} = 0 \qquad \text{or} \qquad \left[k_t - I_{p_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}}k_t\right]\Phi_{z_1} - k_t\Phi_{z_2} = 0$$

which gives relative amplitudes of two discs as

$$\frac{\Phi_{z_1}}{\Phi_{z_2}} = -\frac{I_{p_2}}{I_{p_1}} \qquad (6.13)$$

The second mode shape from equation (6.13) represents the case when both masses oscillate in anti-phase with one another (i.e., the direction of rotation of one disc will also be opposite to the other). Both discs will reach their extreme angular positions simultaneously, and both will reach the static equilibrium (untwisted) position also simultaneously. It should be noted that both the discs have same frequency of oscillation (i.e., *the time period*) but different angular amplitude. Figure 6.7 shows this mode shape of the two-rotor system. From two similar triangles in Figure 6.7(a), we have

$$\frac{\Phi_{z_1}}{l_1} = \frac{\Phi_{z_2}}{l_2} \qquad \Rightarrow \qquad \frac{\Phi_{z_1}}{\Phi_{z_2}} = \frac{l_1}{l_2} \qquad (6.14)$$

where l_1 and l_2 are node position from discs 1 and 2, respectively (Fig. 6.7a). Since both the masses are always vibrating in the opposite direction, there must be a point on the shaft where torsional vibration is not taking place, i.e. where the angular displacement is zero. This point is called a *node*. The location of the node may be established by treating each end of the real system as a separate single-disc cantilever system as shown in Figure 6.7(a). The node is treated as the point, where the shaft is rigidly fixed. Hence, basically we will have two single-DOF overhung rotor systems (Fig. 6.7b) instead of one two-DOF free-free rotor system (Fig. 6.7c). Since value of natural frequency is known (the frequency of oscillation of each of the single-disc overhung system must be same), hence we write

$$\omega_{nf_2}^2 = \frac{k_{t_1}}{I_{p_1}} = \frac{k_{t_2}}{I_{p_2}} \qquad (6.15)$$

where ω_{nf_2} is defined by equation (6.11), k_{t_1} and k_{t_2} are the torsional stiffness of two single-DOF overhung rotor systems, which can be obtained from equation (6.15), as

$$k_{t_1} = \omega_{nf_2}^2 I_{p_1} \quad \text{and} \quad k_{t_2} = \omega_{nf_2}^2 I_{p_2}$$

Lengths l_1 and l_2 then can be obtained as

$$l_1 = \frac{GJ}{k_{t_1}} \qquad \text{and} \qquad l_2 = \frac{GJ}{k_{t_1}} \qquad \text{with} \qquad l_1 + l_2 = l \tag{6.16}$$

which will give the node position. It should be noted that the shear stress would be maximum at the node point being a fixed end of overhung rotor systems.

Example 6.2 Determine natural frequencies and mode shapes for a rotor system as shown in Figure 6.8. Neglect the mass of the shaft and assume that discs as lumped masses. The shaft is 1 m of length, 0.015m of diameter, and 0.8×10^{11} N/m of modulus of rigidity. Discs have polar mass moment of inertia as $I_{p_1} = 0.01$ kg-m^2 and $I_{p_2} = 0.015$ kg-m^2.

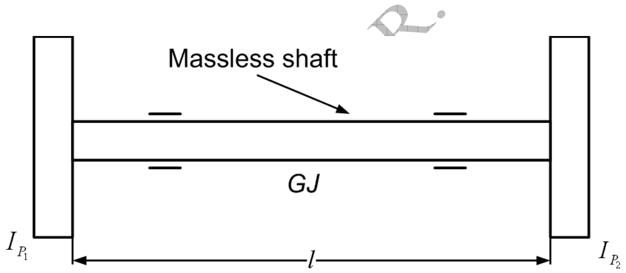


Figure 6.8 A two-disc rotor system

Solution: The stiffness of the shaft can be obtained as

$$k_t = \frac{GJ}{l} = \frac{0.8\times10^{11}\times\pi(0.015)^4/32}{1.0} = 397.61 \text{ Nm/rad}$$

The natural frequency is given as

$$\omega_{nf_1} = 0 \qquad \text{and} \qquad \omega_{nf_2} = \sqrt{\frac{(0.01+0.015)\times397.61}{0.01\times0.015}} = 257.43 \text{ rad/s}$$

The relative displacements would be

$$\frac{\Phi_{z_1}}{\Phi_{z_2}} = -\frac{I_{p_2}}{I_{p_1}} = -\frac{0.015}{0.01} = -1.5$$

which means disc 1 would have 1.5 times angular displacement amplitude as compared to the disc 2, however, in opposite direction. The node position can be obtained as

$$\frac{l_1}{l_2} = \frac{I_{p_2}}{I_{p_1}} = \frac{0.015}{0.01} = 1.5 \; ; \qquad \text{and} \qquad l_1 + l_2 = 1$$

Hence, we get the node location as $l_1 = 0.6$ m (i.e., 0.6 m from disc 1 refer to Fig. 6.7(a)). It can be verified that equivalent two single-mass cantilever rotors will have the same natural frequency, as

$$k_{t_1} = \frac{GJ}{l_1} = \frac{0.8\times 10^{11} \times \pi (0.015)^4 / 32}{0.6} = 662.68 \text{ Nm/rad}$$

and

$$k_{t_2} = \frac{GJ}{l_2} = \frac{0.8\times 10^{11} \times \pi (0.015)^4 / 32}{0.4} = 994.03 \text{ Nm/rad}$$

so that

$$\omega_{nf_2}^{(1)} = \sqrt{\frac{k_{t_1}}{I_{p_1}}} = \sqrt{\frac{662.68}{0.01}} = 257.43 \text{ rad/s}$$

and

$$\omega_{nf_2}^{(2)} = \sqrt{\frac{k_{t_2}}{I_{p_2}}} = \sqrt{\frac{994.03}{0.015}} = 257.43 \text{ rad/s}$$

Answer

6.3 A Two-Disc Rotor System with a Stepped Shaft

Figure 6.9(a) shows a stepped shaft with two large discs at ends with I_p (subscript 1 and 2 represent left and right side disc, respectively) is the polar mass moment of inertia. It is assumed that the rotor has free-free boundary conditions and the polar mass moment of inertia of shaft is negligible as compared to two discs at either ends of the shaft. In such cases the actual shaft should be replaced by an unstepped equivalent shaft for the purpose of the analysis as shown in Fig. 6.9(b). The equivalent shaft diameter may be same as the smallest diameter of the real shaft (or any other diameter). The equivalent shaft must have the same torsional stiffness as the real shaft.

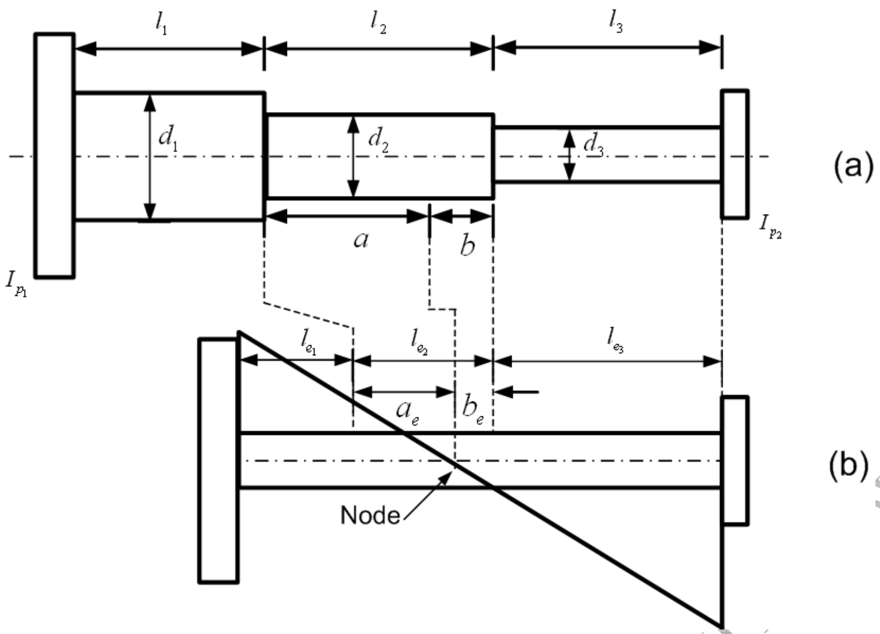


Figure 6.9 Two discs with (a) a stepped shaft (b) an equivalent uniform shaft

Since torsional stiffness corresponding to different shaft segments are connected in series, the equivalent torsional stiffness can be written as

$$\frac{1}{k_{t_e}}=\frac{1}{k_{t_1}}+\frac{1}{k_{t_2}}+\frac{1}{k_{t_3}} \tag{6.17}$$

where k_t is the torsional stiffness, subscripts: 1, 2, 3 represent the shaft segment number and the subscript e represents the equivalent. Nothing equation (6.1), equation (6.17) becomes

$$\frac{l_e}{J_e}=\frac{l_1}{J_1}+\frac{l_2}{J_2}+\frac{l_3}{J_3}$$

where l is the length of the shaft segment and J is the polar moment inertia of the shaft cross-sectional area. Above equation can be written as

$$l_e=l_{e_1}+l_{e_2}+l_{e_3} \tag{6.18}$$

with

$$l_{e_1}=l_1J_e/J_1; \qquad l_{e_2}=l_2J_e/J_2; \qquad l_{e_3}=l_3J_e/J_3$$

where $l_{e_1}, l_{e_2}, l_{e_3}$ are equivalent lengths of shaft segments having the equivalent shaft diameter d_3, and l_e is the total equivalent length of the unstepped shaft as shown in Figure 6.9(b). Let us assume that the node position in the equivalent shaft system comes out in the second shaft segment from the previous section analysis. Noting equations (6.15) and (6.16), the node location in the equivalent shaft from Figure 6.9(b) can be obtained as

$$l_{e_1} + a_e = \frac{GJ_e}{\omega_{nf_2}^2 I_{p_1}} \quad \text{and} \quad l_{e_3} + b_e = \frac{GJ_e}{\omega_{nf_2}^2 I_{p_2}} \tag{6.19}$$

with

$$\omega_{nf_2} = \sqrt{\frac{\left(I_{p_1} + I_{p_2}\right) k_{t_e}}{I_{p_1} I_{p_2}}} \quad \text{and} \quad k_{t_e} = \frac{1}{l_1/(GJ_1) + l_2/(GJ_2) + l_3/(GJ_3)}$$

From equation (6.19), the node position (i.e., a_e or b_e in Fig. 6.9(b)) can be obtained, the corresponding node location in the real shaft system can be obtained as explained below. From equation (6.18), we have

$$l_{e_2} = l_2 \frac{J_e}{J_2}; \qquad J_e = \frac{\pi}{32} d_e^4 = \frac{\pi}{32} d_3^4; \qquad J_2 = \frac{\pi}{32} d_2^4 \tag{6.20}$$

Since equation (6.20) is for the shaft segment in which node is assumed to be present, we can write

$$a_e = a \frac{J_e}{J_2} \quad \text{and} \quad b_e = b \frac{J_e}{J_2} \tag{6.21}$$

where a and b are node position in real system (Fig.6.9(a)). Equation (6.21) can be combined as

$$\frac{a}{b} = \frac{a_e}{b_e} \tag{6.22}$$

So once a_e or b_e is obtained from equation (6.19), the location of the node in the actual shaft can be obtained from equation (6.22). The final location of the node on the shaft in the real system is given in the same proportion as in the shaft of equivalent system in which the node occurs.

Example 6.3 Consider a stepped shaft with two discs as shown in Fig. 6.10. The following shaft dimensions are to be taken: l_1 = 0.5m, l_2 = 0.3m, l_3 = 0.2m, d_1 = 0.015m, d_2 = 0.012m, d_3 = 0.01m. Take the modulus of rigidity of the shaft as 0.8×10^{11} N/m. Discs have polar mass moment of inertia as

$I_{p_1} = 0.015$ kg-m^2 and $I_{p_2} = 0.01$ kg-m^2. Obtain natural frequencies, mode shapes, and the location of the node.

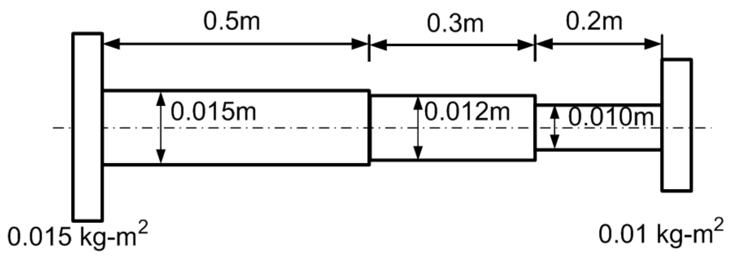


Fig. 6.10 A stepped shaft with two discs

Solution: Let us represent shaft segments towards the left, middle and right sides as 1, 2 and 3, respectively. For the present problem the shaft has following data

$$J_1 = \frac{\pi \times d_1^4}{32} = \frac{\pi \times 0.015^4}{32} = 4.97\times10^{-9}\,\text{m}^4; \qquad J_2 = 2.036\times10^{-9}\,\text{m}^4; \qquad J_3 = 0.982\times10^{-9}\,\text{m}^4;$$

$$k_{t_1} = \frac{GJ_1}{l_1} = \frac{0.8\times10^{11}\times4.97\times10^{-9}}{0.5} = 795.20 \text{ Nm/rad}; \quad k_{t_2} = 542.93 \text{ Nm/rad}; \quad k_{t_3} = 392.80 \text{ Nm/rad}$$

For the stepped shaft the first step would be to obtain the equivalent length with respect to a reference shaft 3 that has diameter of 0.01 m, as

$$l_e = \frac{l_1}{J_1}J_e + \frac{l_2}{J_2}J_e + \frac{l_3}{J_3}J_e = \frac{0.5}{4.97}0.982 + \frac{0.3}{2.036}0.982 + \frac{0.2}{0.982}0.982 = 0.0988 + 0.1447 + 0.2 = 0.4435\text{m}$$

Hence, The equivalent stiffness can be calculated as

$$k_{t_e} = \frac{GJ_e}{l_e} = \frac{0.8\times10^{11}\times0.982\times10^{-9}}{0.4435} = 177.14 \text{ Nm/rad}$$

Hence, $l_{e_1} = 0.0987$ m and $l_{e_2} = 0.1447$ m. The natural frequency of the rotor system can be calculated as

$$\omega_{nf_2} = \sqrt{\frac{(I_{p_1} + I_{p_2})k_{t_e}}{I_{p_1}I_{p_2}}} = \sqrt{\frac{(0.015+0.01)\times177.14}{0.015\times0.01}} = 171.82 \text{ rad/sec}$$

Relative displacements of the rotor system would be

$$\frac{\Phi_{z_1}}{\Phi_{z_2}} = -\frac{I_{p_2}}{I_{p_1}} = -\frac{0.01}{0.015} = -0.667$$

which means disc 1 would have 0.0667 times angular displacement amplitude as compared to the disc 2, however, in opposite direction. It is interesting that relative displacement remains same irrespective of shaft characteristics (i.e., stepped, uniform, etc.) and its stiffness. However, the node position depends upon the shaft characteristics and its stiffness, and can be obtained as for equivalent shaft as

$$\frac{l_{ne_1}}{l_{ne_2}} = \frac{I_{p_2}}{I_{p_1}} = \frac{0.01}{0.015} = 0.667\,; \qquad \text{and} \qquad l_{ne_1} + l_{ne_2} = l_e = 0.4435 \text{ m}$$

Hence, we get the node location as $l_{ne_2} = 0.266$ m (i.e., 0.266 m from disc 2 in the equivalent system see Fig. 6.11). Hence, we have $l_{ne_1} = 0.1775$ m. This means the node will be in second (middle) shaft segment. The location in actual rotor system would be

$$\frac{a}{b} = \frac{a_e}{b_e} = \frac{l_{ne_1} - l_{e_1}}{l_{ne_2} - l_{e_3}} = \frac{0.1775 - 0.0988}{0.266 - 0.2} = \frac{0.0787}{0.066} = 1.1924\,; \qquad \text{and} \qquad a + b = 0.1447 \text{ m}$$

Hence, we have the position of the node in actual system as: b = 0.137m and a = 0.163m (see Fig. 6.9).

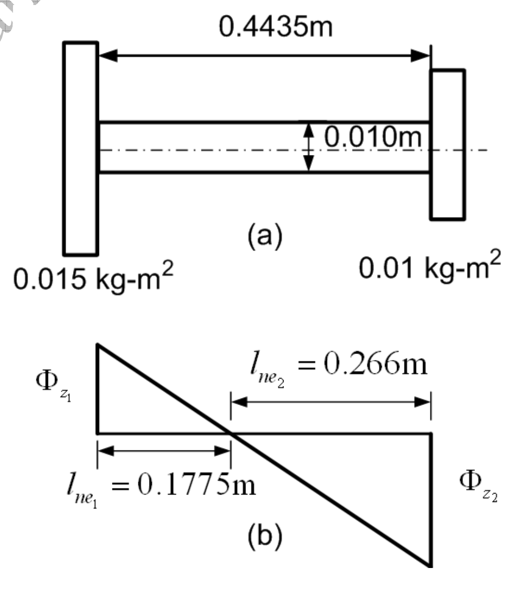


Fig. 6.11 (a) Equivalent system and (b) its mode shape and node position

6.4 A Three-Disc Rotor System

Now the previous section analysis would be extended for the three-disc rotor system having free-free boundary conditions. Two different approaches are applied for the free vibration analysis to get the torsional natural frequnecies and corresponding mode shapes.

6.4.1 A direct approach

A three-disc rotor system is shown in Fig. 6.12. It is assumed that there is no friction at supports and boundary conditions are that of the free-free case. The method using the Newton's second law, with the help of free body diagram (Fig. 6.13), may be applied to analyse the three-mass (or more) rotor system. This method is already demonstrated for the two-mass rotor system in the previous section.

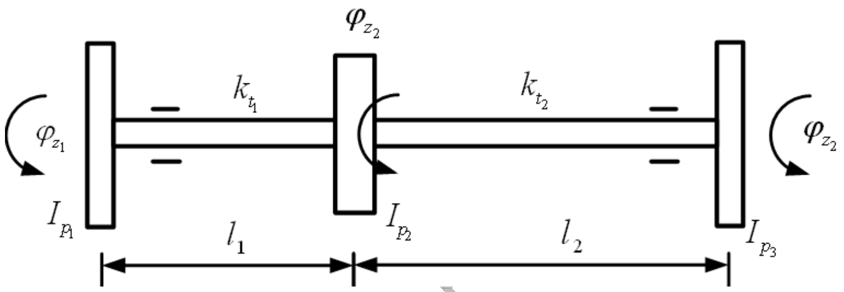


Figure 6.12 A three-disc torsional system

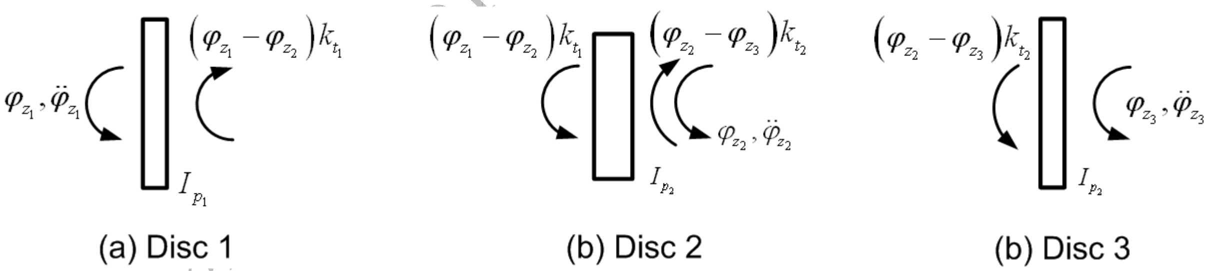


Figure 6.13 Free body diagrams of discs

From free body diagrams of individual discs three equations of motion for free vibrations can be obtained, and in the matrix form it has the following form

$$\begin{bmatrix} I_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & I_{p_3} \end{bmatrix} \begin{Bmatrix} \ddot{\varphi}_{z_1} \\ \ddot{\varphi}_{z_2} \\ \ddot{\varphi}_{z_3} \end{Bmatrix} + \begin{bmatrix} k_{t_1} & -k_{t_1} & 0 \\ -k_{t_1} & \left(k_{t_1}+k_{t_2}\right) & -k_{t_2} \\ 0 & -k_{t_2} & k_{t_2} \end{bmatrix} \begin{Bmatrix} \varphi_{z_1} \\ \varphi_{z_2} \\ \varphi_{z_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.23}$$

For free vibrations, which has the SHM, it takes the form

$$\left(-\omega_{nf}^2\begin{bmatrix} I_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & I_{p_3} \end{bmatrix} + \begin{bmatrix} k_{t_1} & -k_{t_1} & 0 \\ -k_{t_1} & \left(k_{t_1}+k_{t_2}\right) & -k_{t_2} \\ 0 & -k_{t_2} & k_{t_2} \end{bmatrix}\right)\begin{Bmatrix} \varphi_{z_1} \\ \varphi_{z_2} \\ \varphi_{z_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.24}$$

where ω_{nf} is the torsional natural frequency of the rotor system. For finding natural frequencies two methods can be adopted (i) by obtaining characteristic (or frequency) equations, and (ii) by formulating an eigen value problem.

Characteristic (or frequency) equations:

On equating the determinant to zero of the matrix in equation (6.24), we get the characteristic equation of the following form

$$\omega_{nf}^2\left\{\omega_{nf}^4 - \left(k_{t_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}} + k_{t_2}\frac{I_{p_2}+I_{p_3}}{I_{p_2}I_{p_3}}\right)\omega_{nf}^2 + \frac{k_{t_1}k_{t_2}\left(I_{p_1}+I_{p_2}+I_{p_3}\right)}{I_{p_1}I_{p_2}I_{p_3}}\right\} = 0$$

which gives natural frequencies as

$\omega_{nf_1} = 0$;

and

$$\omega_{nf_{2,3}}^2 = \frac{1}{2}\left(k_{t_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}} + k_{t_2}\frac{I_{p_2}+I_{p_3}}{I_{p_2}I_{p_3}}\right) \pm \sqrt{\frac{1}{4}\left(k_{t_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}} + k_{t_2}\frac{I_{p_2}+I_{p_3}}{I_{p_2}I_{p_3}}\right)^2 - \left(\frac{k_{t_1}k_{t_2}\left(I_{p_1}+I_{p_2}+I_{p_3}\right)}{I_{p_1}I_{p_2}I_{p_3}}\right)} \tag{6.25}$$

Mode shapes can be obtained by substituting natural frequencies obtained, one by one, into the equations (6.24) and obtaining relative amplitudes with the help of any two equations (out of three equations), as

$$\left(k_{t_1} - \omega_{nf}^2 I_{p_1}\right)\varphi_{z_1} - k_{t_1}\varphi_{z_2} = 0 \qquad \Rightarrow \frac{\varphi_{z_2}}{\varphi_{z_1}} = \frac{\left(k_{t_1} - \omega_{nf}^2 I_{p_1}\right)}{k_{t_1}} \tag{6.26}$$

and

$$-k_{t_1}\varphi_{z_1} + \left\{\left(k_{t_1}+k_{t_2}\right) - \omega_{nf}^2 I_{p_2}\right\}\varphi_{z_2} - k_{t_2}\varphi_{z_3} = 0 \tag{6.27}$$

On substituting equation (6.26) in equation (6.27), we get

$$-k_{t_1}\varphi_{z_1} + \left\{\left(k_{t_1} + k_{t_2}\right) - \omega_{nf}^2 I_{p_2}\right\}\left\{\left(k_{t_1} - \omega_{nf}^2 I_{p_1}\right)\varphi_{z_1} / k_{t_1}\right\} - k_{t_2}\varphi_{z_3} = 0 \qquad (6.28)$$

which can be simplified to

$$\frac{\varphi_{z_3}}{\varphi_{z_1}} = \frac{\left(I_{p_1} I_{p_2}\right)\omega_{nf}^4 - \left\{\left(I_{p_1} + I_{p_2}\right)k_{t_1} + I_{p_1}k_{t_2}\right\}\omega_{nf}^2 + \left(k_{t_1}k_{t_2}\right)}{k_{t_1}k_{t_2}} \qquad (6.29)$$

It should be noted that from equations (6.26) and (6.29) for $\omega_{nf_1} = 0$, we have $\varphi_{z_2} / \varphi_{z_1} = \varphi_{z_3} / \varphi_{z_1} = 1$ (or $\varphi_{z_1} = \varphi_{z_2} = \varphi_{z_3}$) that belongs to the rigid body mode. Similarly, for the other two natural frequencies relative amplitudes of disc can be obtained by substituting these natural frequencies one by one in equations (6.26) and (6.29).

An eigen value problem:

A more general method of obtaining of natural frequencies and mode shapes is to formulate an eigen value problem and that can relatively easily be solved by computer routines. Eigen values of the eigen value problem of equation (6.24) gives natural frequencies, and eigen vectors represent mode shapes. Equation (6.24) can be written as

$$\left(-\omega_{nf}^2 [M] + [K]\right)\{\Phi\} = \{0\} \qquad (6.30)$$

with

$$[M] = \begin{bmatrix} I_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & I_{p_3} \end{bmatrix}; \qquad [K] = \begin{bmatrix} k_{t_1} & -k_{t_1} & 0 \\ -k_{t_1} & \left(k_{t_1} + k_{t_2}\right) & -k_{t_2} \\ 0 & -k_{t_2} & k_{t_2} \end{bmatrix}; \qquad \{\Phi\} = \begin{Bmatrix} \varphi_{z_1} \\ \varphi_{z_2} \\ \varphi_{z_3} \end{Bmatrix}$$

On multiplying both sides by the inverse of mass matrix in equation (6.30), we get a standard eigen value problem of the following form

$$\left(-\omega_{nf}^2 [I] + [D]\right)\{\Phi\} = \{0\} \qquad (6.31)$$

with

$$[D] = [M]^{-1}[K]$$

The eigen value and eigen vector of the matrix [D] can be obtained conveniently by hand calculations for the matrix size up to 3×3, however, for the larger size matrix from multi-DOF rotor systems any standard software (e.g., MATLAB) could be used. The square root of eigen values will give the natural frequencies and corresponding eigen vectors as mode shapes (i.e., relative amplitudes). These methods would now be illustrated through an example.

Example 6.4 Obtain torsional natural frequencies of a turbine-coupling-generator rotor system as shown in Figure 6.14. The rotor is assumed to be supported on frictionless bearings. The polar mass moment of inertia of the turbine, coupling and generator are I_{p_1} = 25 kg-m^2, I_{p_2} = 5 kg-m^2 and I_{p_3} = 50 kg-m^2, respectively; and these are assumed to be thin discs. Take the modulus of rigidity of the shaft as $G = 0.8 \times 10^{11}$ N/m^2. Assume the shaft diameter uniform throughout and is equal to 0.2 m and the length of shafts between the bearing-turbine-coupling-generator-bearing are 1 m each so that the total span is 4 m. Consider the shaft as massless.

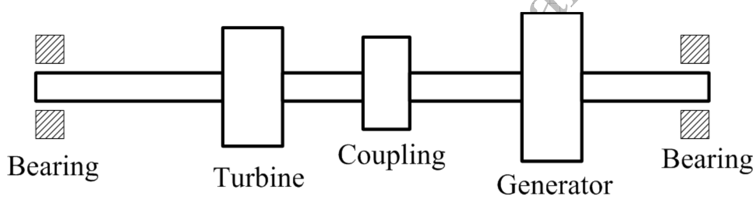


Figure 6.14 A turbine-generator set

Solution: It should be noted that for free-free end conditions both ends of the shaft segments (i.e., between bearing and turbine, and generator and bearing) will not have torsional displacements. Hence, only shaft segments between the turbine and the coupling (let us take it as shaft 1), and between the coupling and the generator (shaft 2) will have torsional stiffness effect. Hence, we have the following data

$$l_1 = l_2 = l = 1 \text{ m}, \qquad J_1 = J_2 = J = \frac{\pi}{32} d^4 = \frac{\pi}{32} 0.2^4 = 1.5708 \times 10^{-4} \text{ m}^4,$$

and

$$k_{t_1} = k_{t_2} = \frac{GJ}{l} = \frac{\left(0.8 \times 10^{11}\right)\left(1.5708 \times 10^{-4}\right)}{1} = 1.257 \times 10^7 \text{ N-rad/m}^2.$$

Natural frequencies of three-disc rotor system are given as (equation (6.32))

$$\omega_{nf_1} = 0$$

and

$$\omega_{nf_{2,3}}^2 = \frac{1}{2}\left(k_{t_1} \frac{I_{p_1} + I_{p_2}}{I_{p_1} I_{p_2}} + k_{t_2} \frac{I_{p_2} + I_{p_3}}{I_{p_2} I_{p_3}} \right) \pm \sqrt{\frac{1}{4}\left(k_{t_1} \frac{I_{p_1} + I_{p_2}}{I_{p_1} I_{p_2}} + k_{t_2} \frac{I_{p_2} + I_{p_3}}{I_{p_2} I_{p_3}} \right)^2 - \left(\frac{k_{t_1} k_{t_2} \left(I_{p_1} + I_{p_2} + I_{p_3} \right)}{I_{p_1} I_{p_2} I_{p_3}} \right)}$$

On substituting values of various parameters of the present problem in above equation, it gives

$\omega_{nf_2} = 0$ rad/s; $\qquad \omega_{nf_2} = 611.56$ rad/s; $\qquad \omega_{nf_3} = 2325.55$ rad/s;

The mode shape (relative angular displacements of various discs) can be obtained as summarised in Table 6.1 (refer equations (6.33) and (6.34)). Fig. 6.15 shows mode shapes with node locations, in drawing T, C and G represent location of the turbine, coupling and generator, respectively.

Table 6.1 Relative angular displacements of various discs

Relative displacement	$\omega_{nf_2} = 0$ rad/s	$\omega_{nf_2} = 611.56$ rad/s	$\omega_{nf_3} = 2325.55$ rad/s
$\dfrac{\varphi_{z_2}}{\varphi_{z_1}} = \dfrac{k_{t_1} - \omega_{nf}^2 I_{p_1}}{k_{t_1}}$	1	0.2563	-9.7600
$\dfrac{\varphi_{z_3}}{\varphi_{z_1}} = \dfrac{\left(I_{p_1} I_{p_2}\right)\omega_{nf}^4 - \left\{\left(I_{p_1} + I_{p_2}\right)k_{t_1} + I_{p_1} k_{t_2}\right\}\omega_{nf}^2 + \left(k_{t_1} k_{t_2}\right)}{k_{t_1} k_{t_2}}$	1	-0.5256	0.4754

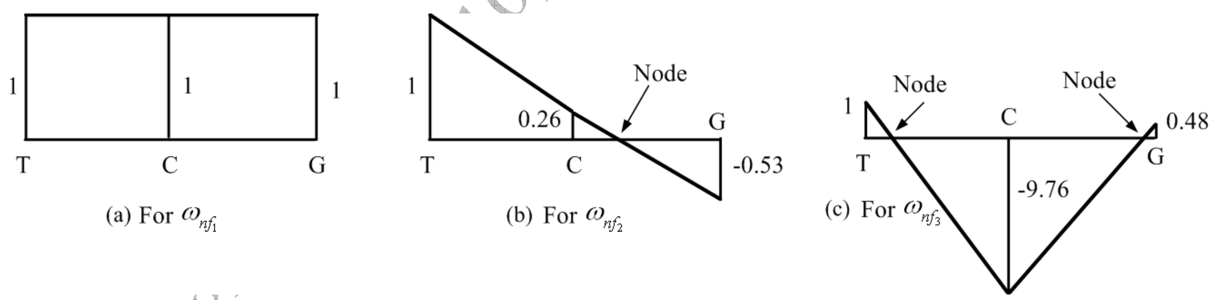


Fig. 6.15 Three mode shapes corresponding to three torsional natural frequencies

Node locations in the second and third modes can be obtained as follows:

Second mode: Only single node (Fig. 6.15b) is present between the coupling and the generator. Hence, from the node position to the generator a single-DOF rotor system can be assumed with the length of shaft as $l_{gn}^{(1)}$ (superscript corresponding to single-node mode and subscript *gn* represent from generator to node) and polar mass moment of inertia as I_{p_3}, this gives

$$\omega_{nf_2}^2 = \frac{k_{t_{gn}}}{I_{p_3}}, \qquad \Rightarrow l_{gn}^{(1)} = \frac{GJ}{I_{p_3}\omega_{nf_2}^2} = \frac{\left(0.8\times10^{11}\right)\left(1.5708\times10^{-4}\right)}{50\times611.56^2} = 0.6723 \text{ m}$$

Third mode: Two nodes are present (Fig. 6.15c), hence the node locations are obtained as

$$l_{tn}^{(2)} = \frac{GJ}{I_{p_1}\omega_{nf_3}^2} = \frac{\left(0.8\times10^{11}\right)\left(1.5708\times10^{-4}\right)}{25\times2325.55^2} = 0.0930 \text{ m}$$

and

$$l_{gn}^{(2)} = \frac{GJ}{I_{p_3}\omega_{nf_3}^2} = \frac{\left(0.8\times10^{11}\right)\left(1.5708\times10^{-4}\right)}{50\times2325.55^2} = 0.0465 \text{ m}$$

where the superscript in the length represent two-node mode and subscript *tn* represents from turbine to nearest node.

Now using the eigen value problem procedure, the above problem will be solved again. This will demonstrate how powerful this procedure is even for multi-DOF systems. The mass and stiffness matrices can be given as

$$[M] = \begin{bmatrix} I_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & I_{p_3} \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{ kg-m}^2,$$

and

$$[K] = \begin{bmatrix} k_{t_1} & -k_{t_1} & 0 \\ -k_{t_1} & \left(k_{t_1}+k_{t_2}\right) & -k_{t_2} \\ 0 & -k_{t_2} & k_{t_2} \end{bmatrix} = \begin{bmatrix} 1.257 & -1.257 & 0 \\ -1.257 & 2.514 & -1.257 \\ 0 & -1.257 & 1.257 \end{bmatrix} \times10^7 \text{ N/ m}^2.$$

Hence, the eigen value problem stiffness matrix becomes

$$[D] = [M]^{-1}[K]$$

$$= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 50 \end{bmatrix}^{-1} \begin{bmatrix} 1.257 & -1.257 & 0 \\ -1.257 & 2.514 & -1.257 \\ 0 & -1.257 & 1.257 \end{bmatrix} \times10^7 = \begin{bmatrix} 0.5028 & -0.5028 & 0 \\ -2.5140 & 5.0280 & -2.5140 \\ 0 & -0.2514 & 0.2514 \end{bmatrix} \times10^6$$

Eigen values and eigen vectors are given as (by the MATLAB of the above matrix)

$$\{\lambda\}=\begin{Bmatrix}5.4082\\0.3740\\0\end{Bmatrix}\times10^6,\ \text{and}\qquad [X]=\begin{bmatrix}-0.1018 & -0.8632 & 0.5774\\ 0.9936 & -0.2212 & 0.5774\\ -0.0484 & 0.4537 & 0.5774\end{bmatrix}$$

Where the columns of matrix [*X*] represent the mode shapes. Hence, natural frequencies are obtained as

$$\{\omega_{nf}\}=\begin{Bmatrix}\omega_{nf_3}\\\omega_{nf_2}\\\omega_{nf_1}\end{Bmatrix}=\{\sqrt{\lambda}\}=\begin{Bmatrix}2325.55\\611.56\\0\end{Bmatrix}\ \text{rad/s}$$

The mode shape can be normalised as (in each column elements is divided by the corresponding first row element, e.g. 0.9936/(-0.018) = -9.76), -0.0484/(-0.018) = 0.48, -0.2212/(-0.8632) = 0.26, etc.)

$$[X]=\left[\begin{Bmatrix}\varphi_{x_T}\\\varphi_{x_C}\\\varphi_{x_G}\end{Bmatrix}_{\omega_{nf3}}\quad\begin{Bmatrix}\varphi_{x_T}\\\varphi_{x_C}\\\varphi_{x_G}\end{Bmatrix}_{\omega_{nf2}}\quad\begin{Bmatrix}\varphi_{x_T}\\\varphi_{x_C}\\\varphi_{x_G}\end{Bmatrix}_{\omega_{nf1}}\right]=\begin{bmatrix}1 & 1 & 1\\-9.76 & 0.26 & 1\\0.48 & -0.53 & 1\end{bmatrix}$$

These mode shapes are exactly same as in Fig. 6.15.

6.4.2 An indirect approach

From the previous method, it is clear that for a particular natural frequency a unique mode shape exists. In the present method, the information regarding the possible mode shapes would be utilised to get the corresponding natural frequencies. In case the shaft has steps then, the first step would be to reduce the actual shaft to an equivalent shaft of uniform diameter as shown in Figure 6.16(a).

For three-disc rotor system, three natural frequencies are expected and correspondingly three natural (or normal) modes of vibrations. Since the free-free boundary conditions one of the modes would be the rigid-body mode, in which all the discs have same motion. Apart from the rigid body mode, there will be two possible natural modes of vibration, in which the rotors all reach their extreme positions at same instant and all pass through their equilibrium position at the same instant. There will be a different natural frequency for each of these normal modes.

In one mode there is a single *node* (a point where there will not be any angular displacement) between discs 1 and 2 or between discs 2 and 3 (see Figure 6.16(b)). It depends upon the relative polar mass moment of inertia of discs, and the stiffness of shaft segments. However, the oscillations of the outside discs 1 and 3 are opposite in phase. The disc 2 will have same or opposite phase with disc 1 (or disc 3) and it depends upon the node position. It is assumed in Figure 6.16(b) that the node lies between discs 2 and 3.

While in the other mode there are two nodes, one between discs 1 and 2 and the other between discs 2 and 3 (as shown in Fig. 6.16(c)). Oscillations of outside discs (1 and 3) are now in phase, while the inside disc will have opposite phase with respect to both discs 1 & 3.

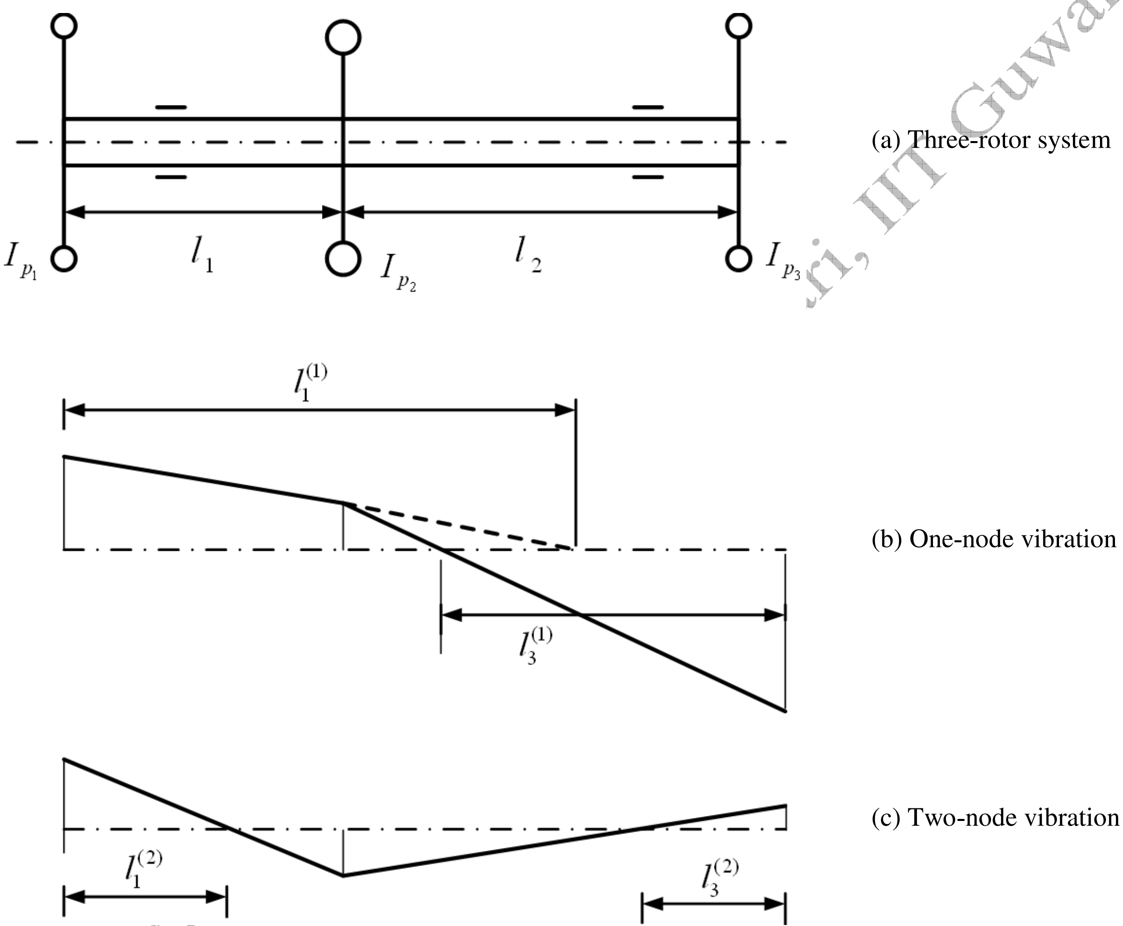


Fig. 6.16 A three-disc rotor system with two possible flexible modes

Let I_{P_1}, I_{P_2} and I_{P_3} be the polar mass moment of inertia of discs 1, 2 and 3, respectively. For two node vibration, let $l_1^{(2)}$ be the distance of one node from disc 1, and $l_3^{(2)}$ the distance of the other node from disc 3 (see Fig. 6.16b). Then the natural frequency of the single-DOF cantilever system with disc 1 is given as

$$\omega_{nf_1}^{(2)} = \sqrt{\frac{k_{t_1}^{(2)}}{I_{p_1}}} = \sqrt{\frac{GJ}{l_1^{(2)}}\frac{1}{I_{p_1}}} \tag{6.35}$$

Similarly, for the single-DOF cantilever system with disc 3 (Figure 6.16c), we have

$$\omega_{nf_3}^{(2)} = \sqrt{\frac{GJ}{l_3^{(2)}}\frac{1}{I_{p_3}}} \tag{6.36}$$

For the single-DOF fixed-fixed system with disc 2 (Fig. 6.16c), we have

$$\omega_{nf_2}^{(2)} = \sqrt{\frac{k_{t_2}^{(2)}}{I_{p_2}}} \tag{6.37}$$

where

$$k_{t_2}^{(2)} = \frac{GJ}{l_1 - l_1^{(2)}} + \frac{GJ}{l_2 - l_3^{(2)}} = GJ\frac{l_1 + l_2 - l_1^{(2)} - l_3^{(2)}}{\left(l_1 - l_1^{(2)}\right)\left(l_2 - l_3^{(2)}\right)} \tag{6.38}$$

where k_{t_b} is the torsional stiffness of a rotor system with fixed-fixed end conditions. On substituting equation (6.38) into equation (6.37), we get

$$\omega_{nf_2}^{(2)} = \sqrt{\frac{GJ}{I_{p_2}}\frac{\left(l_1 + l_2 - l_1^{(2)} - l_3^{(2)}\right)}{\left(l_1 - l_1^{(2)}\right)\left(l_2 - l_3^{(2)}\right)}} \tag{6.39}$$

Since for a particular mode all frequencies $\omega_{nf_1}^{(2)}$, $\omega_{nf_2}^{(2)}$ and $\omega_{nf_3}^{(2)}$ must be equal (superscript represents the two-node mode). This leads to two independent equations to be solved for $l_1^{(2)}$ and $l_3^{(2)}$. Once we know these node positions we could be able to get the natural frequency of the two-node (or one-node) mode. On equating equations (6.35) and (6.36), we get

$$l_1^{(2)} I_{p_1} = l_3^{(2)} I_{p_3} \tag{6.40}$$

Similarly on equating equations (6.35) and (6.39), we get

$$\frac{1}{l_1^{(2)} I_{p_1}} = \frac{1}{I_{p_2}}\frac{\left(l_1 + l_2 - l_1^{(2)} - l_3^{(2)}\right)}{\left(l_1 - l_1^{(2)}\right)\left(l_2 - l_3^{(2)}\right)} \tag{6.41}$$

Equation (6.40) can be used to eliminate $l_1^{(2)}$ from equation (6.41), and it get simplified to

$$\left[\frac{I_{p_2}I_{p_3}}{I_{p_1}}+\frac{I_{p_3}^2}{I_{p_1}}+I_{p_3}\right]\left(l_3^{(2)}\right)^2-\left[\frac{I_{p_3}I_{p_2}l_2}{I_{p_1}}+I_{p_2}l_1+I_{p_3}\left(l_1+l_2\right)\right]l_3^{(2)}+I_{p_2}l_1l_2=0 \tag{6.42}$$

The two roots of $l_3^{(2)}$ from this quadratic give positions of nodes for the one-node and two-node vibration frequencies. The actual frequencies are obtained by substituting the two values of $l_3^{(2)}$ in equation (6.36). From equation (6.40) two values of $l_1^{(2)}$ could be obtained corresponding to two values of $l_3^{(2)}$. Note that only one of these two values of $l_1^{(2)}$ may give the position of a real node, while the other gives the point at which the elastic line between discs 1 and 2, when produced, cuts the axis of the shaft (as shown in Fig 6.16(b) by the dotted line). The above method can be extended for other boundary conditions (fixed-free, fixed-fixed, etc.) and for more number of discs, however, the complexity of handling higher degree of polynomials will be tremendous. The present method is now illustrated through an example.

Example 6.5 Solve the Example 6.4 by the indirect method described in previous section.

Solution: From equation (6.42), we have

$$\left[\frac{I_{p_2}I_{p_3}}{I_{p_1}}+\frac{I_{p_3}^2}{I_{p_1}}+I_{p_3}\right]\left(l_3^{(2)}\right)^2-\left[\frac{I_{p_3}I_{p_2}l_2}{I_{p_1}}+I_{p_2}l_1+I_{p_3}\left(l_1+l_2\right)\right]l_3^{(2)}+I_{p_2}l_1l_2=0$$

On substituting values of physical parameters (Fig. 6.17a), we get

$$\left[\frac{5\times 50}{25}+\frac{50^2}{25}+50\right]\left(l_3^{(2)}\right)^2-\left[\frac{5\times 50\times 1}{25}+5\times 1+50\times 2\right]l_3^{(2)}+5\times 1\times 1=0$$

or

$$160\left(l_3^{(2)}\right)^2-115l_3^{(2)}+5=0$$

which gives two values corresponding to two modes (i.e., the one and two -node modes, Figs. 6.17 b and c), as

$l_3^{(2)}$ =0.6723 m and 0.04648 m.

Two possible values of $l_1^{(2)}$ can be obtained from equation (6.40), as

$$l_1^{(2)} = l_3^{(2)} I_{p_3} / I_{p_1}$$

which gives two values corresponding to two modes (i.e., the one and two -node modes),as

$$l_1^{(2)} = 1.3446 \text{ m and } 0.09297 \text{ m.}$$

Hence, we have two solutions

$$(l_1^{(2)},\ l_3^{(2)}) = (0.09297,\ 0.04648) \text{ m} \qquad \text{and} \qquad (l_1^{(2)},\ l_3^{(2)}) = (1.3446,\ 0.6723)$$

It is clear that two nodes are possible at ($l_1^{(2)}$, $l_3^{(2)}$) = (0.09297, 0.04648) m (Fig. 6.17c). While the single node is possible at ($l_1^{(1)}$, $l_3^{(1)}$) = (1.3446, 0.6723) m out of which both are feasible (Fig. 6.17b), since they represent the same point. Hence, corresponding $l_1^{(1)}$ = 0.6723. It should be noted that mode shapes in Fig. 6.17(band c) are not to the scale; however, qualitative comparison can be made with the previous example. Quantitatively also it can be observed that they are exactly same.

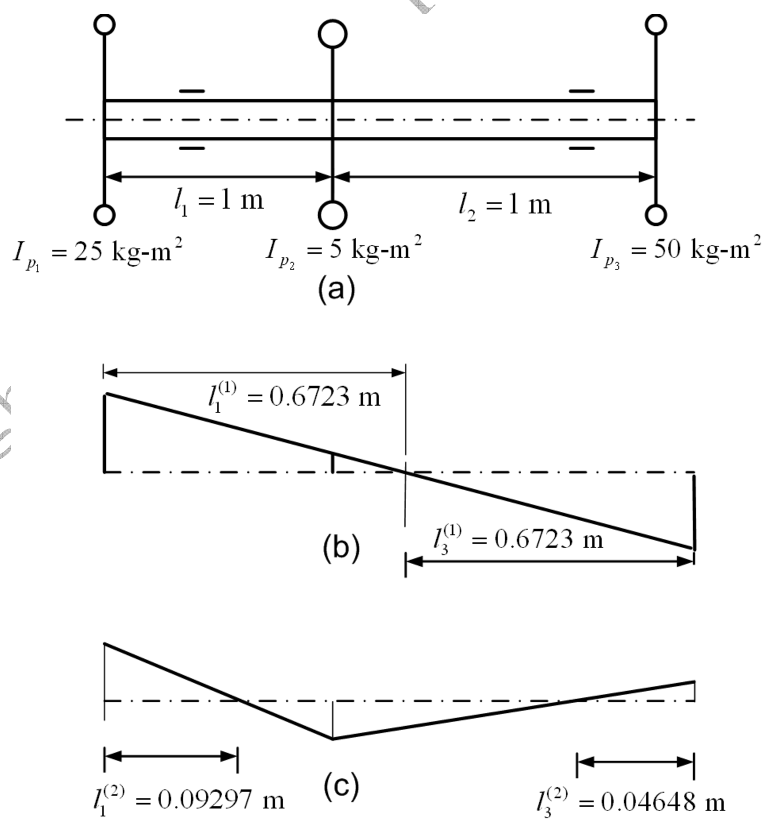


Fig. 6.17 (a) A three-mass rotor system (b) single node mode shape (c) two node mode shape

Now, the natural frequency corresponding to two-node mode can be obtained as

$$\omega_{nf_3}^{(2)}=\sqrt{\frac{GJ}{l_3^{(2)}}\frac{1}{I_{p_3}}}=\sqrt{\frac{\left(0.8\times10^{11}\right)\times\left(1.5708\times10^{-4}\right)}{0.04648}\frac{1}{50}}=2325.34 \text{ rad/s}$$

with

$$J=\frac{\pi}{32}d^4=\frac{\pi}{32}0.2^4=1.5708\times10^{-4}\text{ m}^4.$$

The natural frequency corresponding to single-node mode can be obtained as

$$\omega_{nf_2}^{(1)}=\sqrt{\frac{GJ}{l_3^{(1)}}\frac{1}{I_{p_3}}}=\sqrt{\frac{\left(0.8\times10^{11}\right)\times\left(1.5708\times10^{-4}\right)}{0.6723}\frac{1}{50}}=611.42 \text{ rad/s}$$

It should be noted that these natural frequencies and the node positions are exact same as obtained in example 6.4.

6.5 Transfer Matrix Methods

When there are more than three discs in the rotor system or when the mass of the shaft itself may be significant (i.e., continuous systems, which has infinite-DOFs) so that more number of lumped masses to be considered, then the analysis described in previous sections (i.e., the single, two or three-discs rotor systems) become complicated and inadequate to model such systems. Such rotor systems are called the multi-DOF system. Alternative methods are the transfer matrix method (TMM), continuous systems approach, finite element method (FEM), etc. In present chapter, we will consider TMM in detail and in the next chapter we will consider the continuous system approach and the FEM.

A typical multi-disc rotor system, supported on frictionless supports, is shown in Figure 6.18. The longitudinal axis is taken as z-axis, about which discs have angular displacements, φ_z. For the present analysis discs are considered as rigid and located at a point, and the shaft is treated as flexible and massless. The number of discs is n, the *station number* is designated from 0 to (n+1), and hence the system has total (n+2) stations as shown in Fig. 6.18. The free diagram of a shaft and a disc are shown in Figure 6.19. At particular station in the system, we have two state variables: the angular twist, $\varphi_z(t)$, and the torque, $T(t)$. Now in subsequent sections we will develop relationship of these state variables between two neighbouring stations in terms of physical properties of the disc and the shaft, and which can be used to obtain governing equations of motion of the whole rotor system.

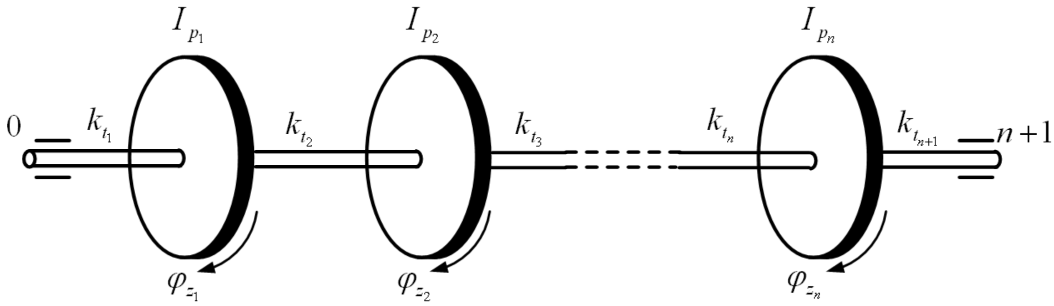


Figure 6.18 A multi-disc rotor system

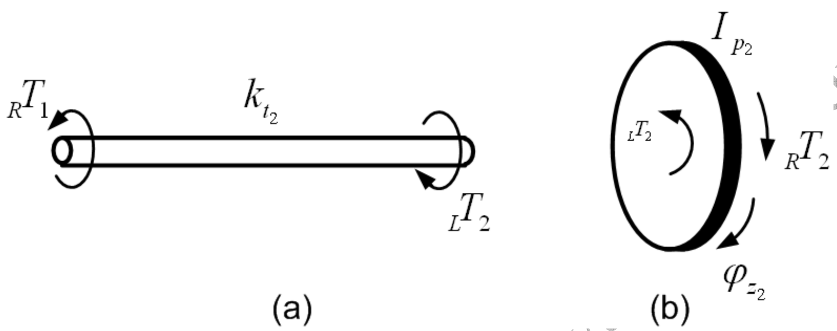


Fig. 6.19(a) A free body diagram of shaft section 2 (b) A free body diagram of rotor section 2

6.5.1 A point matrix: In this subsection we will develop a relationship between state variables at either end (i.e., the right and left sides) of a disc.

The equation of motion for the disc 2 is given by (see Figure 6.19(b))

$$ {}_RT_2 - {}_LT_2 = I_{p_2}\ddot{\varphi}_{z_2} \qquad (6.43)$$

where I_p is the polar mass moment of inertia, back subscripts: R and L represent the *right* and the *left* of a disc, respectively. For free vibrations, the angular oscillation of the disc is given by

$$\varphi_{z_2} = \Phi_{z_2}\sin\omega_{nf}t \qquad \text{so that} \qquad \ddot{\varphi}_{z_2} = -\omega_{nf}^2\Phi_{z_2}\sin\omega_{nf}t = -\omega_{nf}^2\varphi_{z_2} \qquad (6.44)$$

where Φ_z is the amplitude of angular displacement, and ω_{nf} is the torsional natural frequency. On substituting equation (6.44) into equation (6.43), we get

$$ {}_RT_2 - {}_LT_2 = -\omega_{nf}^2 I_{p_2}\varphi_{z_2} \qquad \text{or} \qquad {}_RT_2 = \left(-\omega_{nf}^2 I_{p_2}\right){}_L\varphi_{z_2} + {}_LT_2 \qquad (6.45)$$

Since angular displacements on the either side of the rotor are equal, hence

$$ {}_R\varphi_{z_2} = {}_L\varphi_{z_2} \tag{6.46}$$

Equations (6.45) and (6.46) can be combined as

$$ {}_R\{S\}_2 = [P]_2 \, {}_L\{S\}_2 \tag{6.47}$$

with

$$[P]_2 = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_2} & 1 \end{bmatrix}; \qquad \{S\} = \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}$$

where $\{S\}_2$ is the *state vector* corresponding to station 2, and $[P]_2$ is the *point matrix* for disc 2. Hence in general the point matrix relates a state vector, which is left to a disc, to a state vector right to the disc. When an external torque, $T_E(t)$, is applied to a disc (e.g., the disc as a gear element or a pulley driven by a belt) in the direction of the chosen positive angular displacement direction, then equation (6.47) will be modified as

$$ {}_R\{S\}_2 = [P]_2 \, {}_L\{S\}_2 + \{T_E\}_2 \tag{6.48}$$

with

$$\{T_E\}_2 = \begin{Bmatrix} 0 \\ -T_{E_2} \end{Bmatrix}$$

It will be more convenient to write equation (6.48) in the following form

$$ {}_R\{S^*\}_2 = [P^*]_2 \, {}_L\{S^*\}_2 \tag{6.49}$$

with

$$[P^*]_2 = \begin{bmatrix} 1 & 0 & 0 \\ -\omega_{nf}^2 I_{p_2} & 1 & -T_E \\ 0 & 0 & 1 \end{bmatrix}, \qquad \{S^*\} = \begin{Bmatrix} \varphi_z \\ T \\ 1 \end{Bmatrix}$$

where $[P^*]$ and $[S^*]$ are called the *modified point matrix* and the *modified state vector*, respectively. It should be noted the third equation of expression (6.49) is an identity equation, and it helps in including the external torque in the modified point matrix.

6.5.2 A field matrix: In this subsection we will develop a relationship between state variables at two ends of a shaft segment. For shaft element 2 as shown in Figure 6.19(a), the angle of twist is related to its torsional stiffness, k_t, and to the torque, $T(t)$, which is transmitted through it, as

$$ {}_L\varphi_{z_2} - {}_R\varphi_{z_1} = \frac{{}_R T_1}{k_{t_2}} \qquad \text{or} \qquad {}_L\varphi_{z_2} = {}_R\varphi_{z_1} + \frac{{}_R T_1}{k_{t_2}} \tag{6.50} $$

Since the torque is same at either end of the shaft, hence

$$ {}_L T_2 = {}_R T_1 \tag{6.51} $$

On combining equations (6.50) and (6.51) in the matrix form, we get

$$ {}_L\{S\}_2 = [F]_2\, {}_R\{S\}_1 \tag{6.52} $$

with

$$ [F]_2 = \begin{bmatrix} 1 & 1/k_{t_2} \\ 0 & 1 \end{bmatrix}; \qquad \{S\} = \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix} $$

where $[F]_2$ is the *field matrix* for the shaft element 2. Hence, in general, the field matrix relates a state vector which is one end of a shaft segment to the other end of the shaft segment. It should be noted that equation (6.52) is also valid for a torsional spring (e.g., a flexible coupling between two shaft segments), which has k_t as the torsional stiffness, however, such spring have negligible axial length as compared to the shaft length. Ideally such torsional springs can be considered as a *point spring* (similar to a *point mass*). A flexible coupling between a motor and a shaft or between a turbine and a generator could be modelled by such torsional springs.

Equation (6.52) can be modify to take into account an external toque in the rotor system (it assumed here that the external torque applied at disc locations only), as

$$ {}_L\{S^*\}_2 = [F^*]_2\, {}_R\{S^*\}_1 \tag{6.53} $$

with

$$ [F^*]_2 = \begin{bmatrix} 1 & 1/k_{t_2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \{S^*\} = \begin{Bmatrix} \varphi_z \\ T \\ 1 \end{Bmatrix} $$

where $[F^*]$ is the *modified field matrix*.

On substituting equation (6.52) into equation (6.47), we get

$$_R\{S\}_2 = [U]_2\, _R\{S\}_1$$

with

$$[U]_2 = [P]_2[F]_2 = \begin{bmatrix} 1 & 1/k_{t_2} \\ -\omega_{nf}^2 I_{p_2} & 1-\dfrac{\omega_{nf}^2 I_{p_2}}{k_{t_2}} \end{bmatrix}$$

where $[U]_2$ is the transfer matrix, which relates the state vector at right of station 2 to the state vector at right of station 1, when the external toque is absent. On the same lines, we can write

$$\begin{aligned} &_R\{S\}_1 = [U]_1\{S\}_0 \\ &_R\{S\}_2 = [U]_2\, _R\{S\}_1 = [U]_2[U]_1\{S\}_0 \\ &_R\{S\}_3 = [U]_3\, _R\{S\}_2 = [U]_3[U]_2[U]_1\{S\}_0 \\ &\vdots \\ &_R\{S\}_n = [U]_n\, _R\{S\}_{n-1} = [U]_n[U]_{n-1}\cdots[U]_1\{S\}_0 \\ &_R\{S\}_{n+1} = [U]_{n+1}\, _R\{S\}_n = [U]_{n+1}[U]_n\cdots[U]_1\{S\}_0 = [T]\{S\}_0 \end{aligned} \qquad (6.54)$$

where $\{S\}_0$ is the state vector at 0^{th} station (i.e., for the present case leftmost station of the rotor system), $_R\{S\}_{n+1}$ is the state vector at $(n+1)^{th}$ station (i.e., for the present case rightmost station of the rotor system), and $[T]$ is the *overall system transfer matrix*. Hence, it relates the state vector at far left to the state vector at far right. When the external toque, T_E, is also present then simply the modified point and field matrices should be considered, as

$$_R\{S^*\}_2 = [U^*]_2\, _R\{S^*\}_1$$

with

$$[U^*]_2 = [P^*]_2[F^*]_2 = \begin{bmatrix} 1 & 0 & 0 \\ -\omega_{nf}^2 I_{p_2} & 1 & -T_E \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/k_{t_2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/k_{t_2} & 0 \\ -\omega_{nf}^2 I_{p_2} & \left(-\omega_{nf}^2 I_{p_2}/k_{t_2}+1\right) & -T_E \\ 0 & 0 & 1 \end{bmatrix}$$

where $[U]_2$ is the modified transfer matrix. It should be noted that the size of the overall system transfer matrix remains same as that of the field or the point matrix, i.e. (2×2) for free vibrations; and when the external torque is also considered then the size becomes (3×3). The overall transformation for, free vibrations, can be written as

$$ {}_R\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_{n+1} = \begin{bmatrix} t_{11}\left(\omega_{nf}\right) & t_{12}\left(\omega_{nf}\right) \\ t_{21}\left(\omega_{nf}\right) & t_{22}\left(\omega_{nf}\right) \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_0 \tag{6.55} $$

The overall transfer matrix elements are a function of the natural frequency, ω_{nf}, of the system (or the excitation frequency, ω, for the case when the external toque is present). Now different boundary conditions will be considered to illustrate the application of boundary conditions in the overall transfer matrix equation for obtaining natural frequencies and mode shapes of the system. In all cases number of discs is kept equal to *n* and depending upon the boundary conditions and location of discs the station numbers may change.

(i) *Free-free boundary conditions*: For free-free boundary conditions (Fig. 6.18), at each ends of the rotor system the torque transmitted through the shaft is zero, hence

$$ {}_R T_{n+1} = T_0 = 0 \tag{6.56} $$

On using equation (6.56) into equation (6.55), the second set of equation gives

$$ t_{21}\left(\omega_{nf}\right) {}_R\varphi_{z_o} = 0 \tag{6.57} $$

Since ${}_R\varphi_{z_0} \neq 0$ for a general case, hence from equation (6.57) we must have

$$ t_{21}\left(\omega_{nf}\right) = 0 \tag{6.58} $$

which is satisfied for ω_{nf_i}; $i = 1, 2, \cdots, N$, where *N* is the number of degrees of freedom of the system (for the present case it will be equal to number of disc, *n*, in the system) and these are system natural frequencies. Equation (6.58) is called the *frequency equation* and it has a form of a polynomial in terms of the natural frequency, ω_{nf}. For higher degree polynomials these roots, ω_{nf_i}, may be found by any of root-searching techniques (e.g., Incremental method, Bisection method, Newton-Raphson method, etc.; refer to Press et al., 1998). Briefly, the root searching method is described here for the sake of completeness.

Let us define

$$ f\left(\omega_{nf}\right) = t_{21}\left(\omega_{nf}\right) $$

where *f*( ) is a function of ω_{nf} . If ω_{nf} is the initially guessed of the natural frequency, which is not actual solution. Then, let the next guess value is $\left(\omega_{nf}+\Delta\omega_{nf}\right)$ by which solution is expected to improve. Hence, by using the Taylar series expansion, we have

$$f\left(\omega_{nf}+\Delta\omega_{nf}\right)=f\left(\omega_{nf}\right)+\frac{\partial f}{\partial\omega_{nf}}\Delta\omega_{nf}+\frac{1}{2}\frac{\partial^2 f}{\partial\omega_{nf}^2}\left(\Delta\omega_{nf}\right)^2+\cdots$$

where $\Delta\omega_{nf}$ is the increment in initial guessed value of ω_{nf} . On neglecting higher order terms, it gives

$$\Delta\omega_{nf}=\frac{f\left(\omega_{nf}+\Delta\omega_{nf}\right)-f\left(\omega_{nf}\right)}{\partial f\left(\omega_{nf}\right)/\partial\omega_{nf}}$$

A flow chart of the overall solution algorithm is shown in Fig. 6.20. In the flow chart ε is a small parameter, to be chosen depending upon the function value to be minimised, and the accuracy up to which the solution is desired. It should be noted that using such a numerical analysis for finding the natural frequencies, there is no need to multiply various point and field matrices in the variable form to get the overall transfer matrix, instead it has to be done in the numerical form and that is much easier to handle.

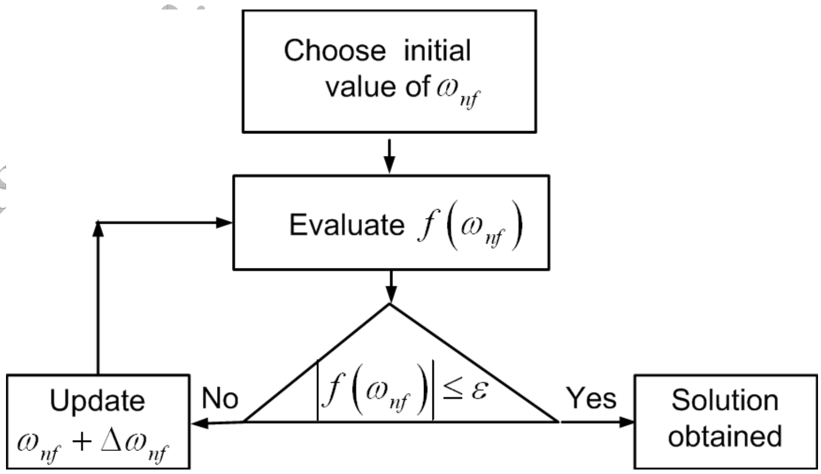


Fig. 6.20 A flow chart of an algorithm for finding roots of a function

Relative angular twists can be determined for each value of ω_{nf_i} . From the first set of equation (6.55), we have

$$ {}_R\varphi_{z_{n+1}} = t_{11}\left(\omega_{nf_i}\right) {}_R\varphi_{z_0} \tag{6.59} $$

Since mode shape is nothing but relative angular displacement between various discs. On taking ${}_R\varphi_{z_0} = 1$ as a reference value for obtaining the mode shape, we get

$$ {}_R\varphi_{z_{n+1}} = t_{11}\left(\omega_{nf_i}\right) \tag{6.60} $$

Equation (6.60) gives ${}_R\varphi_{z_{n+1}}$ for a particular value of the natural frequency ω_{nf_i}, by using equation (6.54) relative displacements of all other stations can be obtained. The mode shape can be plotted with the station number as the abscissa and the angular displacement as the ordinate. The similar process can be repeated to obtain mode shapes corresponding to other values of natural frequencies. In general, for each natural frequency there will be a corresponding distinctive mode shape.

(ii) *Fixed-free boundary conditions*: For fixed-free boundary conditions (Fig. 6.21), at fixed end (let at 0^{th} station) the angular displacement is zero and at the free end (i.e., at n^{th} station) the torque is zero, hence

$$ {}_R\varphi_{z_0} = 0 \qquad \text{and} \qquad {}_RT_n = 0 \tag{6.61} $$

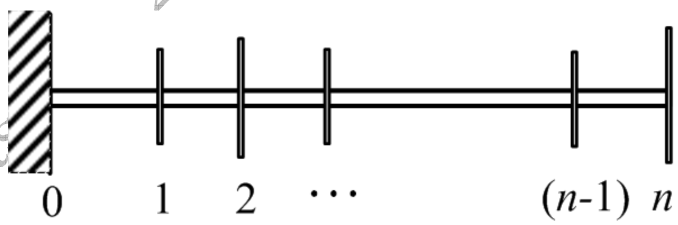


Fig. 6.21 A multi-DOF rotor system with fixed-free boundary conditions

On using equation (6.61) into equation (6.55), the second set of equation gives

$$ t_{22}\left(\omega_{nf}\right) {}_RT_0 = 0 \tag{6.62} $$

Since ${}_RT_0 \neq 0$ for a general case, hence from equation (6.62) we have the frequency equation as

$$ t_{22}\left(\omega_{nf}\right) = 0 \tag{6.63} $$

It should be noted that for the case when the free end is at 0^{th} station (i.e., at the extreme left) and the fixed end is at n^{th} station (i.e., at the extreme right), the frequency equation would be (it is assumed that the free-end and intermediate stations have a disc)

$$t_{11}\left(\omega_{nf}\right) = 0 \tag{6.64}$$

(iii) *Fixed-fixed boundary conditions*: For fixed-fixed boundary conditions (Fig. 6.22), at both fixed ends (at 0^{th} and $(n+1)^{th}$ stations) the angular displacements are zero, hence

$$\varphi_{z_0} = 0 \quad \text{and} \quad \varphi_{z_{n+1}} = 0 \tag{6.65}$$

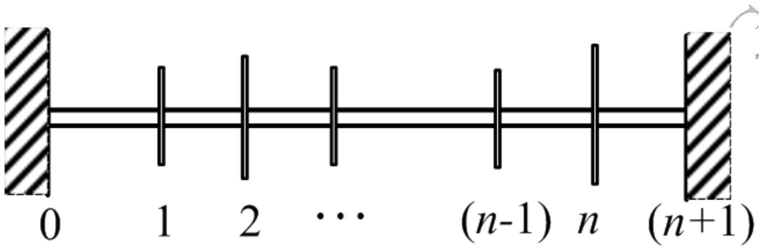


Fig. 6.22 A rotor system with fixed-fixed boundary conditions

On using equation (6.65) into equation (6.55), the second set of equation gives

$$t_{12}\left(\omega_{nf}\right) {}_{R}T_0 = 0 \tag{6.66}$$

Since ${}_{R}T_0 \neq 0$ for a general case, hence from equation (6.66) we have the frequency equation as

$$t_{12}\left(\omega_{nf}\right) = 0 \tag{6.67}$$

Table 6.2 summarises frequency equations and equations for state vector calculation for all the cases discussed above.

Table 6.2 Equations for the calculation of natural frequencies and mode shapes.

S.N.	Boundary conditions	Station numbers	Equations to get natural frequencies	Equations to get mode shapes
1	Free-free	0, n+1: Free ends	$t_{21}\left(\omega_{nf}\right)=0$	${}_R\varphi_{z_{n+1}}=t_{11}\left(\omega_{nf_i}\right)\varphi_{z_0}$
2	Cantilever (Fixed-free)	0: Fixed end, n: free end	$t_{22}\left(\omega_{nf}\right)=0$	${}_R\varphi_{z_n}=t_{12}\left(\omega_{nf_i}\right)T_{z_0}$
		0: Fixed end, n: free end	$t_{11}\left(\omega_{nf}\right)=0$	${}_RT_{z_n}=t_{21}\left(\omega_{nf_i}\right)\varphi_{z_0}$
3	Fixed-fixed	0, n+1: Fixed ends	$t_{12}\left(\omega_{nf}\right)=0$	${}_RT_{z_{n+1}}=t_{22}\left(\omega_{nf_i}\right)T_{z_0}$

In above cases we have considered intermediate supports as frictionless, and no friction of discs with the medium in which these discs are oscillating. In actual practice, we will have supports and discs with friction, and this will produce some frictional (damping) torque on to the shaft or discs. While rotor is rotating with at a certain constant spin speed, these supports and discs frictions would give a constant torque. However, the torque onto the shaft and discs will be function of the spin speed of the rotor. Overall effects of these frictions would be very less on the torsional natural frequencies of rotor systems, and for initial estimates of system dynamic characteristics it can be ignored. Torsional oscillations of the rotor with flexible elements like couplings and torsional dampers will be considered subsequently.

A word of caution regarding the numbering of stations: for the present formation we stick to the numbering scheme the 0^{th} station is assigned to the extreme left side of the station, and subsequent station numbers (i.e., 1, 2, …) are given to the station encountering towards the right. In the case numbering to the station is from extreme right and increases towards left, then the following point and field matrices should be used (which are slightly different as compared to equations (6.68) and (6.69))

$$\left[\vec{P}\right]=\begin{bmatrix}1 & 0\\ \omega_{nf}^2 I_p & 1\end{bmatrix} \quad \text{and} \quad \left[\vec{F}\right]=\begin{bmatrix}1 & -1/k\\ 0 & 1\end{bmatrix} \tag{6.70}$$

where $\left[\vec{P}\right]$ and $\left[\vec{F}\right]$ are the point and field matrices when the transformation of state vector is performed from the right to the left. For example equations (6.47) and (6.52) can be written as

$$_L\{S\}_2=\left[\vec{P}\right]_2\,{}_R\{S\}_2 \quad \text{and} \quad {}_R\{S\}_1=\left[\vec{F}\right]_2\,{}_L\{S\}_2 \tag{6.71}$$

with

$$\left[\vec{P}\right]_2=[P]_2^{-1} \quad \text{and} \quad \left[\vec{F}\right]_2=\left[\vec{F}\right]_2^{-1}$$

It should be noted that these point and field matrices are in fact inverse of the previous matrices. To avoid this confusion in the present text the station number is consistently assigned from the left end to the right end, and the transformation of the state vector is also followed the same sequence (i.e., from the left to the right). To illustrate the TMM now several simple numerical problems will be taken up.

Example 6.6 Obtain torsional natural frequencies of a rotor system as shown in Figure 6.23 by using the transfer matrix method. Assume the shaft as massless. Check the result obtained with the closed form solution available. Take G = 0.8×10^{11} N/m^2.

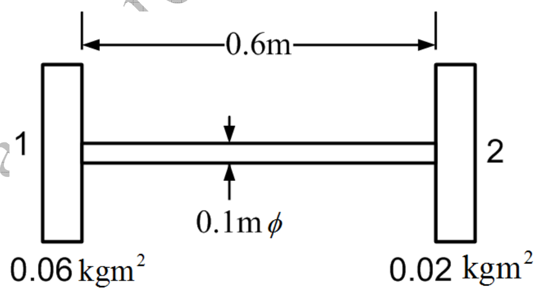


Figure 6.23

Solution: We have following properties of the rotor system

$$G=0.8\times10^{11}\ \text{N/m}^2;\quad l=0.6\ \text{m};\quad J=\frac{\pi}{32}(0.1)^4=9.82\times10^{-6}\text{m}^4$$

The torsional stiffness of the shaft is given as

$$k_t=\frac{GJ}{l}=\frac{0.8\times10^{11}\times9.82\times10^{-6}}{0.6}=1.31\times10^6\ \text{Nm/rad}$$

Analytical method: Natural frequencies in the closed form are given as

$$\omega_{nf_1} = 0; \quad \text{and} \quad \omega_{nf_2} = \sqrt{\frac{(I_{p_1} + I_{p_2})k_t}{I_{p_1} I_{p_2}}} = \sqrt{\frac{(0.06+0.02)1.31\times 10^6}{0.06\times 0.02}} = 9345.23 \text{ rad/sec}$$

Mode shapes (relative amplitudes) are given as

$$\text{for} \quad \omega_{nf_1} = 0, \qquad \frac{\Phi_{z_2}}{\Phi_{z_0}} = 1;$$

and

$$\text{for} \quad \omega_{nf_2} = 9345.23 \text{ rad/s}, \qquad \frac{\Phi_{z_2}}{\Phi_{z_0}} = -\frac{I_{p_1}}{I_{p_2}} = -3;$$

Transfer matrix method: Let the station number be 1 and 2 as shown in Fig. 6.24. State vectors can be related between stations 1 and 2, as

$${}_R\{S\}_1 = [P]_1\, {}_L\{S\}_1$$

and

$${}_R\{S\}_2 = [P]_2 [F]_2\, {}_R\{S\}_1 = [P]_2 [F]_2 [P]_1\, {}_L\{S\}_1$$

The overall transformation of state vectors between 1 & 2 is given as

$${}_R\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_2 = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/k_t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_1} & 1 \end{bmatrix} {}_L\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_1 = \begin{bmatrix} 1 & 1/k_t \\ -\omega_{nf}^2 I_{p_2} & \left(1-\omega_{nf}^2 I_{p_2}/k_t\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_1} & 1 \end{bmatrix} {}_L\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_1$$

$$= \begin{bmatrix} 1-\omega_{nf}^2 I_{p_1}/k_t & 1/k_t \\ \left\{-\omega_{nf}^2 I_{p_2} - \omega_{nf}^2 I_{p_1}\left(1-\omega_{nf}^2 I_{p_2}/k_t\right)\right\} & \left(1-\omega_{nf}^2 I_{p_2}/k_t\right) \end{bmatrix}_L \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_1$$

On substituting values of various rotor parameters, it gives

$${}_R\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_2 = \begin{bmatrix} \left(1-4.58\times 10^{-8}\omega_{nf}^2\right) & 7.64\times 10^{-7} \\ \left(-0.08\omega_{nf}^2 + 9.16\times 10^{-10}\omega_{nf}^4\right) & \left(1-1.53\times 10^{-8}\omega_{nf}^2\right) \end{bmatrix}_L \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_1 \qquad \text{(a)}$$

Since ends of the rotor are free-free type, hence, the following boundary conditions will apply

$$ {}_{L}T_1 = {}_{R}T_2 = 0 \tag{b} $$

On application of boundary conditions (b) in equation (a), we get the following condition

$$ t_{21}\left(\omega_{nf}\right) = \left(-0.08\omega_{nf}^2 + 9.16\times10^{-10}\,\omega_{nf}^4\right)\,{}_{L}\{\varphi_z\}_1 = 0 $$

which gives for the non-trial solution, the following frequency equation

$$ \omega_{nf}^2[9.16\times10^{-10}\,\omega_{nf}^2 - 0.08] = 0 $$

It gives natural frequencies as

$$ \omega_{nf_1} = 0 \ \text{ and } \ \omega_{nf_2} = 9345.23 \text{ rad/sec} $$

which are exactly the same as obtained by the closed form solution. Mode shapes can be obtained by substituting these natural frequencies, one at a time, into the first (or the second) expression of equation (a), as

$$ \frac{\Phi_{z_2}}{\Phi_{z_0}} = \left(1 - 4.58\times10^{-8}\,\omega_{nf}^2\right)\Big|_{\omega_{nf_1}=0} = 1, \qquad \text{rigid body mode} $$

and

$$ \frac{\Phi_{z_2}}{\Phi_{z_0}} = \left(1 - 4.58\times10^{-8}\,\omega_{nf}^2\right)\Big|_{\omega_{nf_2}=9345.23} = -3, \qquad \text{anti-phase mode} $$

which are also exactly the same as obtained by closed form solutions.

Example 6.7 Obtain torsional natural frequency for a cantilever shaft carrying a disc and a spring at free end as shown in Figure 6.24. The disc has the polar mass moment of inertia of 0.02 kg-m^2. The shaft has 0.4 m of the length and 0.015m of the diameter. The spring has torsional stiffness of k_{t_2} = 100 N-m/rad. Take $G = 0.8\times10^{11}$ N/m^2 for the shaft.

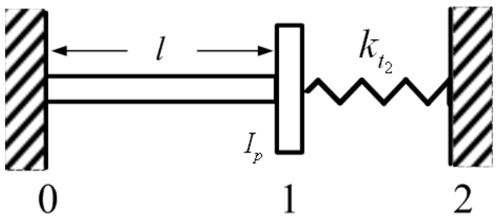


Figure 6.24 A cantilever rotor with a spring at the free end

Solution: Let the fixed end has the station number as 0, the shaft free end has the station number as 1, and spring other end fixed to the fixed support has the station number 2 (Fig. 6.24). The transformation of the state vector from station 0 to station 1 can be written as

$$_R\{S\}_1 = [P]_1 [F]_1 \{S\}_0 = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{l}{GJ} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_0 = \begin{bmatrix} 1 & \dfrac{l}{GJ} \\ -\omega_{nf}^2 I_{p_1} & 1 - \dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_0 \tag{a}$$

The spring at free end can be thought as an equivalent shaft segment with same stiffness that of the spring. The overall transfer matrix for such an idealisation between stations 0 and 2 would be

$$\{S\}_2 = [F]_2 [P]_1 [F]_1 \{S\}_0 = \begin{bmatrix} 1 & \dfrac{1}{k_{t_2}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{l}{GJ} \\ -\omega_{nf}^2 I_{p_1} & 1 - \dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_0 = \begin{bmatrix} 1 - \dfrac{\omega_{nf}^2 I_{p_1}}{k_{t_2}} & \dfrac{l}{GJ} + \dfrac{1}{k_{t_2}} \left(1 - \dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \right) \\ -\omega_{nf}^2 I_{p_1} & 1 - \dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_0 \tag{b}$$

Boundary conditions for the present case would be

$$\varphi_{z_0} = \varphi_{z_2} = 0 \tag{c}$$

On applying boundary conditions to equation (b), from first equation, we get

$$\left\{ \frac{l}{GJ} + \frac{1}{k_{t_2}} \left(1 - \frac{\omega_{nf}^2 I_{p_1} l}{GJ} \right) \right\} T_0 = 0 \tag{d}$$

Since torque T_0 can not be zero, hence we get the natural frequency from equation (d) as

$$\omega_{nf} = \sqrt{\frac{k_{t_2} + (GJ/l)}{I_{p_1}}} \text{ rad/s} \tag{e}$$

with

$$J = \frac{\pi}{32} 0.015^4 = 4.97 \times 10^{-9}\,\text{m}^4,\; I_{p_1} = 0.02\,\text{kg-m}^2,\; \frac{GJ}{l} = 994.02\,\text{Nm/rad}, \quad k_{t_2} = 100\,\text{Nm/rad}$$

Hence, we have natural frequency as

$$\omega_{nf} = 233.88 \text{ rad/s} \qquad \textit{answer}$$

From equation (e), it can be observed that the effect of the spring at the free end is to increase the effective stiffness of the system (i.e., springs connected in parallel with the equivalent stiffness of $k_{t_2} + (GJ/l)$, where GJ/l is the stiffness of the shaft).

Alternatively, the spring can be included as a boundary condition as follows. In this case the transformation equation (a) is valid. The equilibrium equation at the free end would be

$$_{R}T_1 + k_{t_2}\varphi_{z_1} = 0 \tag{f}$$

where $_{R}T_1$ is the reaction toque at the right of disc. Hence, the boundary conditions would be

$$\varphi_{z_0} = 0 \qquad \text{and} \qquad {}_{R}T_1 = -k_{t_2}\varphi_{z_1} \tag{g}$$

On application of boundary conditions (g) in equation (a), we get

$$_{R}\begin{Bmatrix} \varphi_z \\ -k_{t_2}\varphi_z \end{Bmatrix}_1 = \begin{bmatrix} 1 & \dfrac{l}{GJ} \\ -\omega_{nf}^2 I_{p_1} & 1 - \dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \end{bmatrix} \begin{Bmatrix} 0 \\ T \end{Bmatrix}_0 \tag{h}$$

Equation (h) can be split as follows

$$_{R}\varphi_{z_1} = \frac{l}{GJ} T_0 \qquad \text{and} \qquad -k_{t_2}\,{}_{R}\varphi_{z_1} = \left(1 - \frac{\omega_{nf}^2 I_{p_1} l}{GJ}\right) T_0 \tag{i}$$

which gives an eigen value problem of the following form

$$\begin{bmatrix} 1 & \dfrac{-l}{GJ} \\ k_{t_2} & 1-\dfrac{\omega_{nf}^2 I_{p_1} l}{GJ} \end{bmatrix} \begin{Bmatrix} \varphi_{z_1} \\ T_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{j}$$

For the non-trial solution, on taking determinant of the above matrix, it gives the natural frequency exactly same as in equation (e).

Example 6.8 Obtain the torsional frequency response at the disc and the support torque at the fixed end of the shaft of a rotor system shown in Fig. 6.25. An external sinusoidal torque of amplitude $T_E = 10$ Nm is applied with a single frequency, ω. Identify the torsional critical speed of the system from the response so obtained. The disc has polar mass moment of inertia of 0.02 kg-m^2. The shaft has 0.4 m of length and 0.015 m of diameter. Take $G = 0.8\times10^{11}$ N/m^2.

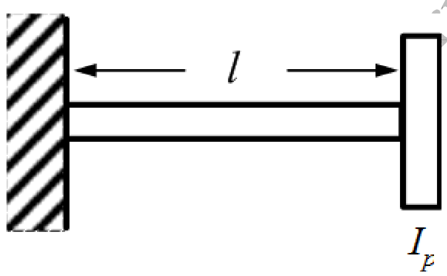


Fig 6.25 A shaft with cantilevered end conditions

Solution: Let the station number of the fixed end is 0 and that of the free end is 1. The transformation of state vector can be written as

$${}_R\{S^*\}_1 = [P^*]_1 [F^*]_1 \{S^*\}_0 = \begin{bmatrix} 1 & 0 & 0 \\ -\omega^2 I_{p_1} & 1 & -T_{E_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/k_{t_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \\ 1 \end{Bmatrix}_0 = \begin{bmatrix} 1 & 1/k_{t_1} & 0 \\ -\omega^2 I_{p_1} & 1-\dfrac{\omega^2 I_{p_1}}{k_{t_1}} & -T_{E_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varphi_z \\ T \\ 1 \end{Bmatrix}_0 \tag{a}$$

where ω is the external excitation frequency and T_E is the external torque amplitude. Boundary conditions of the present problem are

$$\varphi_{z_0} = 0 \qquad \text{and} \qquad {}_RT_1 = 0 \tag{b}$$

On application of boundary conditions (b) in equation (a), we get

$$
{}_R\begin{Bmatrix} \varphi_z \\ 0 \\ 1 \end{Bmatrix}_1 = \begin{bmatrix} 1 & 1/k_{t_1} & 0 \\ -\omega^2 I_{p_1} & 1-\dfrac{\omega^2 I_{p_1}}{k_{t_1}} & -T_{E_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ T \\ 1 \end{Bmatrix}_0 \tag{c}
$$

which gives following equations

$$
{}_R\varphi_{z_1} = \frac{T_0}{k_{t_1}} \quad \text{and} \quad 0 = \left(1-\frac{\omega^2 I_{p_1}}{k_{t_1}}\right) T_0 - T_{E_1} \tag{d}
$$

From above, the frequency response at station 1 would take the following form

$$
{}_R\varphi_{z_1} = \frac{T_{E_1}}{\left(1-\dfrac{\omega^2 I_{p_1}}{k_{t_1}}\right)} = \frac{T_{E_1}}{\left(1-\dfrac{\omega^2}{\omega_{nf}^2}\right)} \quad \text{with} \quad \omega_{nf} = \sqrt{k_t / I_p} \tag{e}
$$

For the present problem $T_{E_1} = 10$ Nm, $J = 4.97\times10^{-9}$ m^4, $I_{p_1} = 0.02$ kg-m^2, $k_{t_1} = \dfrac{GJ}{l} = 994.02$ Nm/rad, and hence $\omega_{nf} = 222.94$ rad/s.

Hence from equations (e) and (d), we have

$$
{}_R\varphi_{z_1} = \frac{10}{\left(1-\dfrac{\omega^2}{4.97\times10^4}\right)} \text{ rad} \quad \text{and} \quad T_0 = \frac{10\times994.02}{\left(1-\dfrac{\omega^2}{4.97\times10^4}\right)} \tag{f}
$$

Equation (f) can be used to plot the amplitudes of the frequency response at the disc and the reactive toque at the fixed support with respect to the excitation frequency, ω. However, it can be seen from the denominator that the resonance takes place when it becomes zero, i.e., $\omega = \omega_{nf} = 222.94$ rad/s, which is the condition of critical speed, $\omega = \omega_{cr} = 222.94$ rad/s.

Example 6.9 Find torsional natural frequencies and mode shapes of a rotor system shown in Figure 6.26. B is a fixed end, and D_1 and D_2 are rigid discs. The shaft is made of steel with the modulus of rigidity $G = 0.8\ (10)^{11}$ N/m^2 and a uniform diameter $d = 10$ mm. Shaft lengths are: $BD_1 = 50$ mm, and $D_1D_2 = 75$ mm. Polar mass moment of inertia of discs are: $I_{p_1} = 0.08$ kg-m^2 and $I_{p_2} = 0.2$ kg-m^2. Consider the shaft as massless and apply (i) the analytical method, and (ii) the transfer matrix method.

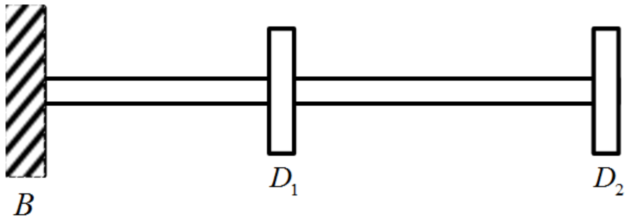


Figure 6.26

Solution: The torsional free vibration would be done by classical analytical method and the TMM to have comparion of results.

Analytical method: From free body diagrams of discs as shown in Figure 6.27, equations of motion for free vibrations can be written as

$$I_{p_1}\ddot{\varphi}_{z_1} + k_1\varphi_{z_1} + k_2(\varphi_{z_1} - \varphi_{z_2}) = 0 \quad \text{and} \quad I_{p_2}\ddot{\varphi}_{z_2} + k_2(\varphi_{z_2} - \varphi_{z_1}) = 0 \tag{a}$$

Equations of motion are homogeneous second order differential equations. In free vibrations, discs will execute simple harmonic motions.

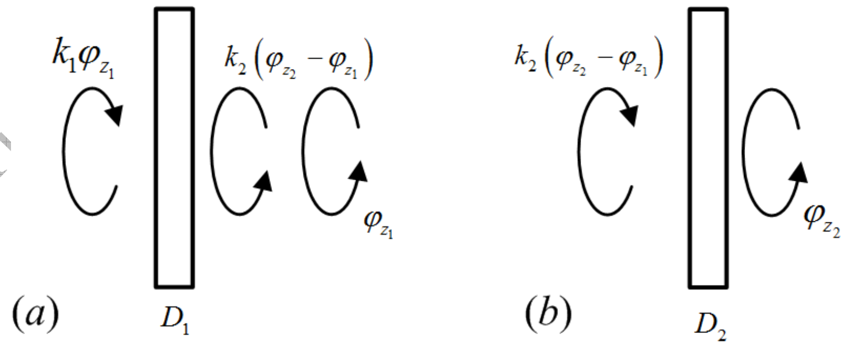


Figure 6.27 Free body diagrams of discs

For the simple harmonic motion, $\ddot{\varphi}_z = -\omega_{nf}^2\varphi_z = -\omega_{nf}^2\Phi_z \sin\omega_{nf}t$, hence equations of motion take the form

$$\begin{bmatrix} k_1 + k_2 - I_{p_1}\omega_{nf}^2 & -k_2 \\ -k_2 & k_2 - I_{p_2}\omega_{nf}^2 \end{bmatrix} \begin{Bmatrix} \Phi_{z_1} \\ \Phi_{z_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{b}$$

On taking determinant of the above matrix, it gives the frequency equation as

$$I_{p_1} I_{p_2} \omega_{nf}^4 - (I_{p_1} k_2 + I_{p_2} k_1 + I_{p_2} k_2)\omega_{nf}^2 + k_1 k_2 = 0 \tag{c}$$

which can be solved for ω_{nf}^2, as

$$\omega_{nf}^2 = \frac{I_{p_1} k_2 + I_{p_2} k_1 + I_{p_2} k_2 \pm \sqrt{\left(I_{p_1} k_2 + I_{p_2} k_1 + I_{p_2} k_2\right)^2 - 4 k_1 k_2 I_{p_1} I_{p_2}}}{2 I_{p_1} I_{p_2}} \tag{d}$$

For the present problem following properties are given

$$J_1 = \frac{\pi}{32} d^4 = \frac{\pi}{32}(0.01)^4 = 9.82 \times 10^{-4}\ \text{m}^4 = J_2$$

$$k_1 = \frac{GJ_1}{l_1} = 1570.79\ \text{N/m} \quad \text{and} \quad k_2 = \frac{GJ_2}{l_2} = 1047.19\ \text{N/m}$$

$$I_{p_1} = 0.08\ \text{kgm}^2 \quad \text{and} \quad I_{p_2} = 0.2\ \text{kgm}^2$$

From equation (d), natural frequencies are obtained as

$$\omega_{nf_1} = 54.17\ \text{rad/s} \quad \text{and} \quad \omega_{nf_2} = 187.15\ \text{rad/s}$$

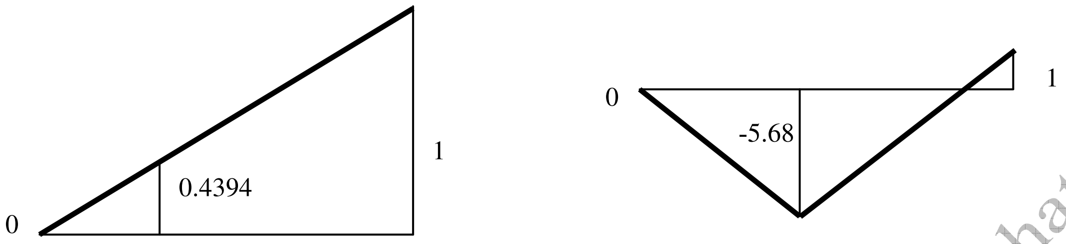
The relative amplitude ratio can be obtained from the first expression of equation (b), as

$$\frac{\Phi_{z_1}}{\Phi_{z_2}} = \frac{k_2}{k_1 + k_2 - I_{p_1}\omega_{nf}^2} = 0.4394 \quad \text{for} \quad \omega_{nf_1}; \quad \text{and} \quad -5.689 \quad \text{for} \quad \omega_{nf_2} \tag{e}$$

Alternatively, the relative amplitude ratio can be obtained from the second expression of equation (b), as

$$\frac{\Phi_{z_1}}{\Phi_{z_2}} = \frac{k_2 - I_{p_2}\omega_{nf}^2}{k_2} = 0.4394 \quad \text{for} \quad \omega_{nf_1}; \quad \text{and} \quad -5.689 \quad \text{for} \quad \omega_{nf_2} \tag{f}$$

As expected it should give the same result as in equation (e). Mode shapes are shown in Figure 6.28. In which the first one is in-phase mode and second is the anti-phase mode. Practically, the anti-phase mode is difficult to excite (more resistive torque) as compared to the in-phase mode, and because of this the natural frequency of the former mode is more than the latter.



(a) For ω_{n_1} (b) For ω_{n_2}

Figure 6.28 Mode shapes

Transfer matrix method

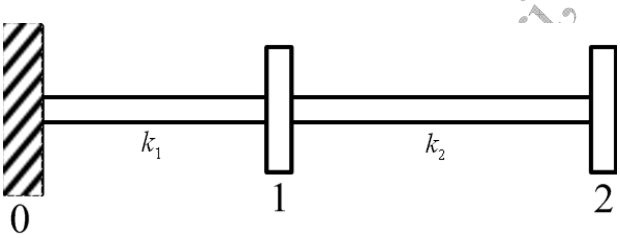


Figure 6.29 Two-discs rotor system with numbering of stations

For Figure 6.29, state vectors between 0[th] and 2[nd] stations can be related as

$$_R\{S\}_2 = [P]_2 [F]_2 [P]_1 [F]_1 \{S\}_0 \tag{g}$$

State vectors at neighbouring stations (i.e., 1 and 2, and 0 and 1) can be related as

$$_R\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_2 = \begin{bmatrix} 1 & 1/k_2 \\ -\omega_{nf}^2 I_{p_2} & \dfrac{-\omega_{nf}^2 I_{p_2}}{k_1}+1 \end{bmatrix} {}_R\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_1 \quad \text{and} \quad {}_R\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_1 = \begin{bmatrix} 1 & 1/k_1 \\ -\omega_{nf}^2 I_{p_1} & \dfrac{-\omega_{nf}^2 I_{p_1}}{k_1}+1 \end{bmatrix} {}_R\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_0 \tag{h}$$

which can be combined to give

$$_R\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_2 = \begin{bmatrix} \left(1-\dfrac{\omega_{nf}^2 I_{p_1}}{k_2}\right) & \dfrac{1}{k_1}\left(1-\dfrac{\omega_{nf}^2 I_{p_1}}{k_2}\right)+\dfrac{1}{k_2} \\ -\omega_{nf}^2 I_{p_2} - \omega_{nf}^2 I_{p_1}\left(\dfrac{-\omega_{nf}^2 I_{p_2}}{k_2}+1\right) & \left(\dfrac{t_{21}}{k_1}-\dfrac{\omega_{nf}^2 I_{p_2}}{k_2}+1\right) \end{bmatrix} \begin{Bmatrix}\varphi_z\\ T\end{Bmatrix}_0 \tag{i}$$

Boundary conditions are: at station 0, $\varphi_{z_0} = 0$; and at right of station 2, ${}_RT_2 = 0$. On application of boundary conditions in equation (i), the second equation gives the frequency equation as

$$t_{22}\left(\omega_{nf}\right) = \frac{1}{k_1}\left[-\omega_{nf}^2 I_{p_2} - \omega_{nf}^2 I_{p_1}\left(\frac{-\omega_{nf}^2 I_{p_2}}{k_2} + 1\right)\right] - \frac{\omega_{nf}^2 I_{p_2}}{k_2} + 1 = 0$$

which can be simplified as

$$I_{p_1} I_{p_2} \omega_{nf}^4 - (I_{p_1} k_2 + I_{p_2} k_1 + I_{p_2} k_2)\omega_{nf}^2 + k_1 k_2 = 0$$

It should be noted that it is same as obtained by the analytical method in equation (c). Hence, natural frequencies by TMM will be also given by equation (d). For obtaining mode shapes from equations (h) and (i), we have

$${}_R\varphi_{z_2} = t_{12} T_0; \qquad {}_R\varphi_{z_2} = {}_R\varphi_{z_1} + \frac{{}_RT_1}{k_2}; \qquad {}_R\varphi_{z_1} = \frac{T_0}{k_1} \tag{j}$$

From equation (j), we have

$$\frac{{}_R\varphi_{z_1}}{{}_R\varphi_{z_2}} = \frac{{}_R\Phi_{z_1}}{{}_R\Phi_{z_2}} = \frac{1}{k_1 t_{12}} = \frac{k_2}{k_1 + k_2 - I_{p_1}\omega_{nf}^2} \tag{k}$$

which is again same as equation (e). Since mode shapes are relative angular displacements of various discs in the rotor system, on assuming one of the angular displacement as unity (i.e., $\varphi_{z_2} = 1$), we can get torque acting at various sections of the shaft from equation (j), as

$$T_0 = \frac{1}{t_{12}} = \frac{k_2}{\omega_{nf}^2\left(k_2 I_{p_1} + k_2 I_{p_2} - I_{p_1} I_{p_2}\omega_{nf}^2\right)} \tag{l}$$

and

$${}_RT_1 = k_2\left({}_R\varphi_{z_2} - {}_R\varphi_{z_1}\right) = k_2\left(1 - \frac{T_0}{k_1}\right) = k_2\left\{1 - \frac{k_2}{\omega_{nf}^2 k_1\left(k_2 I_{p_1} + k_2 I_{p_2} - I_{p_1} I_{p_2}\omega_{nf}^2\right)}\right\} \tag{m}$$

It should be noted that these torques would be produced for a unit angular displacement at disc 2 (i.e., $\varphi_{z_2} = 1$).

Example 6.10 Find torsional natural frequencies and mode shapes of the rotor system shown in Figure 6.30. B_1 and B_2 are frictionless bearings, which provide free-free end condition; and D_1, D_2, D_3 and D_4 are rigid discs. The shaft is made of the steel with the modulus of rigidity G = 0.8 $(10)^{11}$ N/m^2 and a uniform diameter d = 20 mm. Various shaft lengths are as follows: B_1D_1 = 150 mm, D_1D_2 = 50 mm, D_2D_3 = 50 mm, D_3D_4 = 50 mm and D_4B_2 = 150 mm. The mass of discs are: m_1 = 4 kg, m_2 = 5 kg, m_3 = 6 kg and m_4 = 7 kg. Consider the shaft as mass-less. Consider discs as thin and take diameter of discs as $d_1 = 8$ cm, $d_2 = 10$ cm, $d_3 = 12$ cm, and $d_4 = 14$ cm.

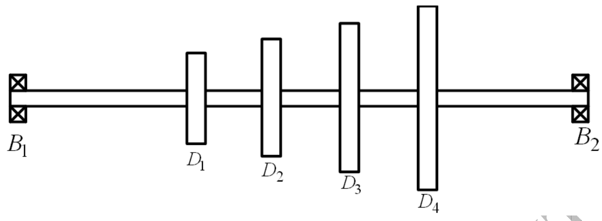


Figure 6.30 A multi-disc rotor system

Solution: The discs have the following data

m_1 = 4 kg, m_2 = 5 kg, m_3 = 6 kg, m_4 = 7 kg

$d_1 = 0.08$ m, $d_2 = 0.1$ m, $d_3 = 0.12$ m, $d_4 = 0.14$ m,

$I_{p_1} = \frac{1}{2} m_1 r_1^2 = \frac{1}{2} \times 4 \times 0.04^2 = 0.0032$ kg-m^2, $I_{p_2} = \frac{1}{2} \times 5 \times 0.05^2 = 0.00625$ kg-m^2,

$I_{p_3} = \frac{1}{2} \times 6 \times 0.06^2 = 0.0108$ kg-m^2, $I_{p_4} = \frac{1}{2} \times 7 \times 0.07^2 = 0.01715$ kg-m^2,

The shaft has GJ = 1256.64 N-m^2 and following dimensions according to station numbers (1, 2, 3 and 4 are given station numbers at disc locations; shaft segments at ends will not contribute in the free vibration for the present case)

l_1 = 50 mm, l_2 = 50 mm, l_3 = 50 mm

Now the overall transformation of the state vector can be written as

$${}_R\{S\}_4 = [T]_L\{S\}_1 \tag{a}$$

with

$$[T]=[P]_4[F]_3[P]_3[F]_2[P]_2[F]_1[P]_1 \qquad \text{(b)}$$

$$[P]_i=\begin{bmatrix}1 & 0\\ -\omega_{nf}^2 I_{p_i} & 1\end{bmatrix}; \qquad [F]_i=\begin{bmatrix}1 & 1/k_{t_i}\\ 0 & 1\end{bmatrix}; \qquad \{S\}=\begin{Bmatrix}\varphi_z\\ T\end{Bmatrix} \qquad \text{(c)}$$

From Table 6.2, for free-free boundary conditions the frequency equation is

$$f\left(\omega_{nf}\right)=t_{2,1}\left(\omega_{nf}\right)=0 \qquad \text{(d)}$$

On solving the roots of above function by the root searching method, it gives the following natural frequencies

$\omega_{nf_1}=$ 0 rad/s, $\omega_{nf_2}=$?? rad/s, $\omega_{nf_2}=$?? rad/s, $\omega_{nf_4}=$?? rad/s,

From Table 6.2, the eigen vector can be obtain from the following equation

$$_R\varphi_{z_4}=t_{11}\left(\omega_{nf_i}\right)\varphi_{z_1} \qquad \text{(e)}$$

Now on choosing $\varphi_{z_1}=1$ as reference value and let us obtain the state vectors corresponding to the second mode, i.e. $\omega_{nf_i}=\omega_{nf_2}$). From equations (e) and noting the boundary condition, we get the state vector at 1st and 4th station as

$$_L\{S\}_1=\begin{Bmatrix}1\\0\end{Bmatrix}; \quad \text{and} \quad _R\{S\}_4=\begin{Bmatrix}t_{11}\left(\omega_{nf_2}\right)\\0\end{Bmatrix}=\begin{Bmatrix}?\\0\end{Bmatrix}$$

At other stations also the state vectors can be obtained as

$$_R\{S\}_1=[P]_1\ {_L\{S\}_1}=\begin{Bmatrix}?\\?\end{Bmatrix}; \qquad _L\{S\}_2=[F]_1\ {_R\{S\}_1}=\begin{Bmatrix}?\\?\end{Bmatrix}$$

and

$$_R\{S\}_2=[P]_2\ {_L\{S\}_2}=\begin{Bmatrix}?\\?\end{Bmatrix}; \qquad _L\{S\}_3=[F]_2\ {_R\{S\}_2}=\begin{Bmatrix}?\\?\end{Bmatrix}$$

$$ {}_R\{S\}_3 = [P]_3 \, {}_L\{S\}_3 = \begin{Bmatrix} ? \\ ? \end{Bmatrix}; \qquad {}_L\{S\}_4 = [F]_3 \, {}_R\{S\}_3 = \begin{Bmatrix} ? \\ ? \end{Bmatrix} $$

Hence, the mode shape (the relative angular displacement) can be drawn as shown in Fig. 8.31 for the second modes? On the same lines other state vectors corresponding to remaining natural frequencies can be obtained to get the related mode shapes as shown in Fig.8.31?.

6.6 Geared Systems

In actual practice, it is rare that the rotor system has a single shaft (with either uniform or stepped cross sections) with multiple discs as we analysed in previous sections. In some machine the shaft may not be continuous from one end of the machine to the other, but may have a gearbox installed at one or more locations. Hence, shafts will be having different angular velocities as shown in Figure 6.30(a). For the purpose of analysis the geared system must be reduced to system with a continuous shaft so that they may be analysed for torsional vibrations by methods as described in preceding sections.

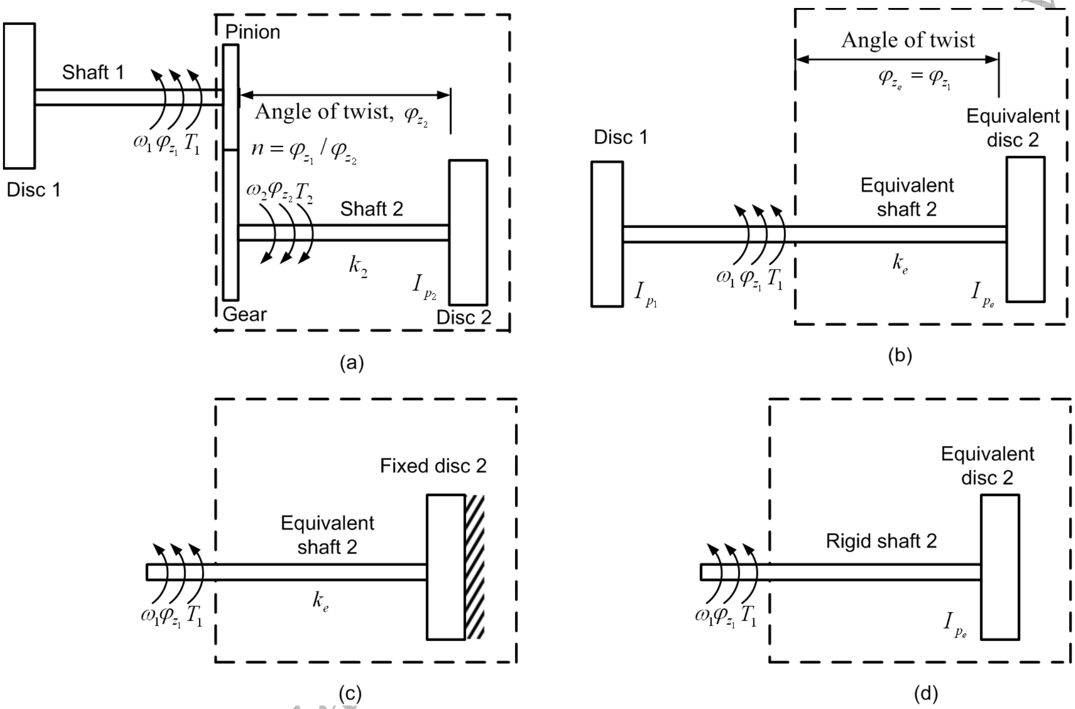


Fig. 6.30 (a) Actual geared system (b) An equivalent system without geared system (c) The equivalent system with disc 2 as fixed and (d) The equivalent system with shaft 2 as rigid

It is assumed that gears and shafts have negligible polar mass moment of inertia as compared to discs in the geared rotor system. In the actual system as shown in Figure 6.30(a), k_2 is the torsional stiffness of the shaft between gear 2 and disc 2, and I_{p_2} is the polar mass moment of inertia of disc 2. Let the equivalent system as shown in Fig. 6.30(b) has the shaft torsional stiffness k_e and the disc mass moment of inertia I_{P_e}. The strain and kinetic energy values must be the same in both the real and dynamically equivalent systems for the theoretical model to be valid.

Equivalent stiffness: Let the disc with the polar mass moment of inertia, I_{p_2}, is imagined to be held rigidly in both the real and equivalent (Fig. 6.30c) systems, while the pinion shaft 1 is rotated through an angle of φ_{z_1} at the input to gearbox (i.e., at the pinion). Shaft 2 is rotated through an angle $\varphi_{z_2} = \varphi_{z_1} / n$ at the gear 2, where n is the *gear ratio*. It is the ratio of the angular speed of the driving gear (pinion) to that of the driven gear, i.e.

$$n = \frac{\omega_1}{\omega_2} = \frac{\varphi_{z_1}(t)}{\varphi_{z_2}(t)} = \frac{N_2}{N_1}$$

where ω is the spin speed of the gear and N is the number of teeth of the gear. The *speed ratio*, *train value*, and *kinematic coefficient* are other terms used for gear ratio, however, these are inverse of the gear ratio, i.e. the ratio of the angular speed of the driven gear to that of the driving gear. Hence, the strain energy stored in shaft 2 of the actual system, for a twist of φ_{z_1} at the input to the gear box, can be written as

$$U_r = \tfrac{1}{2} k_2 \varphi_{z_2}^2 = \tfrac{1}{2} k_2 \left(\frac{\varphi_{z_1}}{n} \right)^2 \tag{6.72}$$

where U_r is the strain energy in the real system. While applying the same input at the gear box to the equivalent system (Fig. 6.30c) results in the stain energy stored in the equivalent shaft, and can be expressed as

$$U_e = \tfrac{1}{2} k_e \varphi_{z_e}^2 = \tfrac{1}{2} k_e \varphi_{z_1}^2 \tag{6.73}$$

where U_e is the strain energy in the equivalent system, and since we have $\varphi_{z_e} = \varphi_{z_1}$. On equating equations (6.72) and (6.73), it gives the equivalent stiffness as

$$k_e = \frac{k_2}{n^2} \tag{6.74}$$

Equivalent polar mass moment of inertia: Now we consider the shaft 2 as a rigid shaft in both the real and equivalent systems (Fig. 6.30d), so that angular motion of gear 2 and disc 2 is same. That means whatever motion at pinion is given to: (i) the real system disc 2 gets same motion as the gear 2, (ii) for the equivalent system disc 2 gets same motion as the pinion itself. Kinetic energies of both the real and equivalent systems must also be equated

$$T_r = \tfrac{1}{2} I_{p_2} \varpi_2^2 \qquad \text{and} \qquad T_e = \tfrac{1}{2} I_{p_e} \varpi_e^2 \tag{6.75}$$

where T_r and T_e is the kinetic energies in the real and equivalent system, respectively; $\bar{\omega}_2$ and $\bar{\omega}_e$ are angular frequencies of disc 2 of the real (I_{p_2}) and equivalent (I_{p_e}) systems, respectively. Equations (6.75) can be equated and is written as

$$\tfrac{1}{2} I_{p_2} \left(\omega_2 + \dot{\varphi}_{z_2}\right)^2 = \tfrac{1}{2} I_{p_e} \left(\omega_1 + \dot{\varphi}_{z_e}\right)^2 \tag{6.76}$$

where φ_{z_2} and φ_{z_e} are the angle of twist of shaft 2 in the actual and equivalent systems, respectively. It can be seen from Figure 6.30(d) that $\varphi_{z_e} = \varphi_{z_1}$ and ω_1 and ω_2 are angular frequencies of the shaft 1 and 2, respectively. We have the following basic relations

$$\varphi_{z_1} = \frac{T_1}{k_e} = \varphi_{z_e} \quad \text{and} \quad \varphi_{z_2} = \frac{T_2}{k_2} = \frac{nT_1}{k_2} \tag{6.77}$$

where T_1 and T_2 are torques at gears 1 and gear 2, respectively, in actual system. Noting equation (6.77), equation (6.76) can be written as

$$\frac{1}{2} I_{p_2} \left\{ \frac{\omega_1}{n} + \frac{d}{dt}\left(\frac{nT_1}{k_2}\right) \right\}^2 = \tfrac{1}{2} I_{p_e} \left\{ \omega_1 + \frac{d}{dt}\left(\frac{T_1}{k_e}\right) \right\}^2 \tag{6.78}$$

where T_1 is the torque input to the pinion (shaft 1). On substituting equation (6.74) in equation (6.78), we get

$$\tfrac{1}{2} I_{p_2} \left\{ \frac{\omega_1}{n} + \frac{d}{dt}\left(\frac{nT_1}{k_2}\right) \right\}^2 = \tfrac{1}{2} I_{p_e} \left\{ \omega_1 + \frac{d}{dt}\left(\frac{n^2 T_1}{k_2}\right) \right\}^2 \quad \Rightarrow \quad \frac{I_{p_2}}{n^2} \left\{ \omega_1 + \frac{d}{dt}\left(\frac{n^2 T_1}{k_2}\right) \right\}^2 = I_{p_e} \left\{ \omega_1 + \frac{d}{dt}\left(\frac{n^2 T_1}{k_2}\right) \right\}^2$$

which simplifies to

$$I_{p_e} = \frac{I_{p_2}}{n^2} \tag{6.79}$$

where k_e and I_{p_e} are, respectively, the equivalent shaft stiffness and the equivalent polar mass moment of inertia of the geared system referred to the ‘reference shaft’ speed, i.e. shaft 1. The general rule, for forming the equivalent system for the purpose of analysis, is to divide all shaft stiffness and rotor polar mass moment of inertia of the geared system by n^2 (where n is the gear ratio). When analysis is completed, it should be remembered that the elastic line of the mode shape of the equivalent system (i.e., the line *abc* in Fig 6.31) is modified for the real system by dividing the displacement amplitudes of the equivalent shaft by the gear ratio n as shown in Figure 6.31 by the line *abde*. It should be noted that angular displacements shown in Fig. 6.31 are now that of discs 1and 2.

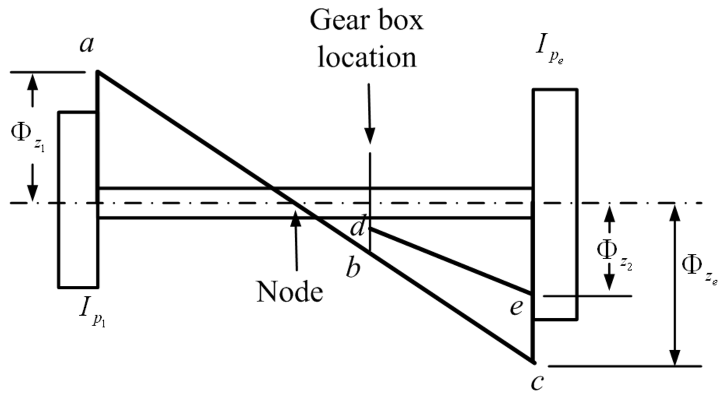


Figure 6.31 The elastic line in the equivalent and original systems

Example 6.11 For a geared system as shown in Figure 6.32, find torsional natural frequencies and mode shapes. Find also the location of the node point on the shaft (i.e., the location of the point where the angular twist during torsional vibrations is zero). The shaft ‘A’ has the diameter of 5 cm and the length of 0.75 m, and the shaft ‘B’ has the diameter of 4 cm and the length of 1.0 m. Take the modulus of rigidity of the shaft G = 0.8 × 10^{11} N/m^2, the polar mass moment of inertia of discs are I_{p_A} = 24 Nm^2 and I_{p_B} = 10 Nm^2. Neglect the inertia of gears.

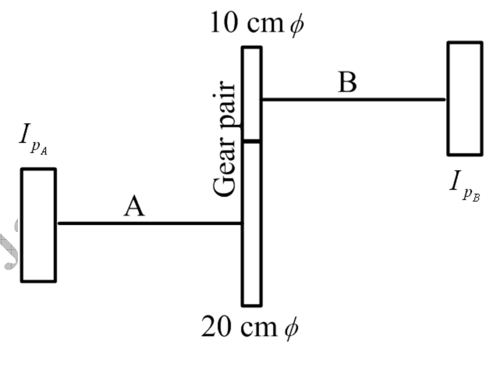


Figure 6.32 A two-disc geared system

Solution: On taking shaft B as the input shaft (or the reference shaft) as shown in Figure 6.33, the gear ratio can be defined as

$$\text{Gear ratio} = n = \frac{\text{Input shaft speed}}{\text{Output shaft speed}} = \frac{\omega_B}{\omega_A} = \frac{D_A}{D_B} = \frac{20}{10} = 2$$

where D is the nominal diameter of the gear.

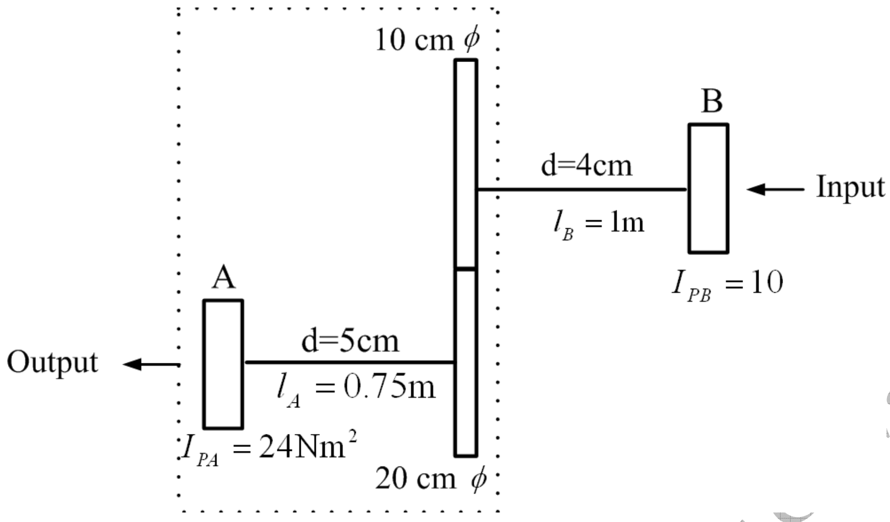


Figure 6.33 A geared system

The polar moment of inertia of the shaft cross-section and the torsional stiffness of the shaft can be obtained as

$$J_A = \frac{\pi}{32} d_A^4 = 6.136 \times 10^{-7} \text{ m}^4; \qquad J_B = \frac{\pi}{32} d_B^4 = 2.51 \times 10^{-7} \text{ m}^4;$$

and

$$k_A = \frac{GJ_A}{l_A} = \frac{0.8 \times 10^{11} \times 6.136 \times 10^{-7}}{0.75} = 6.545 \times 10^4 \text{ Nm/rad}; \qquad k_B = 2.011 \times 10^4 \text{ Nm/rad};$$

On treating as a reference shaft to the shaft B and replacing an equivalent shaft system of shaft A (i.e., the same diameter as that of reference shaft B), the system will become as shown in Figure 6.34. The equivalent system of the shaft system A has the following torsional stiffness and polar mass moment of inertia properties

$$k_{A_e} = \frac{k_A}{n^2} = \frac{6.545 \times 10^4}{2^2} = 1.6362 \times 10^4 \text{ Nm/rad} \qquad \text{and} \qquad I_{P_A} = \frac{I_{P_A}}{n^2} = \frac{24}{2^2} = 6 \text{ Nm}^2$$

which gives the equivalent length of shaft A as (note that now its diameter is that of the reference shat B)

$$l_{A_e} = \frac{GJ_B}{k_{A_e}} = \frac{0.8 \times 10^{11} \times 2.513 \times 10^{-7}}{1.6362 \times 10^4} = 1.229 \text{m}.$$

Hence, the equivalent full shaft length is given as $l_e = l_{A_e} + l_B = 1.229 + 1 = 2.229$ m .

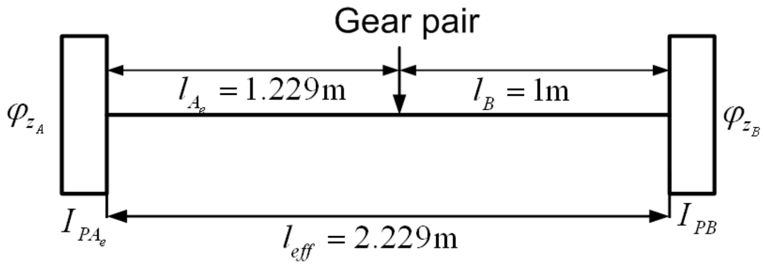


Figure 6.34 An equivalent single-shaft geared system

The equivalent stiffness of the full shaft is given as (Fig. 6.34)

$$\frac{1}{k_e} = \frac{1}{k_{A_e}} + \frac{1}{k_B} = \frac{1}{2.011\times10^4} + \frac{1}{1.6362\times10^4} = 1.1085\times10^{-4}\ \text{rad/Nm}$$

which gives $K_e = 9021.2\ \text{Nm/rad}$. The flexible natural frequency of the equivalent two-disc rotor system as shown in Figure 6.34 is given as

$$\omega_{nf_2} = \sqrt{\frac{(I_{P_{A_e}} + I_{P_B})k_e}{(I_{P_{A_e}} I_{P_B})}} = \sqrt{\frac{(6+10)\times9021.2/9.81}{(6\times10)/9.81^2}} = 153.62\ \text{rad/sec}$$

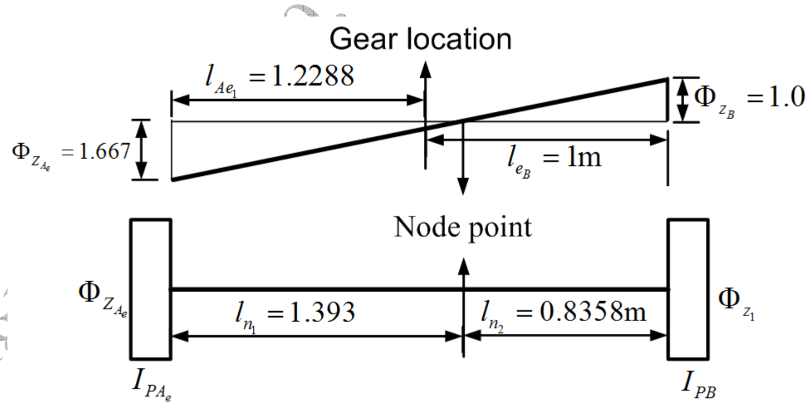


Figure 6.35 Mode shape and nodal point location in the equivalent system

The node location can be obtained from Figure 6.35 as

$$\frac{\Phi_{z_{A_e}}}{l_{n_1}} = \frac{\Phi_{z_B}}{l_{n_2}}$$

which can be written by noting equation (6.13), as

$$\frac{l_{n_1}}{l_{n_2}} = \frac{\Phi_{z_{A_e}}}{\Phi_{z_B}} = -\frac{I_{P_B}}{I_{P_{A_e}}} = -\frac{10}{6} = -1.667$$

The negative sign indicates that both discs are at either end of the node location. The absolute location of the node position is given as

$$l_{n_1} = 1.667\ l_{n_2}$$

Also from Figure 6.35, we have

$$l_{n_1} + l_{n_2} = 2.2288 \qquad \text{which gives} \qquad l_{n_2} = 0.8358 \text{ m}$$

Hence, the node is on shaft B at 0.8356 m from disc B. Alternatively, from similar triangle of the mode shape (Figure 6.35), we have

$$\frac{l_{n_2}}{2.2288 - l_{n_2}} = \frac{\Phi_{z_B}}{\Phi_{z_{A_e}}} = \frac{1}{1.667} \qquad \Rightarrow l_{n_2} = 0.8358 \text{ m}$$

Let $\Phi_{z_B} = 1$ rad, then $\Phi_{z_{A_e}} = -1.667$ rad ; hence, $\Phi_{z_A} = \frac{\Phi_{z_{A_e}}}{n} = -0.8333$ rad

The mode shape and the node location in the actual system are shown in Figure 6.36.

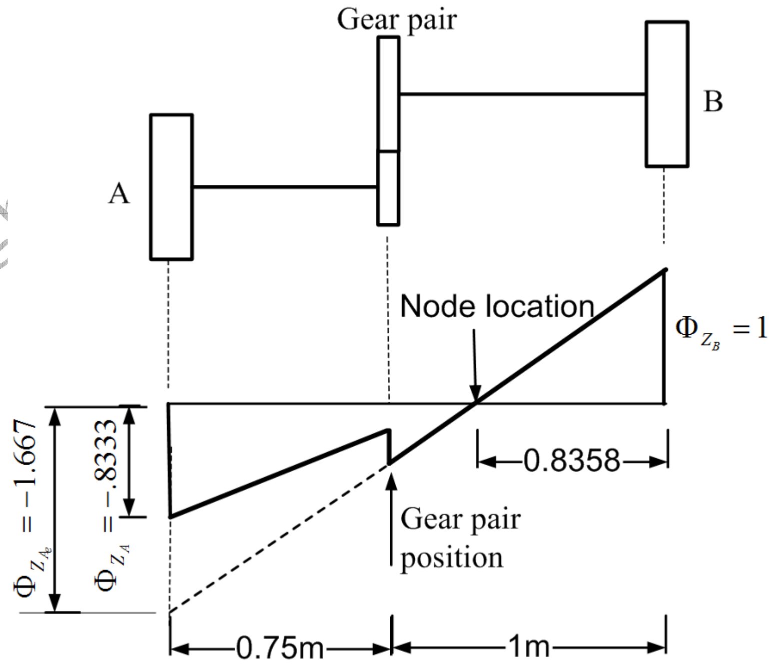


Figure 6.36 The mode shape and the node location in the actual geared system

Alternative way to obtain the natural frequency is to consider the equivalent two-disc rotor system (Figure 6.34) as two single-DOF systems (one such system is shown in Figure 6.37).

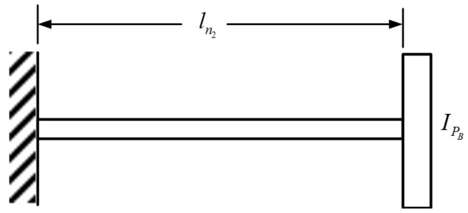


Figure 6.37 A single DOF system

The stiffness and polar mass moment of inertia properties of the system is given as

$$k_{l_{n_2}} = \frac{GJ_B}{l_{n_2}} = \frac{0.8\times10^{11}\times2.513}{0.8358} = 2.435\times10^4 \text{ Nm/rad} \quad \text{and} \quad I_{P_B} = \frac{10}{9.81}\text{kgm}^2$$

It gives the natural frequency as

$$\omega_{nf_2} = \sqrt{\frac{k_{l_{n_2}}}{I_{P_B}}} = \sqrt{\frac{2.435\times10^4\times9.81}{10}} = 153.62 \text{ rad/sec}$$

which is same as obtained earlier. The whole free vibration torsional analysis can be done by taking the speed of shaft A as the reference and converting shaft B by an equivalent system. For completeness some of the basic steps are given as follows.

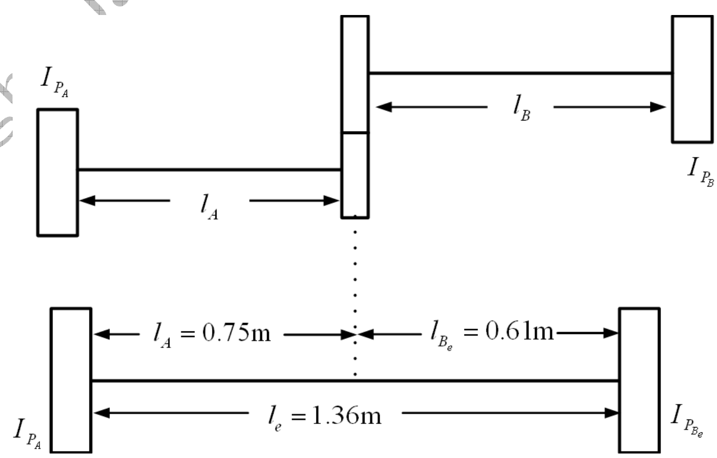


Figure 6.38 Actual and equivalent geared systems

The gear ratio for the present case will be

$$n = \frac{\omega_A}{\omega_B} = \frac{D_B}{D_A} = \frac{10}{20} = 0.5$$

It is assumed that equivalent shaft (i.e., the shaft B) has the diameter same as that of the reference shaft (i.e., the shaft A). The equivalent polar mass moment of inertia and the torsional stiffness can be written as

$$I_{P_{B_e}} = \frac{I_{P_B}}{n^2} = 40 \text{ Nm}^2 \quad \text{and} \quad k_{B_e} = \frac{k_B}{n^2} = \frac{2.011\times 10^4}{(0.5)^2} = 8.044\times 10^4 \text{ Nm/rad}$$

which gives the equivalent length as

$$k_{B_e} = \frac{GJ_A}{l_{B_e}} = 8.044\times 10^4 \text{Nm/rad} \quad \Rightarrow \quad l_{B_e} = \frac{0.8\times 10^{11}\times 6.136\times 10^{-7}}{8.044\times 10^4} = 0.610 \text{ m}$$

The total equivalent length and the equivalent torsional stiffness would be

$$l_e = l_A + l_{B_e} = 0.75 + 0.61 = 1.36 \text{ m}$$

and

$$k_e = \frac{GJ_A}{l_e} = \frac{0.8\times 10^{11}\times 6.136\times 10^{-7}}{1.360} = 3.61\times 10^4 \text{ Nm/rad}$$

Alternatively, the effective stiffness can be obtained as

$$\frac{1}{k_e} = \frac{1}{k_A} + \frac{1}{k_{B_e}} \quad \Rightarrow \quad k_e = \frac{k_A k_{B_e}}{k_A + k_{B_e}} = \frac{6.545\times 10^4 \times 8.044\times 10^4}{6.545\times 10^4 + 8.044\times 10^4} = 3.61\times 10^4 \text{ Nm/rad}$$

Natural frequencies of two mass rotor system are given as

$$\omega_{nf_1} = 0 \quad \text{and} \quad \omega_{nf_2} = \sqrt{\frac{(I_{P_A} + I_{P_{B_e}})}{I_{P_A} I_{P_{B_e}}} k_e} = \sqrt{9.81\times \frac{(24+40)}{24\times 40}\times 3.61\times 10^4} = 153.62 \text{ rad/s}$$

A factor 9.81 is used since I_{P_A} is in Nm^2 and we need in kgm^2.

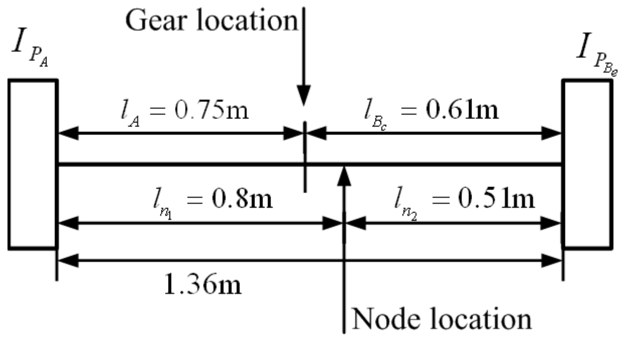


Figure 6.39 Equivalent two mass rotor geared system

The node location can be obtained as

$$\frac{l_{n_1}}{l_{n_2}} = -\frac{I_{P_{Be}}}{I_{P_A}} = -\frac{10}{6} = -1.667 \qquad \text{and} \qquad l_{n_1} + l_{n_2} = l_A + l_{B_e} = 1.36 \text{ m}$$

which gives

$$(1.667 l_{n_1}) + l_{n_2} = 1.36 \quad \Rightarrow \quad l_{n_1} = 0.85 \text{ m} \text{ and } l_{n_2} = 0.51 \text{ m}$$

The stiffness of shaft length equal to l_{n_2} will be (equivalent stiffness corresponding to shaft A speed)

$$k_{l_{n_2}} = \frac{GJ_A}{l_{n_2}}$$

The shaft stiffness corresponding to shaft B speed can be defined in two ways i.e.

$$k_{B_2} = \frac{GJ_B}{l_2} \quad \text{and} \qquad k_{B_2} = n^2 k_{l_{n_2}} = n^2 \frac{GJ_A}{l_{n_2}}$$

On equating above equations the location of the node in the actual system from disc B can be obtained as

$$l_2 = n^2 l_{n_2} \frac{J_B}{J_A} = 0.84$$

which is same as by previous method.

6.7 TMM for Branched Systems

For rolling mills, textile machineries, the marine vessel power transmission shafts, and machine tool drives; there may be many rotor inertias in the system and gear box may be a branch point where more than two shafts are attached. In such cases where there are more than two shafts attached as shown in Fig. 6.40 to the gearbox, the system is said to be *branched*. It has three braches A, B and C; and each branch has multiple discs, e.g. p, q, and r number of discs (including gears) in branches A, B and C, respectively. Such system can not be converted to a single shaft system as we could do to the two-shaft geared system as discussed in previous section. Since now the system contain several discs hence, it is a multi-DOF system and hence the analysis of the branched system would now be done by more general procedure, i.e. the TMM.

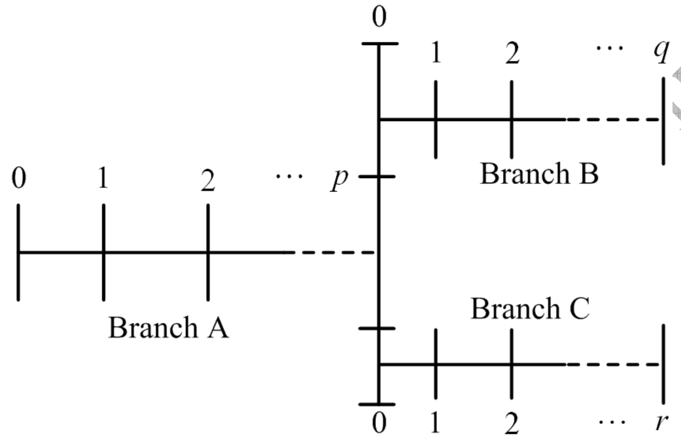


Fig. 6.40 A branched multi-DOF rotor system

For the branched system as shown in Figure 6.40, state vectors for different branches can be written as

$$_R\{S\}_{pA} = [A]\{S\}_{0A}\,; \quad _R\{S\}_{qB} = [B]\{S\}_{0B}\,; \quad _R\{S\}_{rC} = [C]\{S\}_{0C} \tag{6.80}$$

where [A], [B], and [C] are overall transfer matrices for branches A, B, and C; respectively.

Branch A: For branch A, taking $\varphi_{z_{0A}} = 1$ as the reference value for the angular displacement and since the left hand end of branch A is free end, hence for free vibrations we have $T_{0A} = 0$. Equation for branch A takes the form

$$\begin{Bmatrix} \varphi_z \\ T \end{Bmatrix}_{pA} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{6.81}$$

which can be expanded as

$$T_{z_{pA}} = a_{21} \quad \text{and} \quad \varphi_{z_{pA}} = a_{11} \tag{6.82}$$

Branch B: At branch point, between shafts A and B, we have

$$\varphi_{z_{0B}} = \frac{\varphi_{z_{pA}}}{n_{AB}} = \frac{a_{11}}{n_{AB}} \tag{6.83}$$

where n_{AB} is the gear ratio between shafts A and B. For branch B, $T_{qB} = 0$, since the right hand end of the branch is free. For branch B from equation (6.80), and noting condition described by equation (6.83), we have

$$\begin{Bmatrix} \varphi_z \\ 0 \end{Bmatrix}_{qB} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} a_{11}/n_{AB} \\ T \end{Bmatrix}_{0B} \tag{6.84}$$

Equation (6.84) can be expanded as

$$\varphi_{z_{qB}} = b_{11}\frac{a_{11}}{n_{AB}} + b_{12}T_{0B} \tag{6.85}$$

and

$$0 = b_{21}\frac{a_{11}}{n_{AB}} + b_{22}T_{0B} \quad \Rightarrow T_{0B} = -\left(\frac{b_{21}}{b_{22}}\right)\left(\frac{a_{11}}{n_{AB}}\right) \tag{6.86}$$

Branch C and junction point: At branch C, we have the following condition (noting equation (6.82))

$$\varphi_{z_{0C}} = \frac{\varphi_{z_{pA}}}{n_{AC}} = \frac{a_{11}}{n_{AC}} \tag{6.87}$$

where n_{AC} is the gear ratio between shafts A and C. Another condition at the branch to be satisfied regarding work done by the torque transmitted at various branches (assuming negligible friction during transmissions), i.e.

$$\tfrac{1}{2}T_{pA}\varphi_{z_{nA}} = \tfrac{1}{2}T_{0B}\varphi_{z_{0B}} + \tfrac{1}{2}T_{0C}\varphi_{z_{0C}} \quad \Rightarrow T_{pA} = \frac{T_{0B}}{\varphi_{z_{pA}}/\varphi_{z_{0B}}} + \frac{T_{0C}}{\varphi_{z_{pA}}/\varphi_{z_{0C}}} \quad \Rightarrow T_{pA} = \frac{T_{0B}}{n_{AB}} + \frac{T_{0C}}{n_{AC}} \tag{6.88}$$

On substituting equation (6.86) into the third expression of equation (6.88), it can be written as

$$T_{0C} = n_{AC}\left[T_{pA} - \frac{T_{0B}}{n_{AB}}\right] = n_{AC}\left[T_{nA} + \frac{b_{21}}{b_{22}}\frac{a_{11}}{n_{AB}^2}\right]$$

On substituting equation (6.82), we get

$$T_{0C} = n_{AC}\left[a_{21} + \frac{b_{21}}{b_{22}}\frac{a_{11}}{n_{AB}^2}\right] \tag{6.89}$$

Substituting equations (6.87) and (6.89) into equation (6.80), we get

$$\begin{Bmatrix} \varphi_z \\ 0 \end{Bmatrix}_{rc} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} a_{11}/n_{AC} \\ n_{AC}a_{21} + \dfrac{b_{21}a_{11}n_{AC}}{b_{22}n_{AB}^2} \end{Bmatrix} \tag{6.90}$$

where $T_{rc} = 0$ is the boundary condition describing the free right hand end of branch C.

The frequency equation: From equation (6.90), the second equation will give the frequency equation as

$$a_{11}b_{22}c_{21}n_{AB}^2 + a_{11}b_{21}c_{22}n_{AC}^2 + a_{21}b_{22}c_{22}n_{AB}^2n_{AC}^2 = 0 \tag{6.91}$$

where *a*'s, *b*'s and *c*'s are function of the natural frequency, ω_{nf}. The roots of the above equation are system natural frequencies. Angular displacements at the beginning and end of various branches can be summarised as

$$\varphi_{z_{0A}} = 1; \qquad \varphi_{z_{pA}} = a_{11}; \tag{6.92}$$

$$\varphi_{z_{0B}} = \frac{a_{11}}{n_{AB}}; \qquad \varphi_{z_{qB}} = \left(b_{11} - \frac{b_{12}b_{21}}{b_{22}}\right)\frac{a_{11}}{n_{AB}}; \tag{6.93}$$

$$\varphi_{z_{0C}} = \frac{a_{11}}{n_{AC}}; \qquad \varphi_{z_{rC}} = \left(c_{11} - \frac{c_{12}c_{21}}{c_{22}}\right)\frac{a_{11}}{n_{AC}}. \tag{6.94}$$

On substituting one of the value of torsional natural frequencies obtained from equation (6.91) into equations (6.92)-(6.94), angular displacements at the beginning and end of various branches can be obtained. Then these may be substituted back into transfer matrices for each braches considered (i.e., equation (6.80)), where upon the state vector at each station may be evaluated. The plot of angular displacements against shaft positions then indicates the system mode shapes corresponding to the chosen natural frequency. For other natural frequencies also similar steps have to be performed.

Using this method, there will not be any change in the elastic line (mode shape) due to the gear ratio, since these have now already been allowed for in the analysis. Moreover, for the present case we have not gone for the equivalent system at all. For the case when the system can be converted to an equivalent single shaft, the equivalent system approach has the advantage. It should be noted that for the present case the DOF of the rotor system would be ($p + q + r - 2$). The total number of discs (including gears) is ($p + q + r$), however, at junction the DOF of two gears (e.g., at 0^{th} station of braches B and C) is related with the third (at p^{th} station of branch A), hence, we would have two DOF less as compared to the number of discs in the system. Now, through numerical examples the procedure will be illustrated.

Example 6.12 For a geared system as shown in Figure 6.41, find torsional natural frequencies. The shaft 'A' has 5 cm diameter and 0.75 m length, and the shaft 'B' has 4 cm diameter and 1.0 m length. Take the modulus of rigidity of the shaft G equals to 0.8×10^{11} N/m^2, the polar mass moment of inertia of discs and gears are $I_{p_A} = 24\,\text{Nm}^2$, $I_{p_B} = 10\,\text{Nm}^2$, $I_{p_{gA}} = 5\,\text{Nm}^2$, $I_{p_{gB}} = 3\,\text{Nm}^2$.

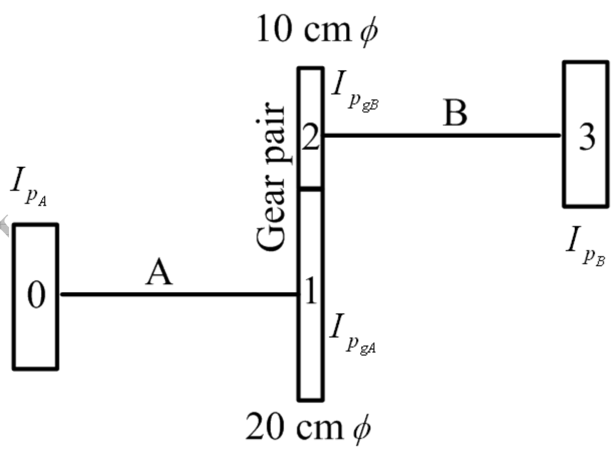


Figure 6.41 Two-discs with a geared system

Solution: Now this problem will be solved using the TMM for illustration of the method to geared system. The pinion and gear have appreciable polar mass moment of inertia. Let us denote the station number of the disc on branch A as 0, the gear as 1 and the station number of the disc on branch B as 2, the gear as 3 (Fig. 6.41).

The state transformation equation for the branch A can be written as

$$ {}_R\{S\}_1 = [A]\,{}_L\{S\}_0 \tag{a} $$

with

$$ [A]=[P]_1[F]_1[P]_0=\begin{bmatrix} 1 & 1/k_A \\ -\omega_{nf}^2 I_{p_{gA}} & 1-\dfrac{\omega_{nf}^2 I_{p_{gA}}}{k_A} \end{bmatrix}_1 \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_{p_A} & 1 \end{bmatrix}_0 $$

$$ =\begin{bmatrix} \left(1-\dfrac{\omega_{nf}^2 I_{p_A}}{k_A}\right) & \dfrac{1}{k_A} \\ \left\{\left(-\omega_{nf}^2 I_{p_{gA}}\right)+\left(-\omega_{nf}^2 I_{p_A}\right)\left(1-\dfrac{\omega_{nf}^2 I_{p_{gA}}}{k_A}\right)\right\} & \left(1-\dfrac{\omega_{nf}^2 I_{p_{gA}}}{k_A}\right) \end{bmatrix} \tag{b} $$

where k_A is the stiffness of shaft A. Similarly, the state transformation equation for the branch B can be written as

$$ {}_R\{S\}_3 = [B]\,{}_L\{S\}_2 \tag{c} $$

with

$$ [B]=[P]_3[F]_3[P]_2 == \begin{bmatrix} \left(1-\dfrac{\omega_{nf}^2 I_{p_{gB}}}{k_B}\right) & \dfrac{1}{k_B} \\ \left\{\left(-\omega_{nf}^2 I_{p_B}\right)+\left(-\omega_{nf}^2 I_{p_{gB}}\right)\left(1-\dfrac{\omega_{nf}^2 I_{p_B}}{k_B}\right)\right\} & \left(1-\dfrac{\omega_{nf}^2 I_{p_B}}{k_B}\right) \end{bmatrix} \tag{d} $$

where k_B is the stiffness of shaft B. At the gear pair, the following conditions hold

$$ \varphi_{x_2} = \frac{\varphi_{x_1}}{n} \qquad \text{and} \qquad T_2 = nT_1 \tag{e} $$

where φ_x is the angular displacement. Equation (e) can be combined as

$$ {}_L\begin{Bmatrix} \varphi_x \\ T \end{Bmatrix}_2 = \begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix}_R \begin{Bmatrix} \varphi_x \\ T \end{Bmatrix}_1 \tag{f} $$

or

$$ {}_L\{S\}_2 = [n]\,{}_R\{S\}_1 \tag{g} $$

with the *gear ratio transformation matrix* is given as

$$[n]=\begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix} \tag{h}$$

Noting equation (g), equation (c) can be written as

$$_R\{S\}_3=[B][n]\,_R\{S\}_1 \tag{i}$$

Noting equation (a), equation (i) can be expressed as

$$_R\{S\}_3=[B][n][A]\,_L\{S\}_0=[T]\,_L\{S\}_0 \tag{j}$$

with the overall transformation matrix is given as

$$[T]=[B][n][A]=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} \dfrac{a_{11}b_{11}}{n}+na_{21}b_{12} & \dfrac{a_{12}b_{11}}{n}+na_{22}b_{12} \\ \dfrac{a_{11}b_{21}}{n}+na_{21}b_{22} & \dfrac{a_{12}b_{21}}{n}+na_{22}b_{22} \end{bmatrix} \tag{k}$$

The overall transformation matrix can also be written as

$$[T]=[B][n][A]=[P]_3[F]_3[P]_2[n][P]_1[F]_1[P]_0 \tag{l}$$

Boundary conditions of the problem are $T_0=T_3=0$ since both ends are free. Equation (j) can be written in expended form as

$$_R\begin{Bmatrix} \varphi_x \\ T \end{Bmatrix}_3=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}_L\begin{Bmatrix} \varphi_x \\ T \end{Bmatrix}_0 \quad \text{or} \quad _R\begin{Bmatrix} \varphi_x \\ 0 \end{Bmatrix}_3=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}_L\begin{Bmatrix} \varphi_x \\ 0 \end{Bmatrix}_0 \tag{m}$$

which gives frequency equation as

$$t_{21}\left(\omega_{nf}\right)=0 \tag{n}$$

From equation (k), frequency equation comes out to be

$$a_{11}b_{21}+n^2a_{21}b_{22}=0 \tag{o}$$

Noting equations (b) and (d), in the expanded form equation (o) can be written

$$\left(1-\frac{\omega_{nf}^2 I_{p_A}}{k_A}\right)\left\{\left(-\omega_{nf}^2 I_{p_B}\right)+\left(-\omega_{nf}^2 I_{p_{gB}}\right)\left(1-\frac{\omega_{nf}^2 I_{p_B}}{k_B}\right)\right\}+n^2\left\{\left(-\omega_{nf}^2 I_{p_{gA}}\right)+\left(-\omega_{nf}^2 I_{p_A}\right)\left(1-\frac{\omega_{nf}^2 I_{p_{gA}}}{k_A}\right)\right\}\left(1-\frac{\omega_{nf}^2 I_{p_B}}{k_B}\right)=0 \quad \text{(p)}$$

For the numerical values of the present problem, equation (p) reduces to

$$\omega_{nf}^2\left\{\omega_{nf}^4-\left(1.665\times10^4\right)\omega_{nf}^2+\left(3.006\times10^7\right)\right\}=0 \quad \text{(q)}$$

From frequency equation (q), the following natural frequencies are obtained

$\omega_{nf_1}=0$, $\omega_{nf_2}=45.46$, $\omega_{nf_3}=119.56$ rad/s,

It should be noted that for the present problem even four discs (polar mass moment of inertia) are present, however only three natural frequencies is obtained. This is due to the fact that gear pair is treated as a single polar mass moment of inertia, so effectively for the present problem only three generalized coordinates are sufficient to describe the motion.

For comparison with Example 6.11, let us put polar mass moment of inertia of the pinion and the gear to zero in equation (p), then we get

$$\omega_{nf}^2\left\{I_{p_A}I_{p_B}\left(n^2k_A+k_B\right)\omega_{nf}^2-\left(n^2I_{p_A}+I_{p_B}\right)k_Ak_B\right\}=0$$

Which gives

$$\omega_{nf_1}=0$$

and

$$\omega_{nf_2}^2=\frac{\left(I_{p_A}+I_{p_B}/n^2\right)\left(n^2k_Ak_B\right)}{\left\{I_{p_A}\left(I_{p_B}/n^2\right)\right\}\left(n^2k_A+k_B\right)}=\frac{\left(I_{p_A}+I_{p_{B_e}}\right)\left\{n^2k_Ak_B/\left(n^2k_A+k_B\right)\right\}}{\left(I_{p_A}I_{p_{B_e}}\right)}$$

Second natural frequency can be simplified as

$$\omega_{nf_2}=\sqrt{\frac{(I_{P_A}+I_{P_{B_e}})k_e}{(I_{P_A}I_{P_{B_e}})}} \quad \text{(q)}$$

with

$$k_{eq}=n^2k_Ak_B/\left(n^2k_A+k_B\right);\qquad I_{p_{B_e}}=I_{p_B}/n^2$$

It should be noted that equation (q) is exactly the same as in previous example (i.e., for equivalent two mass rotor system).

Example 6.13 Obtain torsional natural frequencies and mode shapes of a branched system as shown in Figure 6.42. The polar mass moment of inertia of rotors are: I_{p_A} = 0.01 kg-m^2, I_{p_E} = 0.005 kg-m^2, I_{p_F} = 0.006 kg-m^2, and $I_{p_B} = I_{p_C} = I_{p_D} = 0$. Gear ratios are: n_{BC} = 3 and n_{BD} = 4. Shaft lengths are: l_{AB} = l_{CE} = l_{DF} = 0.25 m, and diameters are d_{AB} = 0.03 m, d_{CE} = 0.02 m and d_{DF} = 0.02 m. Take the modulus of rigidity of the shaft as $G = 0.8 \times 10^{11}$ N/m^2.

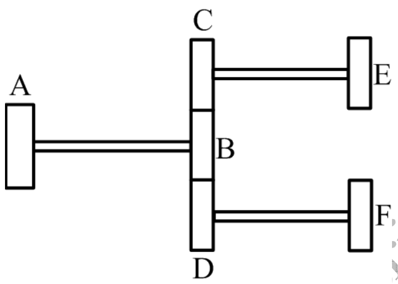


Figure 6.42 A branched rotor system

Solution: The branched system has the following numerical data

$$I_{P_A} = 0.01 \text{ kg-m}^2; \qquad I_{P_E} = 0.005 \text{ kg-m}^2; \qquad I_{P_F} = 0.006 \text{ kg-m}^2$$

$$J_{AB} = \frac{\pi}{32} d_{AB}^4 = \frac{\pi}{32} 0.03^4 = 7.95 \times 10^{-8} \text{ m}^4, \qquad J_{CE} = J_{DF} = \frac{\pi}{32} 0.02^4 = 1.57 \times 10^{-8} \text{ m}^4$$

$$k_{t_{AB}} = \frac{GJ_{AB}}{l_{AB}} = 2.55 \times 10^4 \text{ N/m}, \qquad k_{t_{CE}} = k_{t_{DF}} = 0.50 \times 10^4 \text{ N/m}$$

For branch AB, state vectors at stations are related as

$$_R\{S\}_{n_{AB}} = [A]\{S\}_{0_{AB}}$$

with

$$[A] = [F]_{AB}[P]_A = \begin{bmatrix} 1 & 3.93 \times 10^{-5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.01\omega_{nf}^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 - 3.93 \times 10^{-7} \omega_{nf}^2 & 3.93 \times 10^{-5} \\ -0.01\omega_{nf}^2 & 1 \end{bmatrix}$$

For branch CE, state vectors at stations are related as

$${}_R\{S\}_{n_{CE}} = [C]\{S\}_{0_{CE}}$$

with

$$[C] = [P]_E[F]_{CE} = \begin{bmatrix} 1 & 0 \\ -0.005\omega_{nf}^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2.0\times10^{-4} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2.0\times10^{-4} \\ -0.005\omega_{nf}^2 & 1-1.0\times10^{-6}\omega_{nf}^2 \end{bmatrix}$$

Similarly, for branch DF, we have

$${}_R\{S\}_{n_{DF}} = [D]\{S\}_{0_{DF}}$$

with

$$[D] = [P]_F[F]_{DF} = \begin{bmatrix} 1 & 0 \\ -0.006\omega_{nf}^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2.0\times10^{-4} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2.0\times10^{-4} \\ -0.006\omega_{nf}^2 & 1-1.2\times10^{-6}\omega_{nf}^2 \end{bmatrix}$$

From equation (6.81), the frequency equation can be written as

$$a_{11}c_{22}d_{21}n_{BC}^2 + a_{11}c_{21}d_{22}n_{BD}^2 + a_{21}c_{22}d_{22}n_{BC}^2 n_{BD}^2 = 0$$

On substitution, we get

$$\left(1-3.93\times10^{-7}\omega_{nf}^2\right)\left(1-1.0\times10^{-6}\omega_{nf}^2\right)\left(-0.006\omega_{nf}^2\right)\times9 +$$
$$\left(1-3.93\times10^{-7}\omega_{nf}^2\right)\left(-0.005\omega_{nf}^2\right)\left(1-1.2\times10^{-6}\omega_{nf}^2\right)\times16 +$$
$$\left(-0.01\omega_{nf}^2\right)\left(1-1.0\times10^{-6}\omega_{nf}^2\right)\left(1-1.2\times10^{-6}\omega_{nf}^2\right)\times9\times16 = 0$$

which can be simplified to

$$\omega_{nf}^2\left(1.7685\times10^{-12}\omega_{nf}^4 - 3.3532\times10^{-10}\omega_{nf}^2 + 1.5740\right) = 0$$

Natural frequencies are given as

$\omega_{nf_1} = 0$; $\omega_{nf_2} = 924.4$ rad/s and $\omega_{nf_3} = 1020.6$ rad/s.

It can be seen that the rigid body mode exists since ends of the gear train is free. Mode shapes for each of these natural frequencies can be obtained as follows.

For $\omega_{nf_1} = 0$ with $\varphi_{z_{0AB}} = 1$ as a reference value, angular displacements at various disc locations can be written as

$$\varphi_{z_{0AB}} = 1 ; \qquad \varphi_{z_{nAB}} = a_{11} = 1 ;$$

$$\varphi_{z_{0CE}} = -\frac{a_{11}}{n_{BC}} = -\frac{1}{3} - 0.33 ; \qquad \varphi_{z_{nCE}} = \left(c_{11} - \frac{c_{12}c_{21}}{c_{22}} \right) \varphi_{z_{0CE}} = -0.33 ;$$

$$\varphi_{z_{0DF}} = -\frac{a_{11}}{n_{BD}} = -\frac{1}{4} = -0.25 ; \qquad \varphi_{z_{nDF}} = \left(d_{11} - \frac{d_{12}d_{21}}{d_{22}} \right) \varphi_{z_{0DF}} = -0.25$$

Figure 6.43 shows the mode shape. Similarly, for other natural frequencies displacements can be obtained to get mode shapes as in Figures 6.44 and 6.45.

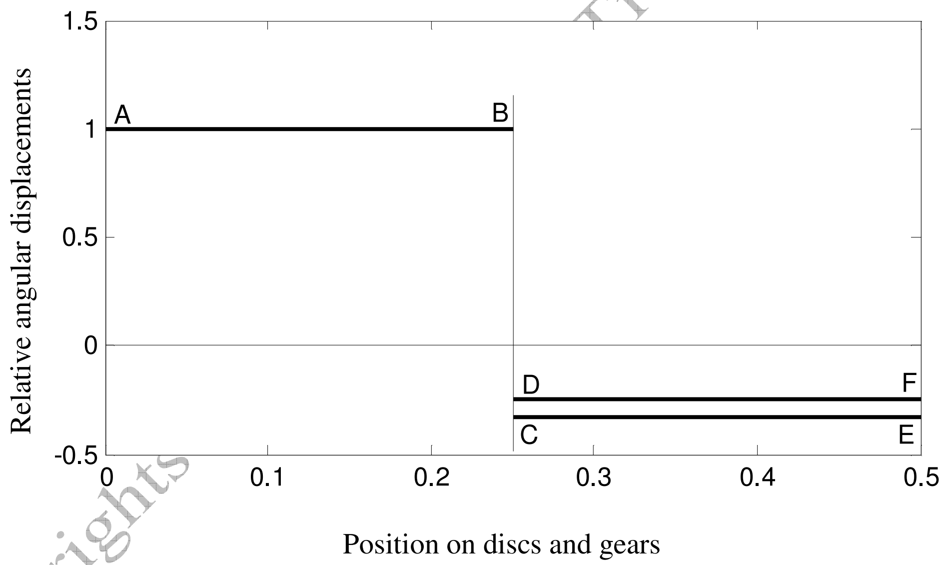


Figure 6.43 Mode shape of the branched system for $\omega_{nf_1} = 0$

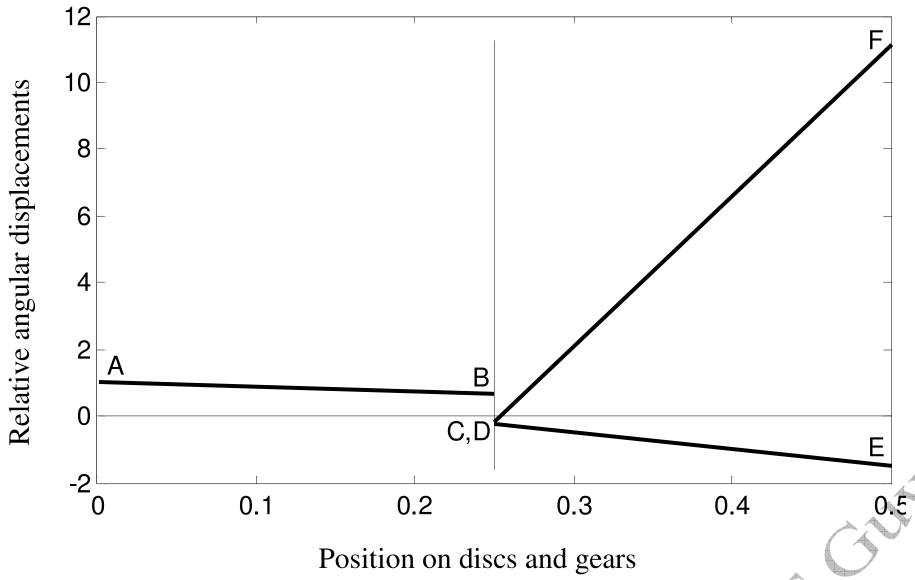


Figure 6.44 Mode shape of the branched system for $\omega_{nf_2} = 924.4$ rad/s

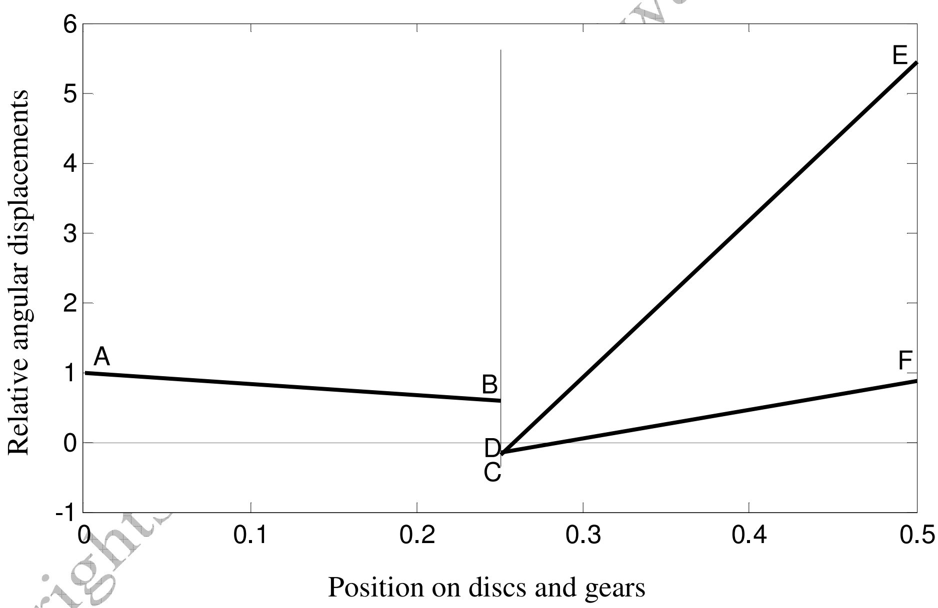


Figure 6.45 Mode shape of the branched system for $\omega_{nf_3} = 1020.6$ rad/s

6.8 TMM for Damped Torsional Vibrations

In any real rotor systems damping is always present. Torsional damping may come from several sources, e.g. the shaft material, bearings, couplings, torsional vibration dampers, aerodynamic damping at discs, rubbing of the rotor with the stator, loose components mounted on shaft, etc. The shaft material or hysteretic damping comes due to intermolecular interaction in the shaft material, which results in increase in the temperature of the shaft material. The torsional vibration damper is a device which may be used to join together two-shaft section as shown in Fig. 6.46. It develops a damping torque, which is dependent upon of the angular velocity on one shaft relative to the other. These types of damping can be considered proportional to the relative angular velocity of discs to which the shaft is connected and it is represented as c_s. The disc aerodynamic (or rubbing) damping, c_d, comes due to interaction of the disc with the working fluid (like steam, gas, air, etc.), lubricant, and coolant; which results in dissipation of the energy in the form of heat. This type of damping is proportional to the angular velocity of the disc itself.

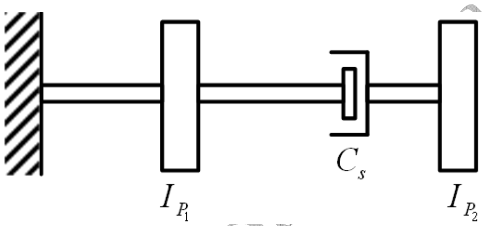


Figure 6.46 A schematic of a torsional vibration damper

Torsional dampers can be used as a means of attenuating (decreasing) system vibrations and to tune system resonant frequencies to suit particular operating conditions. The damping in the system introduces phase lag between the system displacement and torque.

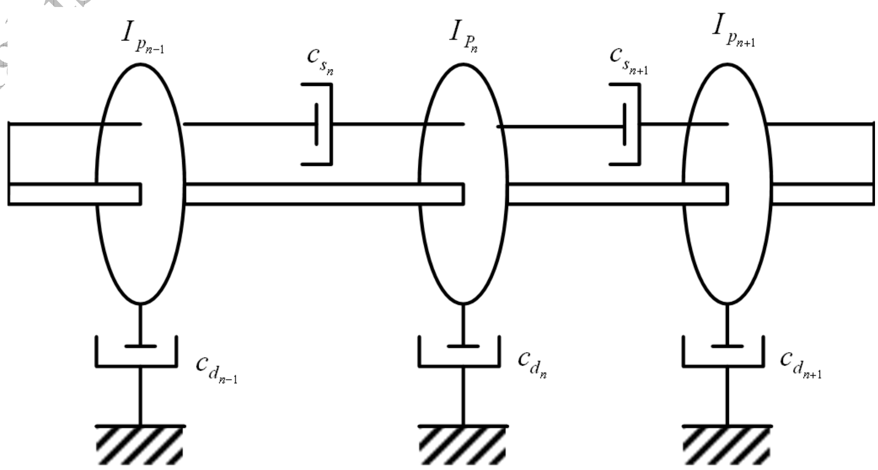


Figure 6.47 General arrangement of multi-DOF rotor system with damping.

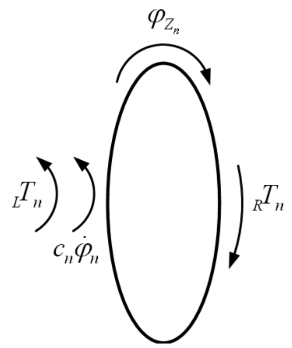


Figure 6.48 A free body diagram of r^{th} rotor.

Figure 6.47 shows a general arrangement of torsional multi-DOF rotor system with the disc and the shaft damping. From the free body diagram of n^{th} rotor (Figure 6.48) the following governing equations can be written as

$$_L\varphi_{z_n} = {}_R\varphi_{z_n} \tag{6.95}$$

and

$$_RT_n - {}_LT_n - c_{d_n}\,{}_L\dot{\varphi}_{z_n} = I_{p_n}\,{}_L\ddot{\varphi}_{z_n} \tag{6.96}$$

For free vibration, torques $_RT_n$ and $_LT_n$ may be written in the form

$$T_n = \bar{T}_n e^{\mathrm{j}\omega_{nf}t} \tag{6.97}$$

where $\bar{T}_n$ is the complex amplitude of the torque at n^{th} disc, and ω_{nf} is the torsional natural frequency of the system. The angular displacement takes the form

$$\varphi_{z_n} = \Phi_{z_n} e^{\mathrm{j}\omega_{nf}t} \tag{6.98}$$

where Φ_{z_n} is the complex amplitude of angular displacement at n^{th} disc. Differentiating equations (6.97) and (6.98) with respect to time and substituting in equations (6.95) and (6.96) leads to

$$_R\begin{Bmatrix}\Phi\\ \bar{T}\end{Bmatrix}_n = \begin{bmatrix}1 & 0\\ -\omega_{nf}^2 I_p + \mathrm{j}\omega_{nf}c_d & 1\end{bmatrix}_n {}_L\begin{Bmatrix}\Phi\\ \bar{T}\end{Bmatrix}_n \tag{6.99}$$

which can be simplified as

$$_R\{S\}_n = [P]_n \, _L\{S\}_n \tag{6.100}$$

with

$$[P]_n = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_p + \mathrm{j}\omega_{nf} c_d & 1 \end{bmatrix}_n ; \qquad _L\{S\}_n = {}_L\begin{Bmatrix} \Phi \\ \bar{T} \end{Bmatrix}_n$$

where $[P]_n$ is a point matrix and $[S]_n$ is a state vector.

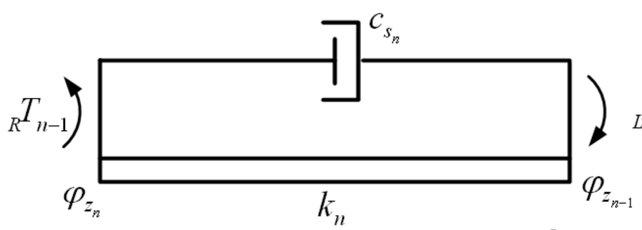


Figure 6.49 A free body diagram of n^{th} shaft segment

The characteristics of the shaft element at station n (Fig. 6.49) are represented in the equation describing the torque applied to the shaft at the location of rotor n, as

$$_LT_n = k_n \left(_L\varphi_{z_n} - _R\varphi_{z_{n-1}} \right) + c_{s_n} \left(_L\dot{\varphi}_{z_n} - _R\dot{\varphi}_{z_{n-1}} \right) \tag{6.101}$$

While the torque transmitted through the shaft is the same at both ends, i.e.,

$$_LT_n = {}_RT_{n-1} \tag{6.102}$$

Substituting equations (6.97) and (6.98), in equations (6.101) and (6.102), we get

$$_R\bar{T}_{n-1} = k_n \left(_L\Phi_{z_n} - _R\Phi_{z_{n-1}} \right) + \mathrm{j}\omega_{nf} c_{s_n} \left(_L\Phi_{z_n} - _R\Phi_{z_{n-1}} \right) \tag{6.103}$$

and

$$_L\bar{T}_n = {}_R\bar{T}_{n-1} \tag{6.104}$$

Combining equations (6.103) and (6.104), we get

$$\begin{bmatrix} k + \mathrm{j}\omega_{nf} c_s & 0 \\ 0 & 1 \end{bmatrix}_n {}_L\begin{Bmatrix} \Phi \\ \bar{T} \end{Bmatrix}_n = \begin{bmatrix} k + \mathrm{j}\omega_{nf} c_s & 1 \\ 0 & 1 \end{bmatrix}_{n-1} {}_R\begin{Bmatrix} \Phi \\ \bar{T} \end{Bmatrix}_{n-1} \tag{6.105}$$

which can be written as

$$[L]_n\,{}_L\{S\}_n = [M]_n\,{}_R\{S\}_{n-1}$$

which can be simplified as

$${}_L\{S\}_n = [L]_n^{-1}[M]_n\,{}_R\{S\}_{n-1} = [F]_n\,{}_R\{S\}_{n-1} \tag{6.106}$$

with $\left[F\right]_n = [L]_n^{-1}[M]_n = \begin{bmatrix} \dfrac{1}{k+\mathrm{j}\omega_{nf}c_s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k+\mathrm{j}\omega_{nf}c_s & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{k+\mathrm{j}\omega_{nf}c_s} \\ 0 & 0 \end{bmatrix}$

where $[F]_n$ is a field matrix at station n. From equations (6.100) and (6.106), we get

$${}_R\{S\}_n = [P]_n\,{}_L\{S\}_n = [P]_n[F]_n\,{}_R\{S\}_{n-1} = [U]_n\,{}_R\{S\}_{n-1} \tag{6.107}$$

with

$$\left[U\right]_n = \begin{bmatrix} 1 & 0 \\ -\omega_{nf}^2 I_p + \mathrm{j}\omega_{nf}c_d & 1 \end{bmatrix}_n \begin{bmatrix} 1 & \dfrac{1}{k+\mathrm{j}\omega_{nf}c_s} \\ 0 & 0 \end{bmatrix}_n = \begin{bmatrix} 1 & \dfrac{1}{k+\mathrm{j}\omega_{nf}c_s} \\ -\omega_{nf}^2 I_p + \mathrm{j}\omega_{nf}c_d & 1+\dfrac{-\omega_{nf}^2 I_p + \mathrm{j}\omega_{nf}c_d}{k+\mathrm{j}\omega_{nf}c_s} \end{bmatrix}_n$$

where $[U]_n$ is a transfer matrix between stations n and (n-1). Once we have the point and field matrices, remaining analysis will remain the same for obtaining natural frequencies, mode shapes, and forced responses. Only difference would be that now we need to handle the complex numbers. Such analysis with damped multi-DOF could be performed relatively simpler way with FEM and it will be discussed subsequently.

6.9 Modelling of Reciprocating Machine Systems

Till now we considered various machines that have components with pure rotary motions. Advantage such rotating machineries are that they do not have as such variable polar moment inertias. Another class of machineries that have possibility of torsional vibrations is reciprocating machines. A multi-cylinder reciprocating machine contains many reciprocating and rotating parts such as pistons, connecting rods, crankshafts, flywheels, dampers, and couplings. The system is so complicated that it is difficult, if not impossible, to undertake an exact analysis of its torsional vibration characteristics. The actual system is characterised by the presence of unpredictable effects like variable inertia, internal dampings, fluid-film bearing forces, misalignments in the transmission units, uneven firing order, etc. (Wilson, 1956, 1963 and 1965; Rao, 1996)

The analysis can be best carried out, by lumping the inertias of rotating and reciprocating parts at discrete points on the main shaft. The problem then reduces to the forced torsional vibration study of an multi-DOF rotor system subjected to varying torques at different cylinder points. The crankshaft and the other drive or driven shafts are generally flexible in torsion, but have low polar moments of inertia, unlike in the case of some large turbines or compressors. On the other hand, parts mounted on the shafting, like the damper, flywheel, generator etc. are rigid and will have very high polar moments of inertia. The system containing the crankshaft, coupling, generator, auxiliary drive shaft, other driven shaft like pumps, and mounted parts can then be reduced to a simple system with a series of rigid rotor (representing the inertias) connected by the massless flexible shafts as shown in Figure 6.50. Now simple procedures will be described to reduce reciprocating inertias to equivalent rotating inertias, the uneven crack shaft geometry to an equivalent uniform shaft system, and the conversion of periodic torque variation to its components.

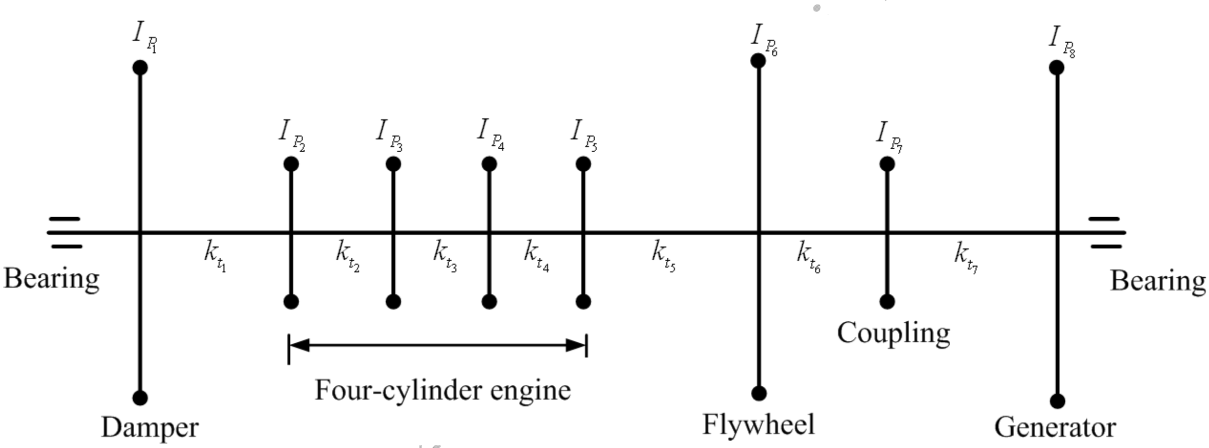


Figure 6.50 A rotor model with *N*-disc of a typical reciprocating engine installation

6.9.1 An equivalent polar mass moment of inertia

Determination of a polar mass moment of inertia is a straightforward matter for rotating parts, however, it is not quite so simple in the case of reciprocating parts. Consider the piston shown in two different positions in Figure 6.51 and let us imagine the crankshaft with a polar mass moment of inertia, $m_{rev}\, r^2$, where m_{rev} is the total revolving mass at the crack radius r (it is also called crack through). It includes all the revolving part of the crank m_{cr}, and the only the revolving part of the connecting rod m_{cn}^{rev} (when its two mass equivalent dynamic system is considered). Let us assume that the crack is not revolving, however, it is executing small torsional oscillations about the mean position shown in diagrams.

In first case (Fig. 6.51a) there is no motion for the piston, with small oscillations of the crank and hence the equivalent polar mass moment of inertia of the piston is zero. Whereas in second case (Fig. 6.51(b)), the piston has practically the same acceleration as that of the crank pin and the equivalent polar moment inertia is $m_{rec}\, r^2$, where m_{rec} is the mass of the reciprocating parts. It includes all the mass of the piston m_p, and only the reciprocating part of the connecting rod m_{cn}^{rec} when it is converted to a two-mass equivalent dynamic system. Hence, the total polar mass moment of inertia varies from $m_{rev}\, r^2$ to ($m_{rev}\, r^2 + m_{rec}\, r^2$), when the crankshaft is rotating.

The inertia of connecting rod can be obtained by considering a two-mass equivalent dynamic system with mass one at piston, m_{cn}^{rec} , and other mass at crank pin, m_{cn}^{rev} . With some approximation (for more accuracy refer to Bevan, 1984) the mass of the connecting rod m_{cn} can be considered as two mass system one at the piston of magnitude $m_{cn}^{rec} = m_{cn} a / l$, and another at the crack radius of magnitude $m_{cn}^{rev} = m_{cn} c / l$, where l is the length of the connecting rod, c is the distance from the piston pin to the center of gravity of the connecting rod, and a is the distance from the crack pin to the center of gravity of the connecting rod (i.e., $l = a + c$). It will contribute to both m_{rev} and m_{rec} by small amount. We consider as an approximation in the system to have an average inertia given by

$$I_p = m_{rev} r^2 + 0.5 m_{rec} r^2 \tag{6.108}$$

with

$$m_{rev} = m_{cr} + m_{cn}^{rev} \qquad \text{and} \qquad m_{rec} = m_p + m_{cn}^{rec} \tag{6.109}$$

where r is crank radius.

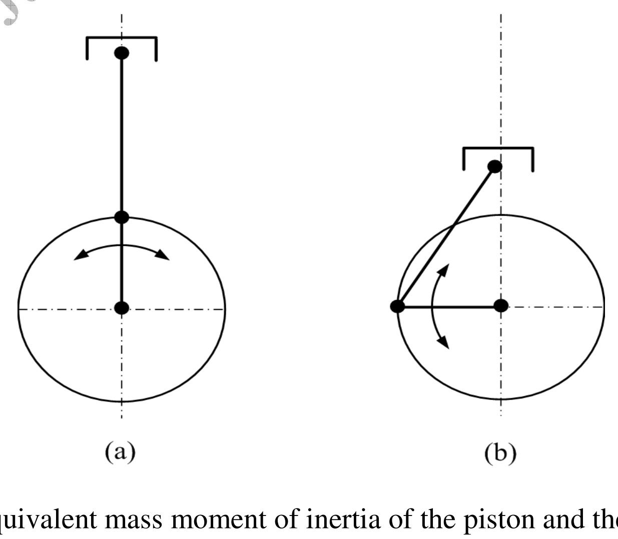


Figure 6.51 An equivalent mass moment of inertia of the piston and the connecting rod

6.9.2 Equivalent torsional stiffness of crack shafts

In determining the torsional stiffness of shafts connecting rotors, the main difficulty arises from the crank webs. Considering a crank shaft into an equivalent ordinary shaft having the same flexibility as the original one, as shown in Figure 6.52. Through this idealisation is physically possible, but the calculations involved are extremely difficult. This is because the crank webs are subjected to bending and the crank pin to twisting, when the main shaft is subjected to twisting. Moreover, the beam formulae, if used will not very accurate, because of short stubs involved rather than long beams usually considered. Further torques applied at the free end also give rise to sidewise displacement, i.e. coupled beading-torsion exists; which is prevented in the machine. For high-speed lightweight engines, the crank webs are no more rectangular blocks and application of the theory becomes extremely difficult. Because of this uncertainties in analytical calculation to estimate the torsional stiffness of crank throws, several experiments have been carried out on a number of crank shafts of large slow speed engines, which have shown that the equivalent length l_e is nearly equal to the actual length, if the diameter of main shaft is equal to the crank pin diameter.

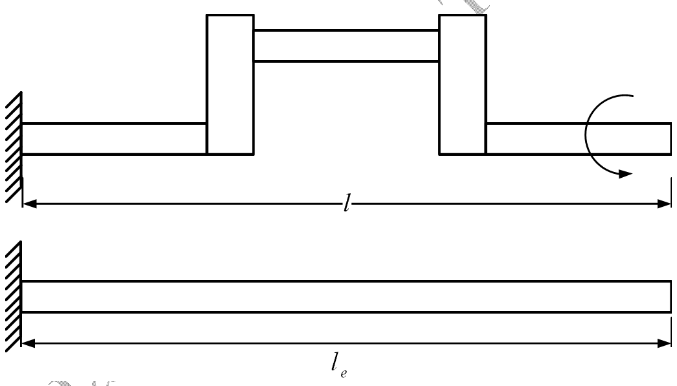


Figure 6.52 Equivalent length of a crank

In general the procedure that is applied to reduce the reciprocating machine system to a mathematical model, is to use a basic diameter, which corresponding to the journal diameter of the crankshaft. The torsional stiffness is all calculated based on the basic diameter, irrespective of their actual diameter. For the end rotors (i.e., the generator rotor) compute the stiffness of the shaft from the coupling up to the point of rigidity. In case where one part of the system is connected to the other part through gears, or other transmission units, it is convenient to reduce all the inertias and stiffness to one reference speed. Finite element methods can be used to obtain equivalent stiffness of the crackshafts. Once the mathematical model is developed, it can be used in illustrating the critical speed calculations, and forced vibration responses.

6.9.3 Torque variations in a reciprocating machinery

Torsional oscillation in the crankshaft and in the shafting of driven machinery is vibration phenomenon of practical importance in the design of reciprocating engines. The average torque delivered by a cylinder in a reciprocating machine, is a small fraction of the maximum torque, which occurs during the firing period. Even though the torque is periodic the fact that it fluctuates so violently within the period, constitutes one of the inherent disadvantages of a reciprocating machine, from the dynamics point of view, as compared with a turbine where the torque is practically uniform. It is possible to express the torque by a reciprocating engine into its harmonic components of several orders of the engine speed, and these harmonic components can excite the engine driven installations into forced torsional vibrations. The engine and the driven unit such as generator or a pump are normally connected by a flexible coupling and thus the total installation has fairly low natural frequencies falling in the speed range of the engine and the harmonics of different order. It is a commonly known fact that failures can occur in reciprocating machine installations, when the running speed of the engine is at or near a dominant torsional critical speed of the system. High dynamic stresses can occur in the main shafting of such engine installations and to avoid these conditions, it is essential that the torsional vibration characteristics of the entire installation be analysed before the unit is put into operation. Any analysis of torsional vibration characteristics of reciprocating machinery should finally predict the maximum dynamic stresses or torque developed in the shafting and couplings of the system, as accurately as possible, so that they can be compared with the permissible values, to check the safety of installation.

Example 6.14 A marine reciprocating engine, flywheel and propeller are approximately equivalent to the following three-rotor system. The engine has a crack 50 cm long and a connecting rod 250 cm long. The engine revolving parts are equivalent to 50 kg at crank radius, and the piston and pin masses are 41 kg. The connecting rod mass is 52 kg and its center of gravity is 26 cm from the crankpin center. The mass of the flywheel is 200 kg with the radius of gyration of 25 cm. The propeller has the polar mass moment of inertia of 6 kg-m^2. The equivalent shaft between the engine masses and the flywheel is 38 cm diameter and 5.3 m long and that between the flywheel and the propeller is 36 cm diameter and 1.5 m long. Find the natural frequencies of the torsional vibrations of the system.

Solution: The main aim of the present solution procedure would be to first find the equivalent rotating mass of the reciprocating engine, once it has been done then the problem will reduce to obtaining the natural frequencies of a three-rotor system as shown in Figure 6.53. The three revolving masses are corresponding to the reciprocating engine, flywheel, and propeller.

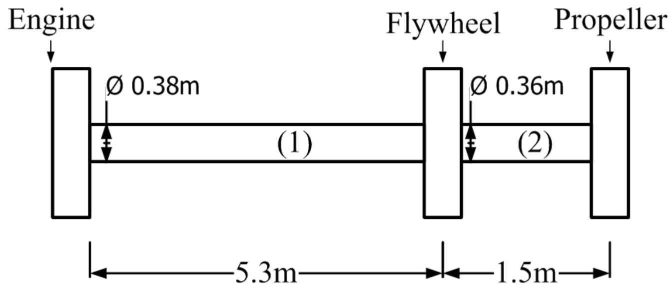


Figure 6.53 A three-disc model of the engine, flywheel and propeller

The equivalent rotating part of the engine can be obtained as follows. We have the following engine data:

Crank radius, r = 0.5 m, Mass of crank revolving parts, m_{cr} = 50 kg at the crank radius

Length of connecting rod l = 2.5 m, Mass of the connecting rod, m_{cn} = 52 kg,

Distance of center of gravity of the connecting rod from the crack pin, a = 0.26 m

Distance of center of gravity of the connecting rod from the piston pin, c = 2.24 m

Piston and pin masses, m_p = 41 kg

Hence, the equivalent (approximate) reciprocating and revolving masses of the connecting rod would be

$$m_{cn}^{rec} = m_{cn} a/l = 52 \times 0.26/2.5 = 5.41 \text{ kg}; \qquad m_{cn}^{rev} = m_{cn} c/l = 52 \times 2.24/2.5 = 46.59 \text{ kg}$$

The equivalent revolving and reciprocating masses are given as

$$m_{rev} = m_{cr} + m_{cn}^{rev} = 50 + 46.59 = 96.59 \text{ kg}$$

and

$$m_{rec} = m_p + m_{cn}^{rec} = 41 + 5.41 = 46.41 \text{kg}$$

Hence, the equivalent polar mass moment of inertia of the engine is obtained as

$$I_{p_e} = (m_{rev} + 0.5 m_{rec}) r^2 = (96.59 + 0.5 \times 46.41) \times 0.5^2 = 29.95 \text{kg-m}^2$$

Now, the polar mass moment of inertia of the flywheel $I_{p_f} = m r_f^2 = 200 \times (0.25)^2 = 12.5$ kg-m^2

For the propeller, the polar mass moment of inertia is given as $I_{p_p} = 6$ kg-m^2.

The torsional stiffness of shaft segments (1) and (2) are given as

$$k_{t_1}=\frac{G_1 J_1}{l_1}=\frac{78.9\times10^9\times\pi\times0.38^4}{32\times5.3}=3\times10^7 \text{ N-m/rad}$$

and

$$k_{t_2}=\frac{G_2 J_2}{l_2}=\frac{78.9\times10^9\times\pi\times0.36^4}{32\times1.5}=8.67\times10^7 \text{ N-m/rad}$$

Natural frequencies of three-disc rotor system (with $I_{p_e}=I_{p_1}$, $I_{p_f}=I_{p_2}$ and $I_{p_p}=I_{p_3}$) are given as (equation (6.110))

$$\omega_{nf_1}=0$$

and

$$\omega_{nf_{2,3}}^2=\frac{1}{2}\left(k_{t_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}}+k_{t_2}\frac{I_{p_2}+I_{p_3}}{I_{p_2}I_{p_3}}\right)\pm\sqrt{\frac{1}{4}\left(k_{t_1}\frac{I_{p_1}+I_{p_2}}{I_{p_1}I_{p_2}}+k_{t_2}\frac{I_{p_2}+I_{p_3}}{I_{p_2}I_{p_3}}\right)^2-\left(\frac{k_{t_1}k_{t_2}\left(I_{p_1}+I_{p_2}+I_{p_3}\right)}{I_{p_1}I_{p_2}I_{p_3}}\right)}$$

which gives

$$\omega_{nf_1}=0 \qquad \omega_{nf_2}=1.58\times10^3 \text{ rad/s} \qquad \omega_{nf_3}=4.72\times10^3 \text{ rad/s}$$

Finding of mode shapes and the position of nodes is left to the reader as a practice problem.

Concluding remarks

To summarise, now we have clear idea about torsional natural frequencies and mode shapes for simple rotor systems. We have obtained torsional natural frequencies and mode shapes using Newton's second law of motion, and using the systematic transfer matrix method (TMM). The TMM is found to be quite versatile and easy in application especially for the multi-DOF rotor systems. The TMM is also developed for rotor system with damping in the disc due to aerodynamic forces and in the shaft due to material damping (both the damping models are taken as viscous damping). Apart form these simple rotor systems, we considered the geared and branched systems for obtaining torsional natural frequencies. For the multi-DOF geared and branched systems, the TMM is applied because of its simplicity in the application. At the end the procedure of obtaining the equivalent rotor system from multi-cylinder reciprocating engines is briefly discussed and for detailed analysis a brief literature review is given.

Exercise Problems

Exercise 6.1 Find torsional natural frequencies and mode shapes of the two-disc rotor system shown in Figure E6.1 by using the transfer matrix method. B_1 and B_2 are frictionless bearings, and D_1 and D_2 are rigid discs. The shaft is made of steel with the modulus of rigidity G = 0.8(10)^{11} N/m^2, and a uniform diameter d = 10 mm. Various shaft lengths are as follows: B_1D_1 = 50 mm, D_1D_2 = 75 mm, and D_2B_2 = 50 mm. The polar mass moment of inertia of discs is: I_{p_1} = 0.0008 kg-m2 and I_{p_2} = 0.002 kg-m^2. Consider the shaft as massless. [Answer: $\omega_{nf_1} = 0$, $\omega_{nf_2} = 1354$ rad/s]

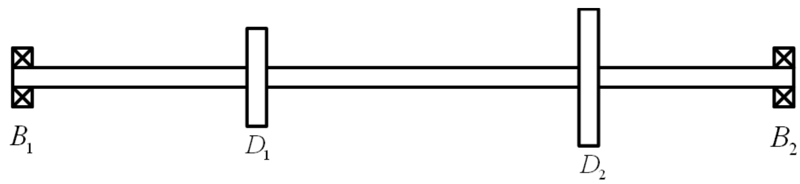


Figure E6.1

Exercise 6.2 Obtain the torsional natural frequency of an overhung rotor system as shown in Figure E6.2. The end B_1 of the shaft has a fixed end condition. The shaft diameter is 10 mm, and total length of the shaft is 0.2 m. The polar mass moment of inertia equal to 0.02 kg-m^2. Neglect the mass of the shaft. Use the transfer matrix method.

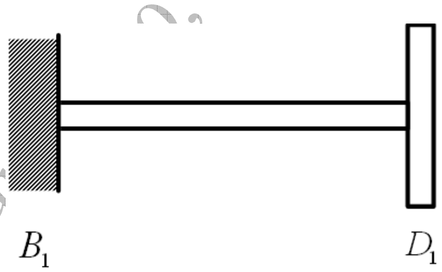


Figure E6.2

Exercise 6.3 Find the torsional natural frequencies and mode shapes of a rotor system as shown in Figure E6.3 by using the transfer matrix method. B_1 and B_2 are fixed supports, and D_1 and D_2 are rigid discs. The shaft is made of steel with the modulus of rigidity of G = 0.8 (10)^{11} N/m^2 and has uniform diameter of d = 10 mm. Different shaft lengths are as follows: B_1D_1 = 50 mm, D_1D_2 = 75 mm, and D_2B_2 = 50 mm. The polar mass moment of inertia of discs is: I_{p_1} = 0.08 kg-m^2 and I_{p_2} = 0.2 kg-m^2. Consider the shaft as massless. [*Answer*: $\omega_{nf_1} = 100.29$ rad/s, $\{\varphi_z\}_1 = \{1 \;\; 1.73\}^T$; $\omega_{nf_2} = 189.05$ rad/s, $\{\varphi_z\}_2 = \{1 \;\; 1/4.33\}^T$]

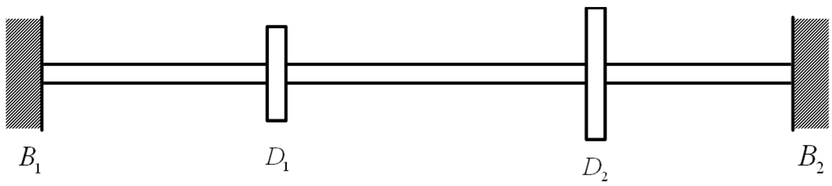


Figure E6.3

Exercise 6.4 Find all torsional natural frequencies and draw corresponding mode shapes of the rotor system shown in Figure E6.4. B_1 is fixed supported (with zero angular displacement about shaft axis) and B_2 and B_3 are simply supported (with non-zero angular displacements). The shaft is made of steel with G = 0.8×10[11] N/m[2], and uniform diameter d = 10mm. The various shaft lengths are as follows: $B_1D_1 = 50\text{mm}$; $D_1B_2 = 50\text{mm}$; $B_2D_2 = 25\text{mm}$; $D_2B_3 = 25\text{mm}$; $B_3D_3 = 30\text{mm}$. The polar mass moment of inertia of discs is: $I_{p_1} = 0.002$ kg-m^2; $I_{p_2} = 0.001$ kg-m^2, and $I_{p_3} = 0.008$ kg-m^2; Use the transfer matrix method. Give all detailed steps involved in obtaining the final system of equation and application of boundary conditions. Consider the shaft as mass-less and discs are lumped masses.

[Answer: $\omega_{nf_1} = 518.1\text{Hz}$, $\omega_{nf_2} = 1184.6\text{Hz}$, $\omega_{nf_3} = 1977\text{Hz}$

$$\begin{Bmatrix} \varphi_{z0} \\ \varphi_{z1} \\ \varphi_{z2} \\ \varphi_{z3} \end{Bmatrix}_{\omega_{nf_1}} = \begin{Bmatrix} 0 \\ 1 \\ 1.984 \\ 2.344 \end{Bmatrix}, \quad \begin{Bmatrix} \varphi_{z0} \\ \varphi_{z1} \\ \varphi_{z2} \\ \varphi_{z3} \end{Bmatrix}_{\omega_{nf_2}} = \begin{Bmatrix} 0 \\ 1.000 \\ -1.791 \\ -0.835 \end{Bmatrix}, \quad \begin{Bmatrix} \varphi_{z0} \\ \varphi_{z1} \\ \varphi_{z2} \\ \varphi_{z3} \end{Bmatrix}_{\omega_{nf_3}} = \begin{Bmatrix} 0 \\ 1.000 \\ -2.95 \\ 2.49 \end{Bmatrix}]$$

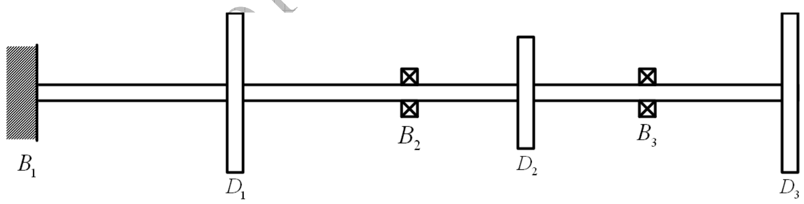


Figure E6.4 A multi-support multi-disc rotor system

Exercise 6.5 Obtain torsional natural frequencies of a turbine-coupling-generator rotor system as shown in Figure E6.5 by the transfer matrix method. The rotor is assumed to be supported on frictionless bearings. The polar mass moment of inertia of the turbine, coupling and generator is I_{p_T} = 25 kg-m^2, I_{p_C} = 5 kg-m^2 and I_{p_G} = 50 kg-m^2, respectively. Take the modulus of rigidity of the shaft as $G = 0.8 \times 10^{11}$ N/m^2. Assume the shaft diameter throughout equal to 0.2 m, and lengths of shafts between the bearing-turbine-coupling-generator-bearing are 1 m each so that the total span is 4 m. The coupling also gives a point flexibility (inverse of stiffness) equivalent to 5 times that of a shaft with 1 m of length and 0.2m of diameter. Consider the shaft as massless.

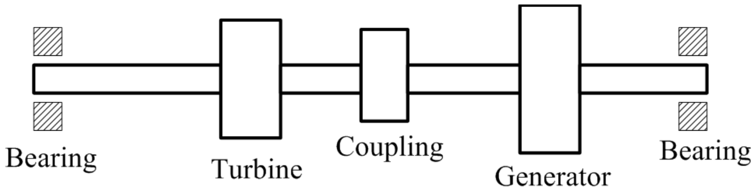


Figure E6.5 A turbine-generator set

Exercise 6.6 Obtain the torsional natural frequency of an overhung rotor system as shown in Figure E6.6 Take the polar mass moment of inertia of the disc as, I_p = 0.04 kg-m^2. The massless shaft has following properties: lengths are a = 0.3 m, b = 0.7 m, the uniform diameter is 10 mm, and the modulus of rigidity $G = 0.8 \times 10^{11}$ N/m^2. Bearing 'A' is flexible and provides a torsional restoring toque with its torsional stiffness equal to 5 percent of the torsional stiffness of the shaft segment having length a. Consider bearing B is a fixed bearing. Use both the direct and transfer matrix methods. [Hint: We need to find the effective torsional stiffness of the rotor system].

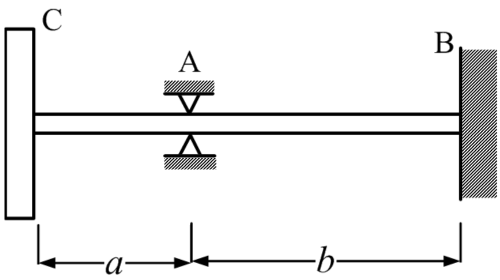


Figure E6.6 An overhung rotor system with an intermediate support

Exercise 6.7 A small electric motor drives another through a long coil spring (n turns, wire diameter d, coil diameter D). The two motor rotors have inertias I_{p_1} and I_{p_2}. Calculate torsional natural frequencies of the set-up. Assuming the ends of the spring to be "built-in" to the shafts. [Hint: Consider the system as a two-mass rotor system and the stiffness of a spring is given as $k_t = \dfrac{Gd^4}{8D^3 n}$, *Answer*: $\omega_{nf_1} = 0$ and $\omega_{nf_2} = \sqrt{\dfrac{k_t\left(I_{p_1} + I_{p_2}\right)}{I_{p_1} I_{p_2}}}$].

Exercise 6.8 For a rotor system with a stepped circular shaft as shown in Figure E6.8 obtain the torsional natural frequencies, mode shapes, and nodal positions. Consider the free-free end conditions. Neglect the polar mass moment of inertia of the shaft and take $G = 0.8 \times 10^{11}$ N/m^2. Use the direct method, the indirect method (based on the node location information of mode shapes), and the transfer matrix method.

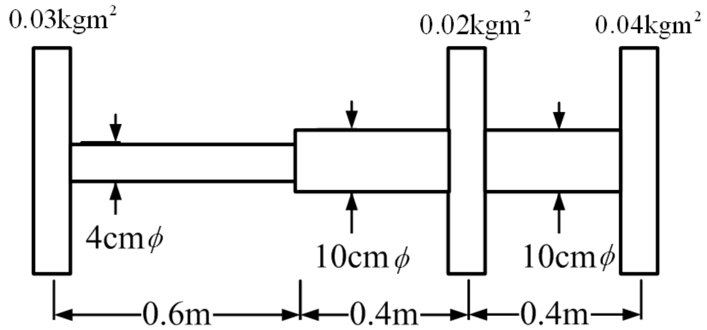


Figure E6.8

Exercise 6.9 A marine reciprocating engine, flywheel, and propeller are approximately equivalent to the following three-rotor system. The engine has a crack 50 cm long and a connecting rod 250 cm long. The engine revolving parts are equivalent to 50 kg at crank radius and the piston and pin masses are 41 kg. The connecting rod mass is 52 kg and its center of gravity is 26 cm from the crankpin center. The mass of the flywheel is 200 kg with the radius of gyration of 25 cm. The propeller has the polar mass moment of inertia of 6 kg-m^2. The equivalent shaft between the engine masses and the flywheel is 38 cm diameter and 5.3 m long, and that between the flywheel and the propeller is 36 cm diameter and 11.5 m long. Find torsional natural frequencies of the rotor system and the position of the nodes.

Exercise 6.10 For a geared system as shown in Figure E6.10 find the torsional natural frequencies and mode shapes. Find also the location of nodal point on the shaft (if any). The shaft 'A' has 1.5 cm diameter and 0.3 m length and the shaft 'B' has 1 cm diameter and 0.4 m length. Take modulus of rigidity of the shaft G equals to 0.8×10^{11} N/m^2, the polar mass moment of inertia of discs and gears are $I_{p_A} = 0.1\,\text{Nm}^2$, $I_{p_B} = 0.08\,\text{Nm}^2$, $I_{p_{gA}} = 0.003\,\text{Nm}^2$, $I_{p_{gB}} = 0.002\,\text{Nm}^2$. Use (i) equivalent system approach and (ii) transfer matrix method.

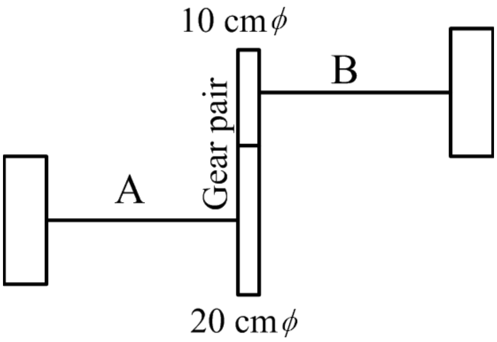


Figure E6.10 A geared rotor system

Exercise 6.11 Obtain torsional natural frequencies and mode shapes of an epi-cyclic gear train as shown in Figure E6.11. Find also the location of nodal point on the shaft. The gear mounted on shaft 'B' is a planetary gear and the gear on shaft 'A' is a sun gear. Consider the polar mass moment of inertia of the shaft, the arm and gears as negligible. Shaft 'A' has 5 cm of diameter and 0.75 m of length and shaft 'B' has 4 cm of diameter and 1.0 m of length. Angular speeds of shaft A and the arm are 300 rpm and 100 rpm, respectively. Take the modulus of rigidity of the shaft G equals to 0.8 × 10^{11} N/m^2, the polar mass moment of inertia of discs are $I_{p_A} = 24$ Nm^2 and $I_{p_B} =$ 10 Nm^2. State the assumptions made in the analysis. [n_{AB} = 1, $\omega_{nf_1} = 0$, $\{u\}_1 = \{1\ \ 1\ \ -1\}^{\mathrm{T}}$; $\omega_{nf_2} = 53.34$ rad/s, $\{u\}_2 = \{1\ \ 0.04\ \ 3.37\}^{\mathrm{T}}$; node location from left hand side x = 0.72 m].

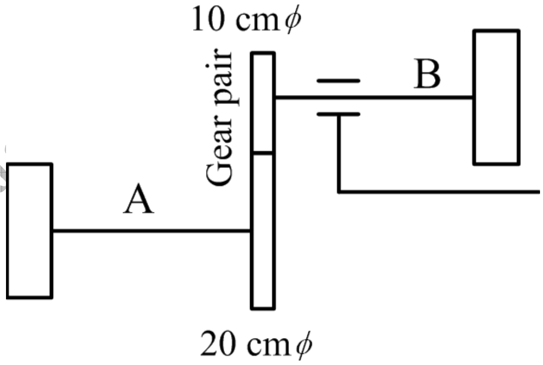
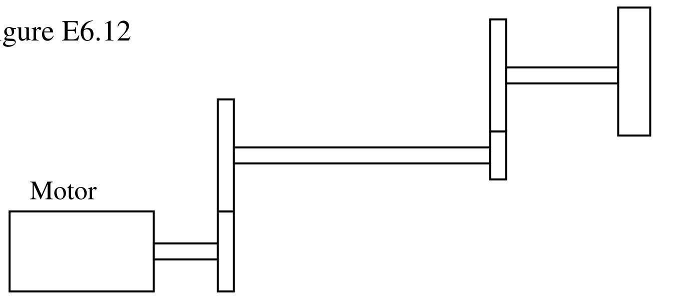


Figure E6.11 An epi-cyclic geared system

Exercise 6.12 For a gear train as shown in Figure E6.12, obtain torsional natural frequencies and the location of the node. Dimensions of shafts are as follows (i) motor shaft: 0.20 m length and 0.015 m diameter, (ii) flywheel shaft: 0.2 m of length and 0.01 m of diameter, and (iii) intermediate shaft: 0.4 m of length and 0.012 m of diameter. The polar mass moment of inertia of the motor and flywheel are

$I_{p_m} = 0.01$ kg-m^2 and $I_{p_m} = 0.04$ kg-m^2, respectively. Gear ratio of the first and second gear pairs are 3 and 4, respectively. Neglect inertias of gears and mass of shafts. Assume the free-free end conditions and all shafts are mounted on frictionless bearings. Take $G = 0.8\times10^{11}$ N/m^2. Use the TMM.

Figure E6.12



Exercise 6.13 A stepped-shaft consists of three segments with lengths of 40 cm, 30 cm and 40 cm; and corresponding diameters as, *d* cm, 13 cm and *d* cm, where *d* is an unknown. The shaft has two flywheels ($I_{p_1} = 11$ kg-m^2 and $I_{p_2} = 11$ kg-m^2, with radius of gyration of both flywheel equals to 0.5 m) at the ends, and the shaft is supported on two frictionless rolling bearings at 20 cm away from the either ends. The operating speed of the shaft is 1500 rpm and due to rotation of the shaft it has external torque impulses such that it has period corresponding to the quarter of the shaft rotation. Obtain the diameter, *d*, such that the torsional critical speed may be 20% above the external torque frequency (fundamental harmonics). Obtain the transverse natural frequency of rotor system, so designed based on the dynamics of the rotor in torsion. Neglect the mass of the shaft, and take $G = 0.8\times10^{11}$ N/m^2, and $E = 2.1\times10^{11}$ N/m^2.

Exercise 6.14 A motor has rotating masses of the polar mass moment of inertia of 58 kg-m^2, which is connected to one end of a shaft of 6 cm diameter and 2.30 m long. At the other end a flywheel and pinion are attached, with the effective polar mass moment of inertia of 220 kg-m^2. The pinion is connected to a gear with a gear ratio of 4 and of the polar mass moment of inertia of 70 kg-m^2, which drives a pump. The measured torsional vibration frequency of the rotor system is 60 Hz. Find the effective polar mass moment of inertia of the pump impeller and entrained water. Take $G = 0.8\times10^{11}$ N/m^2.

Exercise 6.15 A cantilever shaft of 1 m length (l), and 30 mm diameter (d) has a thin disc of 5 kg mass (m) attached at its free end, with the disc radius of 5 cm. The shaft has a through hole parallel to the shaft axis of diameter 3 mm (d_i), which is vertically below the shaft center, with the distance between the centres of the shaft and the hole as 6 mm (e). Consider no warping of the plane; and obtain the torsional natural frequencies of the shaft system. Consider the shaft as massless and modulus of rigidity G = 0.8× 10^{11} N/m^2. [*Hint*: Find the equivalent stiffness of the shaft and then obtain natural frequencies: $\omega_{nf} = \sqrt{k_{t_{eq}} / I_p}$ $k_{t_{eq}} = \frac{GJ}{l}$, $J = I_1 + I_2 = \frac{\pi}{32}\left(d^4 - d_i^4 - 8d_i^2 e^2\right) = 7.926 \times 10^{-8}$ m^4; $\omega_{nf} = 1007.23$ rad/s]

Exercise 6.16 Find torsional natural frequencies and mode shapes of the rotor system shown in Figure E6.16. B is a fixed bearing, which provide fixed support end condition; and D_1, D_2, D_3 and D_4 are rigid discs. The shaft is made of the steel with the modulus of rigidity G = 0.8 $(10)^{11}$ N/m^2 and the uniform diameter d = 20 mm. Various shaft lengths are as follows: D_1D_2 = 50 mm, D_2D_3 = 50 mm, D_3D_4 = 50 mm and D_4B_2 = 150 mm. The mass of discs are: m_1 = 4 kg, m_2 = 5 kg, m_3 = 6 kg and m_4 = 7 kg. Consider the shaft as mass-less. Consider discs as thin and take diameter of discs as $d_1 = 12$ cm, $d_2 = 6$ cm, and $d_3 = 12$ cm, $d_4 = 14$ cm.

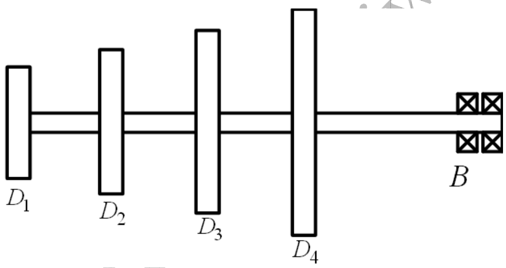


Figure E6.16 A multi-disc overhung rotor

Exercise 6.17 Find torsional natural frequencies and mode shapes of the rotor system shown in Figure E6.17. B is a fixed bearing, which provide fixed support end condition; and D_1, D_2, D_3, D_4 and D_5 are rigid discs. The shaft is made of the steel with the modulus of rigidity G = 0.8 $(10)^{11}$ N/m^2 and the uniform diameter d = 20 mm. Various shaft lengths are as follows: D_1D_2 = 50 mm, D_2D_3 = 50 mm, D_3D_4 = 50 mm, D_4D_5 = 50 mm, and D_5B_2 = 50 mm. The mass of discs are: m_1 = 4 kg, m_2 = 5 kg, m_3 = 6 kg, m_4 = 7 kg, and m_4 = 8 kg. Consider the shaft as massless. Tow cases to be considered (i) Consider the disc as point masses, i.e., neglect the diametral and polar mass moment of inertia of all discs; (ii) consider discs as thin and take diameter of discs as $d_1 = 12$ cm, $d_2 = 6$ cm, and $d_3 = 12$ cm, $d_4 = 14$ cm, and $d_5 = 16$, however, neglect the gyroscopic effects.

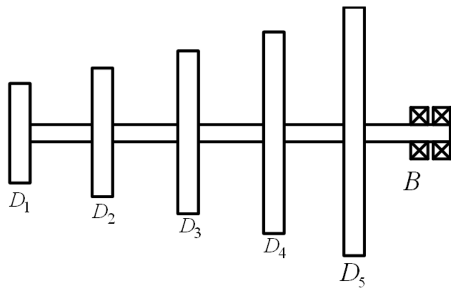


Figure E6.17 A multi-disc overhung rotor

Exercise 6.18 Find torsional natural frequencies and mode shapes of the rotor system shown in Figure E6.18. B_1 and B_2 are bearings, which provide free-free end condition and D_1, D_2, D_3, D_4 and D_5 are rigid discs. The shaft is made of the steel with the modulus of rigidity $G = 0.8\ (10)^{11}$ N/m^2 and a uniform diameter d = 20 mm. Various shaft lengths are as follows: B_1D_1 = 150 mm, D_1D_2 = 50 mm, D_2D_3 = 50 mm, D_3D_4 = 50 mm, D_4D_5 = 50 mm, and D_5B_2 = 150 mm. The mass of discs are: m_1 = 4 kg, m_2 = 5 kg, m_3 = 6 kg, m_4 = 7 kg, and m_5 = 8 kg. Consider the shaft as massless. Consider discs as thin and take diameter of discs as $d_1 = 8$ cm, $d_2 = 10$ cm, $d_3 = 12$ cm, $d_4 = 14$ cm, and $d_5 = 16$.

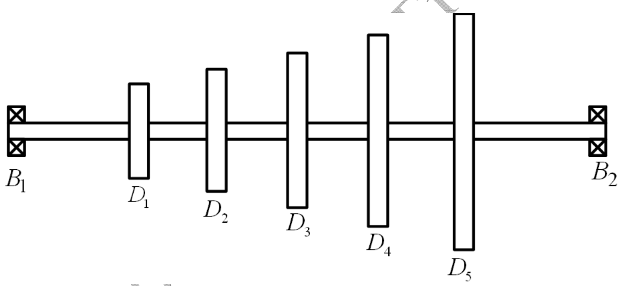


Figure E6.18 A multi-disc rotor system with simply supported end conditions

Exercise 6.19 Find torsional natural frequencies of an overhung rotor system as shown in Figure E6.19. Consider the shaft as massless and is made of steel with the modulus of rigidity of 0.8(10)^{11} N/m^2. A disc is mounted at the free end of the shaft with the polar mass moment of inertia 0.01 kg-m^2. In the diagram all dimensions are in cm. Use the TMM.

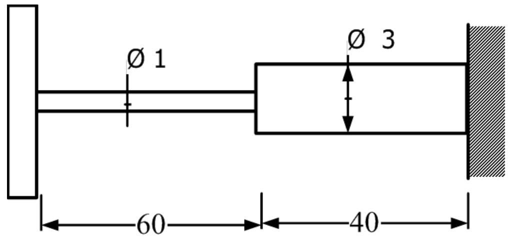


Figure E6.19

Exercise 6.20 Obtain torsional natural frequencies of a geared rotor system shown in Fig. E6.20. Consider the left hand side of shaft A and the right hand side of shaft B have rigid supports. Assume gears as thin discs with mass of 3 kg and 2 kg for gears of diameters 20 cm and 10 cm, respectively. The shaft A and B are respectively 2 cm and 1.5 cm diameters, respectively; with a length of 40 cm each. Neglect the inertia of shafts. The gear tooth-pair provides an effective torsional stiffness of 1 kN-m/rad between gears. Take $G = 0.8\times10^{11}$ N/m^2. Use TMM.

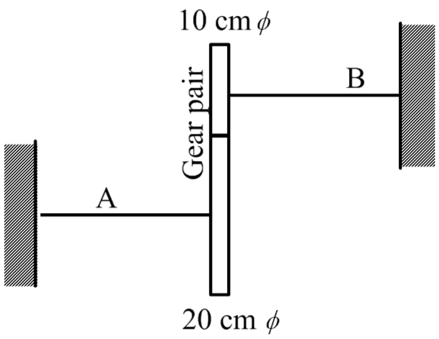


Figure E6.20

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